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Lyapunov-type inequalities for $2M$th order equations under clamped-free boundary conditions

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Abstract
This paper generalizes the well-known Lyapunov-type inequalities for second-order linear differential equations to certain $2M$th order linear differential equations

$$(-1)^M u^{(2M)}(x) - r(x)u(x) = 0 \quad (-s \le x \le s)$$

under clamped-free boundary conditions. The usage of the best constant of some kind of a Sobolev inequality helps clarify the process for obtaining the result.

1 Introduction

Let us consider the second-order linear differential equation

$$\begin{cases} u''(x) + r(x)u(x) = 0 & (-s \le x \le s), \\ u(\pm s) = 0, \end{cases} \tag{1}$$

where $r \in C((-s,s),\mathbb{R})$. It is well known that the Lyapunov inequality

$$\int_{-s}^{s} r^+(x)\,dx > \frac{2}{s} \tag{2}$$

gives a necessary condition for the existence of non-trivial classical solutions of (1), where $r^+(x) = (r(x) + |r(x)|)/2$. There are various extensions and applications for the above result; see, for example, surveys of Brown and Hinton [1] for relations to other fields and Tiryaki [2] for recent developments. Extensions to higher-order equations

$$\begin{cases} u^{(n)}(x) + r(x)u(x) = 0 & (-s \le x \le s), \\ \text{Boundary Conditions}, \end{cases} \tag{3}$$

will be one important aspect. The first result for the high-order equation (3) is due to Levin [3], which states without proof:

Theorem A *Let $n = 2M$, and a non-trivial solution of* (3) *satisfies the clamped boundary condition, $u^{(i)}(\pm s) = 0$ $(i = 0, 1, \dots, M-1)$. Then it holds that*

$$\int_{-s}^{s} r^{+}(x)\,dx > \frac{2^{2M-1}(2M-1)\{(M-1)!\}^2}{s^{2M-1}}.$$

Later, Das and Vatsala [4] gave the proof and extended the result by constructing the Green function. Other interesting developments for higher-order equations are seen in [5–9]. For example, as shown in Yang [8], Lyapunov-type inequalities can be obtained under the following conditions:

$$\begin{cases} \text{(a)} & n = 2M+1, u^{(i)}(\pm s) = 0 \ (i = 0, \dots, M-1), \\ & u^{(2M)}(d) = 0 \ (-s < d < s) \ \text{(see also [10])} \\ \text{(b)} & n \geq 2, u(-s) = u(t_2) = \cdots = u(t_{n-1}) = u(s) = 0, \\ & \text{where, } -s = t_1 < t_2 < \cdots < t_{n-1} < t_n = s \ \text{(see also [10])} \\ \text{(c)} & n = 2M, u^{(2i)}(\pm s) = 0 \ (i = 0, \dots, M-1) \ \text{(see also [5])} \\ \text{(d)} & n \geq 2, u^{(i)}(-s) = 0 \ (i = 0, \dots, k-1), u^{(j)}(s) = 0 \ (j = 0, \dots, n-k-1) \\ & \text{where } k \text{ runs over the range } (1 \leq k \leq n). \end{cases}$$

Here we note for the condition (c), very recently Çakmak [11], He and Tang [12], He and Zang [13] and [14] improved and extended the results of [5] and [8]. This paper considers the necessary condition for the existence of a non-trivial solution of the $2M$th order linear differential equation

$$(-1)^M u^{(2M)}(x) - r(x)u(x) = 0 \quad (-s \leq x \leq s) \tag{4}$$

under yet another boundary condition:

Clamped-free boundary condition

$$u^{(i)}(-s) = 0, \qquad u^{(M+i)}(s) = 0 \quad (i = 0, \dots, M-1).$$

The main result is as follows.

Theorem 1 *Suppose a non-trivial solution u of* (4) *exists under the clamped-free boundary condition, then it holds*

$$\int_{-s}^{s} r^{+}(x)\,dx > \frac{\{(M-1)!\}^2(2M-1)}{(2s)^{2M-1}}. \tag{5}$$

Moreover, the estimate is sharp in the sense that there exists a function $r(x)$, and for this $r(x)$, the solution u of (4) *exits such that the right-hand side is arbitrarily close to the left-hand side.*

The result is obtained using Takemura [15, Theorem 1], which computes the best constant of some kind of a Sobolev inequality. In Section 4, we give a concise proof for an L^p extension of Theorem 1 of [15].

2 Proof of Theorem 1

Now, let us introduce the following L^p-type Sobolev inequality:

$$\Bigl(\sup_{-s\le x\le s}\bigl|u^{(m)}(x)\bigr|\Bigr)^p \le C\int_{-s}^{s}\bigl|u^{(M)}(x)\bigr|^p\,dx, \tag{6}$$

where u belongs to

$$W(M,p) := \bigl\{u \,|\, u^{(M)}\in L^p(-s,s), u^{(i)}(-s)=0\ (i=0,\dots,M-1)\bigr\},$$

$1<p$, m runs over the range $0\le m\le M-1$, and $u^{(i)}$ is the ith derivative of u in a distributional sense. We denote by $C_{CF}(M,m,p)$ the best constant of the above Sobolev inequality (6). Here, we note that in [15], Takemura obtained the best constant for $p=2$, $m=0$ by constructing the Green function of the clamped-free boundary value problem. Although, for the proof of Theorem 1, we simply need the value $C_{CF}(M,0,2)$, we would like to compute $C_{CF}(M,m,p)$ for general p and m since the proof presented in Section 4 does not depend on special values of p and m and quite simplifies the proof of Theorem 1 of [15]. Now, we have the following propositions.

Proposition 1 *The best constant of* (6) *is*

$$C_{CF}(M,m,p) = \frac{1}{\{(M-m-1)!\}^p}\left(\frac{(p-1)(2s)^{\frac{p(M-m)-1}{p-1}}}{p(M-m)-1}\right)^{p-1}, \tag{7}$$

and it is attained by

$$u_*(x) = \int_{-s}^{x}\frac{(x-t)^{M-1}}{(M-1)!}\cdot\left\{\frac{(s-t)^{M-1-m}}{(M-1-m)!}\right\}^{p-1}dt. \tag{8}$$

Proposition 2 *Suppose a* $C^{2M}[-s,s]$ *solution of* (4) *with the clamped-free boundary condition exists, then it holds that*

$$\int_{-s}^{s} r^+(x)\,dx > \frac{1}{C_{CF}(M,0,2)}. \tag{9}$$

Moreover, the estimate is sharp.

Proof of Theorem 1 Clearly, Theorem 1 is obtained from Propositions 1 and 2. □

Thus, all we have to do is to show Propositions 1 and 2. Before proceeding with the proof of these propositions, we would like to show a corollary obtained from Proposition 1.

Corollary 1 *Suppose a non-trivial solution u of the non-linear equation*

$$(-1)^M u(x)u^{(2M)}(x) - r(x)\bigl(u^{(m)}(x)\bigr)^2 = 0 \tag{10}$$

exists under the clamped-free boundary condition, where m satisfies $(1\le m\le M-1)$, *then it holds*

$$\int_{-s}^{s} r^+(x)\,dx > \frac{\{(M-1-m)!\}^2(2(M-m)-1)}{(2s)^{2(M-m)-1}}. \tag{11}$$

The following are the examples of Theorem 1 and Corollary 1.

Example 1 The following example corresponds to the case $M = 1$ and $r(x) = -6/(-11s^2 + 2sx + x^2)$ of (4) with the clamped-free boundary condition

$$\begin{cases} -u''(x) + \frac{6}{-11s^2+2sx+x^2} u(x) = 0, \\ u(-s) = u'(s) = 0. \end{cases}$$

It is easy to see that $u(x) = -(s + x)(11s^2 - 2sx - x^2)$ is the solution of the above equation. Moreover, it holds that

$$\int_{-s}^{s} r^+(x)\,dx = \int_{-s}^{s} -\frac{6}{-11s^2 + 2sx + x^2}\,dx = \frac{\sqrt{3}\log(2+\sqrt{3})}{2s} > \frac{1}{C_{CF}(1,0,2)} = \frac{1}{2s}.$$

Example 2 The following example corresponds to the case $M = 2$, $m = 1$ and $r(x) = (3(17s^2 - 6sx + x^2))/(2(7s^2 - 4sx + x^2)^2)$ of (10) with the clamped-free boundary condition

$$\begin{cases} u(x)u^{(4)}(x) - \frac{3(17s^2-6sx+x^2)}{2(7s^2-4sx+x^2)^2} (u'(x))^2 = 0, \\ u(-s) = u'(-s) = u''(s) = u'''(s) = 0. \end{cases}$$

It is easy to see that $u(x) = (s + x)^2(17s^2 - 6sx + x^2)$ is the solution of the above equation. Moreover, it holds that

$$\int_{-s}^{s} r^+(x)\,dx = \int_{-s}^{s} \frac{3(17s^2 - 6sx + x^2)}{2(7s^2 - 4sx + x^2)^2}\,dx = \frac{3 + 2\sqrt{3}\pi}{12s} > \frac{1}{C_{CF}(2,1,2)} = \frac{1}{2s}.$$

3 Proof of Proposition 2

Assuming Proposition 1, we first prove Proposition 2.

Proof of Proposition 2 Let u be a solution of equation (4). Since u satisfies the clamped-free boundary condition, multiplying (4) by u and integrating it over $[-s, s]$, we have

$$\begin{aligned} \int_{-s}^{s} \left(u^{(M)}(x)\right)^2 dx &= \int_{-s}^{s} (-1)^M u(x)u^{(2M)}(x)\,dx = \int_{-s}^{s} r(x)\left(u(x)\right)^2 dx \\ &\le \left(\sup_{-s\le x\le s} \left|u(x)\right|\right)^2 \int_{-s}^{s} r^+(x)\,dx \\ &\le C(M,0,2) \int_{-s}^{s} \left(u^{(M)}(x)\right)^2 dx \int_{-s}^{s} r^+(x)\,dx. \end{aligned} \tag{12}$$

Here, if $u^{(M)} \equiv 0$, then there exists $a_i \in \mathbb{R}$ $(i = 0, \ldots, M-1)$ such that $u(x) = \sum_{i=0}^{M-1} a_i x^i$. Since u satisfies the clamped boundary condition at $x = -s$, we have $u \equiv 0$. This contradicts the assumption that u is a non-trivial solution of (4). So, canceling $\|u^{(M)}\|_2^2$, we obtain

$$\int_{-s}^{s} r^+(x)\,dx \ge \frac{1}{C_{CF}(M,0,2)}. \tag{13}$$

Next, we show that the inequality (13) is strict. To see this, we note that in (12), if the equality holds for the first inequality, then u is a constant. But, again from the clamped boundary condition at $x = -s$, we have $u \equiv 0$. Thus, the inequality is strict. Finally, we see (5) is sharp. For this purpose, let us define the functional

$$J(\phi) := \frac{\int_{-s}^{s} |\phi^{(M)}|^2\,dx}{\int_{-s}^{s} \tilde{r}|\phi|^2\,dx} \quad \bigl(\phi \in W(M,2), \phi \not\equiv 0\bigr),$$

where $\tilde{r} \in C([-s,s],\mathbb{R}_+)$ is defined later. By the standard argument of the variational method, J has the minimizer $u \in W(M,2)$ (see, for example, [16, Lemma 3]), *i.e.*,

$$\lambda_1 := \min_{\phi \in W(M,2), \phi \not\equiv 0} J(\phi) = J(u).$$

Hence, it satisfies the Euler-Lagrange equation (as a classical solution by the regularity argument)

$$(-1)^M u^{(2M)}(x) = \lambda_1 \tilde{r}(x) u(x) \quad (-s \leq x \leq s). \tag{14}$$

Further, it holds that

$$\begin{aligned} \lambda_1 &= \min_{\phi \in W(M,2), \phi \not\equiv 0} \frac{\int_{-s}^{s} |\phi^{(M)}|^2\,dx}{\int_{-s}^{s} \tilde{r}|\phi|^2\,dx} > \frac{\int_{-s}^{s} |\phi^{(M)}|^2\,dx}{(\sup_{-s \leq x \leq s} |\phi|)^2 \int_{-s}^{s} \tilde{r}\,dx} \\ &\geq \frac{1}{C_{CF}(M,0,2) \int_{-s}^{s} \tilde{r}\,dx}. \end{aligned} \tag{15}$$

Here, let us fix $\tilde{r}$ as

$$\tilde{r}(x) := \begin{cases} \frac{x-s}{\delta} + 1 & (s - \delta < x \leq s), \\ 0 & (-s \leq x \leq s - \delta). \end{cases}$$

For such $\tilde{r}$, let us substitute $\phi = u_*$ (of Proposition 1) into (15). It is easy to see that u_* takes its maximum at $x = s$, hence by taking δ sufficiently small, we see that the right-hand side of (15) can be arbitrarily close to the left-hand side, *i.e.*, for a small positive ϵ_1, λ_1 can be written as

$$\lambda_1 = \frac{1}{C_{CF}(M,0,2) \int_{-s}^{s} \tilde{r}\,dx} + \epsilon_1. \tag{16}$$

Putting $r = \lambda_1 \tilde{r}$, we see from (14) a solution u of

$$(-1)^M u^{(2M)}(x) = r(x) u(x) \quad (-s \leq x \leq s) \tag{17}$$

exists, and from (16) r satisfies

$$\int_{-s}^{s} r(x)\,dx = \frac{1}{C(M,0,2)} + \epsilon_1 \int_{-s}^{s} \tilde{r}\,dx = \frac{1}{C(M,0,2)} + \epsilon_2. \tag{18}$$

Hence, (5) is sharp. □

Proof of Corollary 1 Integrating equation (10), we have

$$\int_{-s}^{s}\left(u^{(M)}(x)\right)^2 dx = \int_{-s}^{s} r(x)\left(u^{(m)}(x)\right)^2 dx \le \Bigl(\sup_{-s\le x\le s}\left|u^{(m)}(x)\right|\Bigr)^2 \int_{-s}^{s} r^+(x)\,dx$$
$$\le C(M,m,2)\int_{-s}^{s}\left(u^{(M)}(x)\right)^2 dx \int_{-s}^{s} r^+(x)\,dx. \tag{19}$$

As in the proof of Proposition 2, by canceling $\|u^{(M)}\|_2^2$, we have

$$\int_{-s}^{s} r^+(x)\,dx \le \frac{1}{C_{CF}(M,m,2)}.$$

Next, we show that the inequality (11) is strict. To see this, we note that in (19), the equality holds for the first inequality if and only if $u^{(m)}$ is a constant. Hence, from the clamped boundary condition at $x = -s$, we have $u^{(m)} \equiv 0$. So, there exists $a_i \in \mathbb{R}$ ($i = 0,\dots,m-1$) such that $u(x) = \sum_{i=0}^{m-1} a_i x^i$. But, again from the clamped boundary condition at $x = -s$, we have $u \equiv 0$. Thus, inequality (11) is strict. □

4 Proof of Proposition 1

We prepare the following lemmas for the proof of Proposition 1.

Lemma 1 *Suppose there exists a function $u_* \in W(M,p)$ which attains the best constant $C(M,n,p)$ of* (6), *then it holds that*

$$\max_{-s\le x\le s}\left|u_*^{(m)}(x)\right| = \left|u_*^{(m)}(s)\right|.$$

Proof Suppose it holds that

$$\max_{-s\le x\le s}\left|u_*^{(m)}(x)\right| = \left|u_*^{(m)}(a)\right|,$$

where $a \not\equiv s$. Further, let us define

$$\tilde{u}(x) := \begin{cases} 0 & (-s \le x \le -a), \\ u_*(x+a-s) & (-a \le x \le s). \end{cases}$$

Then it holds $\tilde{u} \in W(M,p)$ and

$$\max_{-s\le x\le s}\left|\tilde{u}^{(m)}(x)\right| = \max_{-s\le x\le s}\left|u_*^{(m)}(x)\right| = \left|u_*^{(m)}(a)\right|$$

and $\|\tilde{u}^{(M)}\|_{L^p(-s,s)} < \|u_*^{(M)}\|_{L^p(-s,s)}$. Hence,

$$C(M,m,p) = \frac{(\max_{-s\le x\le s}|u_*^{(m)}(x)|)^p}{\|u_*^{(M)}\|_{L^p(-s,s)}} < \frac{(\max_{-s\le x\le s}|\tilde{u}^{(m)}(x)|)^p}{\|\tilde{u}^{(M)}\|_{L^p(-s,s)}}.$$

This contradicts the assumption that $C(M,m,p)$ is the best constant of (6). □

Lemma 2 *Let*

$$H_m(x) := \frac{(-1)^{M-1-m}(x-s)^{M-1-m}}{(M-1-m)!},$$

then for $u \in W(M,p)$ *it holds that*

$$u^{(m)}(s) = \int_{-s}^{s} u^{(M)}(x) H_m(x)\,dx. \tag{20}$$

Proof Integrating by parts, we obtain the result. □

Proof of Proposition 1 From Lemma 2, we see that if the function attains the best constant $C(M,m,p)$, it belongs to $W_*(M,m,p) \subset W(M,p)$:

$$W_*(M,m,p) = \left\{ u \in W(M,p) \Big| \max_{-s \le x \le s} \left| u^{(m)}(x) \right| = \left| u^{(m)}(s) \right| \right\}.$$

Let $u \in W_*(M,m,p)$. Then applying Hölder's inequality to (20), we have

$$\max_{-s \le x \le s} \left| u^{(m)}(x) \right| = \left| u^{(m)}(s) \right| \le \| H_m \|_{L^q(-s,s)} \left\| u^{(M)} \right\|_{L^p(-s,s)}, \tag{21}$$

where q satisfies $1/p + 1/q = 1$. Hence, if there exists the function $u_* \in W_*(M,m,p)$ which attains the equality of (21), it holds that $C(M,m,p) = \|H_m\|^p_{L^q(-s,s)}$. On the contrary, we see that the equality holds for (21) if and only if u satisfies

$$u^{(M)}(x) = \left(\operatorname{sgn} H_m(x) \right) \left| H_m(x) \right|^{q-1}. \tag{22}$$

It is easy to see that

$$u_*(x) = \int_{-s}^{x} \frac{(x-t)^{m-1}}{(M-1)!} \left(\operatorname{sgn} H_m(t) \right) \left| H_m(t) \right|^{q-1} dt = \int_{-s}^{x} \frac{(x-t)^{m-1}}{(M-1)!} \left\{ \frac{(s-t)^{M-1-m}}{(M-1-m)!} \right\}^{\frac{1}{p-1}} dt$$

satisfies (22) and belongs to $W_*(M,m,p)$. Thus, we have shown $C(M,m,p) = \|H_m\|^p_{L^q(-s,s)}$. Now, we compute $\|H_m\|^p_{L^q(-s,s)}$. It is

$$\begin{aligned} \|H_m\|^p_{L^q(-s,s)} &= \frac{1}{\{(M-1-m)!\}^p} \left\{ \int_{-s}^{s} (s-x)^{q(M-1-m)}\,dx \right\}^{\frac{p}{q}} \\ &= \frac{1}{\{(M-1-m)!\}^p} \left\{ \frac{(p-1)(2s)^{\frac{p(M-m)-1}{p-1}}}{p(M-m)-1} \right\}^{p-1}. \end{aligned}$$

This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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