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Solvability and iterative algorithms for a system of generalized nonlinear mixed quasivariational inclusions with (H_i, η_i)-monotone operators

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Abstract
In this paper, we introduce and discuss a new system of generalized nonlinear mixed quasivariational inclusions with (H_i, η_i)-monotone operators in Hilbert spaces, which includes several systems of variational inequalities and variational inclusions as special cases. By employing the resolvent operator technique associated with (H_i, η_i)-monotone operators, we suggest two iterative algorithms for computing the approximate solutions of the system of generalized nonlinear mixed quasivariational inclusions. Under certain conditions, we obtain the existence of solutions for the system of generalized nonlinear mixed quasivariational inclusions and prove the convergence of the iterative sequences generated by the iterative algorithms. The results presented in this paper extend, improve and unify many known results in recent literature.

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1 Introduction

Variational inclusions, as important extensions of the classical variational inequalities, provide us with simple, natural, general and unified frameworks in the study of many fields including mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, as well as engineering sciences. Owing to their wide applications, a lot of existence results and iterative algorithms of solutions for various variational inclusions have been studied in recent years. For details, we refer to [1–20] and the references therein. Among these methods, the resolvent operator techniques to solve various variational inclusions are interesting and important.

In the past years, Agarwal-Huang-Tan [1] introduced a system of generalized nonlinear mixed quasivariational inclusions and investigated the sensitivity analysis of solutions for the system of generalized nonlinear mixed quasivariational inclusions in Hilbert spaces. Kazmi-Bhat [8] introduced a system of nonlinear variational-like inclusions and gave an iterative algorithm for finding its approximate solution. Fang-Huang [4], Huang-Fang [6] and Liu-Kang-Ume [13] introduced some new systems of variational inclusions in Hilbert spaces using the resolvent operator associated with H-monotone operators, maximal η-monotone operators, maximal monotone operators and A-monotone operators, respectively. Afterward, Fang-Huang-Thompson [5] studied a class of variational inclusions with (H,η)-monotone operators, which provides a unifying framework for the classes of maximal monotone operators, maximal η-monotone operators and H-monotone operators. Jin-Liu-Lee [7] studied a class of generalized nonlinear mixed quasivariational inclusions including relaxed Lipschitz and relaxed monotone operators.

Motivated and inspired by the above works, the goal of this paper is as follows. First, a new system of generalized nonlinear mixed quasivariational inclusions with (H_i,η_i)-monotone operators is introduced and studied in Hilbert spaces. Secondly, by utilizing some properties of the resolvent operators with (H_i,η_i)-monotone operators, two new iterative algorithms for approximating solutions of the system of generalized nonlinear mixed quasivariational inclusions are constructed. Finally, we prove the existence of solutions for the system of generalized nonlinear mixed quasivariational inclusions and show the convergence of the iterative sequences generated by the iterative algorithms in Hilbert spaces. Our results generalize, improve and unify a lot of previously known results in this area.

2 Preliminaries

In this paper, we assume that $\mathcal{H}$ is a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $CB(\mathcal{H})$ denote the family of all nonempty closed bounded subsets of $\mathcal{H}$ and $\hat{\mathcal{H}}(\cdot,\cdot)$ denote the Hausdorff metric on $CB(\mathcal{H})$ defined by

$$\hat{\mathcal{H}}(A,B)=\max\Big\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(A,b)\Big\},\quad \forall A,B\in CB(\mathcal{H}),$$

where $d(a,B)=\inf_{b\in B}\|a-b\|$, $d(A,b)=\inf_{a\in A}\|a-b\|$. Set $\mathbb{R}=(-\infty,+\infty)$, ω and $\mathbb{N}$ denote the sets of all nonnegative and positive integers, respectively.

In what follows, we recall some basic concepts, assumptions and results, which will be used in the sequel.

Definition 2.1 Let $\eta:\mathcal{H}\times\mathcal{H}\to\mathcal{H}$, $H:\mathcal{H}\to\mathcal{H}$ be two operators and $M:\mathcal{H}\to 2^{\mathcal{H}}$ be a set-valued operator. M is said to be

(1) *η-monotone* if $\langle x-y,\eta(u,v)\rangle\geq 0$, $\forall u,v\in\mathcal{H}$, $x\in Mu$, $y\in Mv$;
(2) *(H,η)-monotone* if M is η-monotone and $(H+\lambda M)(\mathcal{H})=\mathcal{H}$ for all $\lambda>0$.

Remark 2.1 The class of (H,η)-monotone operators is much more universal than all maximal η-monotone, maximal monotone, H-monotone and η-monotone operators.

Definition 2.2 An operator $\eta:\mathcal{H}\times\mathcal{H}\to\mathcal{H}$ is said to be Lipschitz continuous if there exists a constant $\tau>0$ such that

$$\big\|\eta(u,v)\big\|\leq\tau\|u-v\|,\quad \forall u,v\in\mathcal{H}.$$

Definition 2.3 Let $g,m:\mathcal{H}\to\mathcal{H}$, $\eta:\mathcal{H}\times\mathcal{H}\to\mathcal{H}$ be operators. The operator g is said to be

(1) *Lipschitz continuous* if there exists a constant $s > 0$ such that

$$\|g(u) - g(v)\| \leq s\|u - v\|, \quad \forall u, v \in \mathcal{H};$$

(2) *strongly monotone* if there exists a constant $r > 0$ satisfying

$$\langle g(u) - g(v), u - v\rangle \geq r\|u - v\|^2, \quad \forall u, v \in \mathcal{H};$$

(3) *η-monotone* if $\langle g(u) - g(v), \eta(u, v)\rangle \geq 0$, $\forall u, v \in \mathcal{H}$;

(4) *strictly η-monotone* if g is η-monotone and $\langle g(u) - g(v), \eta(u, v)\rangle = 0$ if and only if $u = v$;

(5) *strongly η-monotone* if there exists a constant $\sigma > 0$ fulfilling

$$\langle g(u) - g(v), \eta(u, v)\rangle \geq \sigma\|u - v\|^2, \quad \forall u, v \in \mathcal{H};$$

(6) *m-relaxed monotone* if there exists a constant $k > 0$ such that

$$\langle g(u) - g(v), m(u) - m(v)\rangle \geq -k\|u - v\|^2, \quad \forall u, v \in \mathcal{H}.$$

Definition 2.4 A set-valued operator $A : \mathcal{H} \to CB(\mathcal{H})$ is said to be *$\hat{\mathcal{H}}$-Lipschitz continuous* if there exists a constant $l_A > 0$ such that

$$\hat{\mathcal{H}}(A(u), A(v)) \leq l_A\|u - v\|, \quad \forall u, v \in \mathcal{H}.$$

Definition 2.5 ([5]) Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be an operator, $H : \mathcal{H} \to \mathcal{H}$ be a strictly η-monotone operator and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be an (H, η)-monotone operator. Then the *resolvent operator* $J^{H,\eta}_{M,\lambda} : \mathcal{H} \to \mathcal{H}$ is defined by

$$J^{H,\eta}_{M,\lambda}(u) = (H + \lambda M)^{-1}(u), \quad \forall u \in \mathcal{H}.$$

Lemma 2.1 ([5]) *Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a Lipschitz continuous operator with a constant $\tau > 0$, $H : \mathcal{H} \to \mathcal{H}$ be a strongly η-monotone operator with a constant $\sigma > 0$ and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be an (H, η)-monotone operator. Then the resolvent operator $J^{H,\eta}_{M,\lambda} : \mathcal{H} \to \mathcal{H}$ is Lipschitz continuous with a constant $\frac{\tau}{\sigma}$, that is,*

$$\|J^{H,\eta}_{M,\lambda}(u) - J^{H,\eta}_{M,\lambda}(v)\| \leq \frac{\tau}{\sigma}\|u - v\|, \quad \forall u, v \in \mathcal{H}.$$

Lemma 2.2 ([21]) *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \in \omega,$$

where $\{t_n\}_{n\in\omega} \subset [0,1]$, $\sum_{n=0}^{\infty} t_n = +\infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < +\infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([22]) *Let (E, d) be a metric space and $T : E \to CB(E)$ be a set-valued mapping. Then for any given $\epsilon > 0$ and any given $x, y \in E$, $u \in Tx$, there exists $v \in Ty$ such that*

$$d(u, v) \leq (1 + \epsilon)H(Tx, Ty),$$

where H is the Hausdorff metric on $CB(E)$.

In what follows, unless specified otherwise, we always assume that for each $i \in \{1,2\}$, $\mathcal{H}_i$ is a real Hilbert space with norm $\|\cdot\|_i$, inner product $\langle\cdot,\cdot\rangle_i$ and Hausdorff metric $\hat{\mathcal{H}}_i$.

Definition 2.6 Let $H:\mathcal{H}_1 \to \mathcal{H}_1$, $F:\mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$ be two operators. The operator F is said to be

(1) *Lipschitz continuous* in the first argument if there exists a constant $\beta > 0$ such that

$$\big\|F(u,z) - F(v,z)\big\|_1 \le \beta\|u-v\|_1, \quad \forall u,v \in \mathcal{H}_1, z \in \mathcal{H}_2;$$

(2) *H-relaxed monotone* in the first argument if there exists a constant $\alpha > 0$ satisfying

$$\big\langle F(u,z) - F(v,z), H(u) - H(v)\big\rangle_1 \ge -\alpha\|u-v\|_1^2, \quad \forall u,v \in \mathcal{H}_1, z \in \mathcal{H}_2.$$

In a similar way, we can define the corresponding concepts for the operator F in the second argument.

3 A system of generalized nonlinear mixed quasivariational inclusions and iterative algorithms

In this section, we shall introduce a new system of generalized nonlinear mixed quasivariational inclusions including (H_i,η_i)-monotone operators in Hilbert spaces, and construct a new iterative algorithm for solving the system of generalized nonlinear mixed quasivariational inclusions. Furthermore, a more general and unified iterative algorithm called the Mann perturbed iterative algorithm with mixed errors is constructed as well in Hilbert spaces.

For each $i \in \{1,2\}$, let $H_i, g_i, m_i : \mathcal{H}_i \to \mathcal{H}_i$, $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i$ be operators, $M_i : \mathcal{H}_1 \times \mathcal{H}_2 \to 2^{\mathcal{H}_i}$ be set-valued operator such that for each fixed $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $M_1(\cdot,y):\mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is (H_1,η_1)-monotone and $M_2(x,\cdot):\mathcal{H}_2 \to 2^{\mathcal{H}_2}$ is (H_2,η_2)-monotone, $F, N_1 : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$, $G, N_2 : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$ be operators and $A, C : \mathcal{H}_1 \to CB(\mathcal{H}_1)$, $B, D : \mathcal{H}_2 \to CB(\mathcal{H}_2)$ be four set-valued operators. Let ρ and ν be two positive constants. For any $(f,g) \in \mathcal{H}_1 \times \mathcal{H}_2$, we consider the following problem:

Find (x,y,u,v,w,z) such that $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $u \in A(x)$, $v \in B(y)$, $w \in C(x)$, $z \in D(y)$, $(g_1 - m_1)(x) \in \operatorname{dom}(M_1(\cdot,y))$, $(g_2 - m_2)(y) \in \operatorname{dom}(M_2(x,\cdot))$ and

$$\begin{aligned} &f \in N_1(u,v) + \rho\big(F(x,y) + M_1\big((g_1 - m_1)(x), y\big)\big), \\ &g \in N_2(w,z) + \nu\big(G(x,y) + M_2\big(x, (g_2 - m_2)(y)\big)\big), \end{aligned} \tag{3.1}$$

where $(g_i - m_i)(x') = g_i(x') - m_i(x')$, $\forall x' \in \mathcal{H}_i$, $i \in \{1,2\}$. The problem (3.1) is called *a system of generalized nonlinear mixed quasivariational inclusions*.

Below are some special cases of the problem (3.1):

(A) If $\rho, \nu = 1$, $f = g = 0$, $M_1((g_1 - m_1)(x), y) = M_1(g_1(x))$ and $M_2(x, (g_2 - m_2)(y)) = M_2(g_2(y))$ for all $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2$, then the problem (3.1) is equivalent to finding (x,y,u,v,w,z) such that $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $u \in A(x)$, $v \in B(y)$, $w \in C(x)$, $z \in D(y)$, $g_1(x) \in \operatorname{dom}(M_1)$, $g_2(y) \in \operatorname{dom}(M_2)$ and

$$\begin{aligned} &0 \in N_1(u,v) + F(x,y) + M_1\big(g_1(x)\big), \\ &0 \in N_2(w,z) + G(x,y) + M_2\big(g_2(y)\big), \end{aligned} \tag{3.2}$$

which is called the system of generalized mixed quasivariational inclusions introduced and studied by Peng-Zhu [16], and the problem (3.2) includes those systems of variational inequalities and variational inclusions in [2, 4, 5, 18] as special cases;

(B) If $N_1 = N_2 = 0$, $g_1 = I_1$ (the identity map on $\mathcal{H}_1$), $g_2 = I_2$ (the identity map on $\mathcal{H}_2$), $\eta_1(a,x) = a - x$ for every $a, x \in \mathcal{H}_1$, $\eta_2(b,y) = b - y$ for any $b, y \in \mathcal{H}_2$, $M_1(x) = \partial\varphi(x)$ and $M_2(y) = \partial\psi(y)$, where $\varphi : \mathcal{H}_1 \to \mathbb{R} \cup \{+\infty\}$ and $\psi : \mathcal{H}_2 \to \mathbb{R} \cup \{+\infty\}$ are two proper, convex, lower semi-continuous functionals, $\partial\varphi(x)$ is the subdifferential of φ at x, $\partial\psi(y)$ is the subdifferential of ψ at y, then the problem (3.2) reduces to the following system of nonlinear variational inequalities introduced by Cho-Fang-Huang-Hwang [2]: seek $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2$ such that

$$\begin{aligned} &\langle F(x,y), a - x\rangle_1 + \varphi(a) - \varphi(x) \geq 0, \\ &\langle G(x,y), b - y\rangle_2 + \psi(b) - \psi(y) \geq 0. \end{aligned} \tag{3.3}$$

In brief, for suitable choices of the mappings presented in the problem (3.1) and the constants ρ, ν, one can obtain various new and previously known systems of variational inequalities and variational inclusions as special cases of the problem (3.1).

Lemma 3.1 *Let λ and μ be two positive constants. For each $i \in \{1,2\}$, let $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i$ be an operator, $H_i : \mathcal{H}_i \to \mathcal{H}_i$ be a strictly η_i-monotone operator, $M_i : \mathcal{H}_1 \times \mathcal{H}_2 \to 2^{\mathcal{H}_i}$ be a set-valued operator such that for each $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $M_1(\cdot,y) : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is (H_1,η_1)-monotone and $M_2(x,\cdot) : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ is (H_2,η_2)-monotone. Then (x,y,u,v,w,z) with $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $u \in A(x)$, $v \in B(y)$, $w \in C(x)$, $z \in D(y)$ is a solution of the problem* (3.1) *if and only if*

$$\begin{aligned} &(g_1 - m_1)(x) = J^{H_1,\eta_1}_{M_1(\cdot,y),\lambda\rho}\big[H_1(g_1 - m_1)(x) + \lambda f - \lambda N_1(u,v) - \lambda\rho F(x,y)\big], \\ &(g_2 - m_2)(y) = J^{H_2,\eta_2}_{M_2(x,\cdot),\mu\nu}\big[H_2(g_2 - m_2)(y) + \mu g - \mu N_2(w,z) - \mu\nu G(x,y)\big], \end{aligned}$$

where $J^{H_1,\eta_1}_{M_1(\cdot,y),\lambda\rho} = (H_1 + \lambda\rho M_1(\cdot,y))^{-1}$, $J^{H_2,\eta_2}_{M_2(x,\cdot),\mu\nu} = (H_2 + \mu\nu M_2(x,\cdot))^{-1}$ and λ, μ are both positive constants.

Proof Observe that

$$\begin{aligned} &f \in N_1(u,v) + \rho\big(F(x,y) + M_1\big((g_1 - m_1)(x), y\big)\big) \\ &\quad \Leftrightarrow \quad H_1(g_1 - m_1)(x) + \lambda f \in \lambda N_1(u,v) + \lambda\rho F(x,y) + H_1(g_1 - m_1)(x) \\ &\qquad\qquad + \lambda\rho M_1\big((g_1 - m_1)(x), y\big) \\ &\quad \Leftrightarrow \quad (g_1 - m_1)(x) = J^{H_1,\eta_1}_{M_1(\cdot,y),\lambda\rho}\big[H_1(g_1 - m_1)(x) + \lambda f - \lambda N_1(u,v) - \lambda\rho F(x,y)\big]. \end{aligned}$$

Analogously, we obtain directly that

$$(g_2 - m_2)(y) = J^{H_2,\eta_2}_{M_2(x,\cdot),\mu\nu}\big[H_2(g_2 - m_2)(y) + \mu g - \mu N_2(w,z) - \mu\nu G(x,y)\big].$$

This completes the proof. □

Based on Lemma 2.3 and Lemma 3.1, we suggest the following two sorts of iterative algorithms for the system of generalized nonlinear mixed quasivariational inclusions (3.1).

Algorithm 3.1 Let λ and μ be two positive constants. For each $i \in \{1,2\}$, let $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i$ be an operator, $H_i : \mathcal{H}_i \to \mathcal{H}_i$ be a strictly η_i-monotone operator, $M_i : \mathcal{H}_1 \times \mathcal{H}_2 \to 2^{\mathcal{H}_i}$ be a set-valued operator such that for each $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $M_1(\cdot,y) : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is (H_1,η_1)-monotone and $M_2(x,\cdot) : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ is (H_2,η_2)-monotone.

Step 0: Choose $(x_0,y_0) \in \mathcal{H}_1 \times \mathcal{H}_2$ and take $u_0 \in A(x_0)$, $v_0 \in B(y_0)$, $w_0 \in C(x_0)$, $z_0 \in D(y_0)$. Set $n = 0$.

Step 1: For any $n \in \omega$, compute $\{(x_n,y_n,u_n,v_n,w_n,z_n)\}_{n\in\omega}$ by the iterative schemes

$$\begin{aligned} x_{n+1} &= x_n - (g_1 - m_1)(x_n) \\ &\quad + J^{H_1,\eta_1}_{M_1(\cdot,y_n),\lambda\rho}\big[H_1(g_1 - m_1)(x_n) + \lambda f - \lambda N_1(u_n,v_n) - \lambda\rho F(x_n,y_n)\big], \\ y_{n+1} &= y_n - (g_2 - m_2)(y_n) \\ &\quad + J^{H_2,\eta_2}_{M_2(x_n,\cdot),\mu\nu}\big[H_2(g_2 - m_2)(y_n) + \mu g - \mu N_2(w_n,z_n) - \mu\nu G(x_n,y_n)\big], \end{aligned} \tag{3.4}$$

$$\begin{aligned} &u_n \in A(x_n), \quad \|u_{n+1} - u_n\|_1 \le \left(1 + \frac{1}{n+1}\right)\hat{\mathcal{H}}_1\big(A(x_{n+1}),A(x_n)\big), \\ &v_n \in B(y_n), \quad \|v_{n+1} - v_n\|_2 \le \left(1 + \frac{1}{n+1}\right)\hat{\mathcal{H}}_2\big(B(y_{n+1}),B(y_n)\big), \\ &w_n \in C(x_n), \quad \|w_{n+1} - w_n\|_1 \le \left(1 + \frac{1}{n+1}\right)\hat{\mathcal{H}}_1\big(C(x_{n+1}),C(x_n)\big), \\ &z_n \in D(y_n), \quad \|z_{n+1} - z_n\|_2 \le \left(1 + \frac{1}{n+1}\right)\hat{\mathcal{H}}_2\big(D(y_{n+1}),D(y_n)\big). \end{aligned} \tag{3.5}$$

Step 2: If x_{n+1}, y_{n+1}, u_{n+1}, v_{n+1}, w_{n+1}, z_{n+1} satisfy Lemma 3.1 to sufficient accuracy, stop; otherwise, set $n := n + 1$ and return to Step 1.

Algorithm 3.2 Let λ and μ be two positive constants. For each $i \in \{1,2\}$ and $n \in \omega$, let $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i$ be an operator, $H_i : \mathcal{H}_i \to \mathcal{H}_i$ be a strictly η_i-monotone operator, $M_i, M_i^n : \mathcal{H}_1 \times \mathcal{H}_2 \to 2^{\mathcal{H}_i}$ be set-valued operators such that for each $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $M_1(\cdot,y), M_1^n(\cdot,y) : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is (H_1,η_1)-monotone and $M_2(x,\cdot), M_2^n(x,\cdot) : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ is (H_2,η_2)-monotone. Given $(x_0,y_0) \in \mathcal{H}_1 \times \mathcal{H}_2$, compute $\{(x_n,y_n,u_n,v_n,w_n,z_n)\}_{n\in\omega}$ by the iterative schemes

$$\begin{aligned} x_{n+1} &= \big(1 - \alpha_n^*\big)x_n + \alpha_n^*\big[x_n - (g_1 - m_1)(x_n) \\ &\quad + J^{H_1,\eta_1}_{M_1^n(\cdot,y_n),\lambda\rho}\big(H_1(g_1 - m_1)(x_n) + \lambda f - \lambda N_1(u_n,v_n) - \lambda\rho F(x_n,y_n)\big)\big] + e_n, \\ y_{n+1} &= \big(1 - \alpha_n^*\big)y_n + \alpha_n^*\big[y_n - (g_2 - m_2)(y_n) \\ &\quad + J^{H_2,\eta_2}_{M_2^n(x_n,\cdot),\mu\nu}\big(H_2(g_2 - m_2)(y_n) + \mu g - \mu N_2(w_n,z_n) - \mu\nu G(x_n,y_n)\big)\big] + f_n, \end{aligned} \tag{3.6}$$

where $\{(u_n,v_n,w_n,z_n)\}_{n\in\omega}$ satisfies (3.5), $\{\alpha_n^*\}_{n\in\omega}$ is a real sequence in $[0,1]$ and $\{e_n\}_{n\in\omega}$, $\{f_n\}_{n\in\omega}$ are two sequences of the elements of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, which are introduced to take into account possible inexact computation, satisfying

(a) $\sum_{n=0}^{\infty} \alpha_n^* = +\infty$;
(b) $e_n = e_n' + e_n''$, $\|e_n''\| = \xi_n^*\alpha_n^*$, $\forall n \in \omega$, $\sum_{n=0}^{\infty} \|e_n'\| < +\infty$, $\lim_{n\to\infty} \xi_n^* = 0$;
(c) $f_n = f_n' + f_n''$, $\|f_n''\| = \varsigma_n^*\alpha_n^*$, $\forall n \in \omega$, $\sum_{n=0}^{\infty} \|f_n'\| < +\infty$, $\lim_{n\to\infty} \varsigma_n^* = 0$.

Remark 3.1 Algorithm 3.2 includes several known algorithms in [2–6, 13, 15–18] as special cases.

4 Existence and convergence theorems

At present, we seek those conditions which ensure the existence of solutions for the problem (3.1) and the convergence of the iterative sequence generated by Algorithm 3.1. Furthermore, based on the existence of the solutions for the problem (3.1), the convergence of the Mann perturbed iterative sequence generated by Algorithm 3.2 is discussed.

Theorem 4.1 *Let ρ and ν be two positive constants. For $i \in \{1,2\}$, let $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i$ be Lipschitz continuous with a constant τ_i, $H_i : \mathcal{H}_i \to \mathcal{H}_i$ be strongly η_i-monotone and Lipschitz continuous with constants σ_i and δ_i, respectively, $g_i, m_i : \mathcal{H}_i \to \mathcal{H}_i$ be Lipschitz continuous with constants s_i and t_i, respectively, g_i be m_i-relaxed monotone with a constant k_i, $g_i - m_i$ be strongly monotone with a constant r_i, $M_i : \mathcal{H}_1 \times \mathcal{H}_2 \to 2^{\mathcal{H}_i}$ be set-valued operators such that for each $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $M_1(\cdot,y) : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is (H_1,η_1)-monotone and $M_2(x,\cdot) : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ is (H_2,η_2)-monotone. Let $A, C : \mathcal{H}_1 \to CB(\mathcal{H}_1)$ be $\hat{\mathcal{H}}_1$-Lipschitz continuous with constants l_A, l_C, respectively, and $B, D : \mathcal{H}_2 \to CB(\mathcal{H}_2)$ be $\hat{\mathcal{H}}_2$-Lipschitz continuous with constants l_B, l_D, respectively. Let $F : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$ be $H_1(g_1 - m_1)$-relaxed monotone in the first argument with a constant α_1, Lipschitz continuous in the first and second arguments with constants β_1 and γ_1, respectively. Let $G : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$ be $H_2(g_2 - m_2)$-relaxed monotone in the second argument with a constant α_2, Lipschitz continuous in the first and second arguments with constants β_2 and γ_2, respectively. Assume that $N_1 : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$ is Lipschitz continuous in the first and second arguments with constants b_1 and c_1, respectively, $N_2 : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$ is Lipschitz continuous in the first and second arguments with constants b_2 and c_2, respectively. Assume that there exist positive constants λ_1, λ_2, λ and μ satisfying*

$$
\begin{aligned}
&\big\| J^{H_1,\eta_1}_{M_1(\cdot,y_1),\lambda\rho}(k) - J^{H_1,\eta_1}_{M_1(\cdot,y_2),\lambda\rho}(k) \big\|_1 \\
&\quad \le \lambda_1 \|y_1 - y_2\|_2, \quad \forall (y_1,y_2,k) \in \mathcal{H}_2 \times \mathcal{H}_2 \times \mathcal{H}_1, \\
&\big\| J^{H_2,\eta_2}_{M_2(x_1,\cdot),\mu\nu}(m) - J^{H_2,\eta_2}_{M_2(x_2,\cdot),\mu\nu}(m) \big\|_2 \\
&\quad \le \lambda_2 \|x_1 - x_2\|_1, \quad \forall (x_1,x_2,m) \in \mathcal{H}_1 \times \mathcal{H}_1 \times \mathcal{H}_2;
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
\max\bigg\{ &\theta_1 + \frac{\tau_1}{\sigma_1}\Big(\sqrt{\theta_2 + 2\lambda\rho\alpha_1 + \lambda^2\rho^2\beta_1^2} + \lambda\theta_4\Big) + \mu\theta_3 + \lambda_2, \\
&\theta_1' + \frac{\tau_2}{\sigma_2}\Big(\sqrt{\theta_2' + 2\mu\nu\alpha_2 + \mu^2\nu^2\beta_2^2} + \mu\theta_4'\Big) + \lambda\theta_3' + \lambda_1 \bigg\} < 1,
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
&\theta_1 = \sqrt{1 - 2r_1 + s_1^2 + 2k_1 + t_1^2}, \qquad \theta_2 = \delta_1^2\big(s_1^2 + 2k_1 + t_1^2\big), \\
&\theta_1' = \sqrt{1 - 2r_2 + s_2^2 + 2k_2 + t_2^2}, \qquad \theta_2' = \delta_2^2\big(s_2^2 + 2k_2 + t_2^2\big); \\
&\theta_3 = \frac{\tau_2}{\sigma_2}(\nu\gamma_2 + c_2 l_D), \qquad \theta_4 = b_1 l_A, \\
&\theta_3' = \frac{\tau_1}{\sigma_1}(\rho\gamma_1 + c_1 l_B), \qquad \theta_4' = b_2 l_C.
\end{aligned}
\tag{4.3}
$$

Then the problem (3.1) *admits a solution* $(x,y,u,v,w,z) \in \mathcal{H}_1 \times \mathcal{H}_2 \times A(x) \times B(y) \times C(x) \times D(y)$ *and the sequence* $\{(x_n,y_n,u_n,v_n,w_n,z_n)\}_{n\in\omega}$ *generated by Algorithm* 3.1 *converges to* (x,y,u,v,w,z) *as* $n \to \infty$.

Proof Set

$$a_n^* = H_1(g_1 - m_1)(x_n) + \lambda f - \lambda N_1(u_n,v_n) - \lambda\rho F(x_n,y_n), \quad \forall n \in \omega.$$

Let $n \in \mathbb{N}$. In view of (3.4), (4.1) and Lemma 2.1, we arrive at

$$\begin{aligned}
&\|x_{n+1} - x_n\|_1 \\
&\quad\le \big\|x_n - x_{n-1} - \big[(g_1 - m_1)(x_n) - (g_1 - m_1)(x_{n-1})\big]\big\|_1 \\
&\qquad + \big\|J^{H_1,\eta_1}_{M_1(\cdot,y_n),\lambda\rho}\big(a_n^*\big) - J^{H_1,\eta_1}_{M_1(\cdot,y_n),\lambda\rho}\big(a_{n-1}^*\big)\big\|_1 \\
&\qquad + \big\|J^{H_1,\eta_1}_{M_1(\cdot,y_n),\lambda\rho}\big(a_{n-1}^*\big) - J^{H_1,\eta_1}_{M_1(\cdot,y_{n-1}),\lambda\rho}\big(a_{n-1}^*\big)\big\|_1 \\
&\quad\le \big\|x_n - x_{n-1} - \big[(g_1 - m_1)(x_n) - (g_1 - m_1)(x_{n-1})\big]\big\|_1 \\
&\qquad + \frac{\tau_1}{\sigma_1}\big\|a_n^* - a_{n-1}^*\big\|_1 + \lambda_1\|y_n - y_{n-1}\|_2.
\end{aligned} \tag{4.4}$$

By the Lipschitz continuity of g_1 and m_1, strong monotonicity of $g_1 - m_1$ and m_1-relaxed monotonicity of g_1, we obtain that

$$\begin{aligned}
&\big\|x_n - x_{n-1} - \big[(g_1 - m_1)(x_n) - (g_1 - m_1)(x_{n-1})\big]\big\|_1^2 \\
&\quad= \|x_n - x_{n-1}\|_1^2 - 2\big\langle x_n - x_{n-1}, (g_1 - m_1)(x_n) - (g_1 - m_1)(x_{n-1})\big\rangle_1 + \big\|g_1(x_n) - g_1(x_{n-1})\big\|_1^2 \\
&\qquad - 2\big\langle g_1(x_n) - g_1(x_{n-1}), m_1(x_n) - m_1(x_{n-1})\big\rangle_1 + \big\|m_1(x_n) - m_1(x_{n-1})\big\|_1^2 \\
&\quad\le \theta_1^2\|x_n - x_{n-1}\|_1^2.
\end{aligned} \tag{4.5}$$

Note that

$$\begin{aligned}
&\big\|a_n^* - a_{n-1}^*\big\|_1 \\
&\quad\le \big\|H_1(g_1 - m_1)(x_n) - H_1(g_1 - m_1)(x_{n-1}) - \lambda\rho\big[F(x_n,y_n) - F(x_{n-1},y_n)\big]\big\|_1 \\
&\qquad + \lambda\rho\big\|F(x_{n-1},y_n) - F(x_{n-1},y_{n-1})\big\|_1 + \lambda\big\|N_1(u_n,v_n) - N_1(u_{n-1},v_{n-1})\big\|_1.
\end{aligned} \tag{4.6}$$

Since H_1 is Lipschitz continuous, F is relaxed monotone with respect to $H_1(g_1 - m_1)$ in the first argument and Lipschitz continuous in the first argument, respectively, g_1, m_1 are both Lipschitz continuous, and g_1 is m_1-relaxed monotone, we derive that

$$\begin{aligned}
&\big\|H_1(g_1 - m_1)(x_n) - H_1(g_1 - m_1)(x_{n-1}) - \lambda\rho\big[F(x_n,y_n) - F(x_{n-1},y_n)\big]\big\|_1^2 \\
&\quad= \big\|H_1(g_1 - m_1)(x_n) - H_1(g_1 - m_1)(x_{n-1})\big\|_1^2 \\
&\qquad - 2\lambda\rho\big\langle F(x_n,y_n) - F(x_{n-1},y_n), H_1(g_1 - m_1)(x_n) - H_1(g_1 - m_1)(x_{n-1})\big\rangle_1 \\
&\qquad + \lambda^2\rho^2\big\|F(x_n,y_n) - F(x_{n-1},y_n)\big\|_1^2 \\
&\quad\le \big(\theta_2 + 2\lambda\rho\alpha_1 + \lambda^2\rho^2\beta_1^2\big)\|x_n - x_{n-1}\|_1^2.
\end{aligned} \tag{4.7}$$

Since F is Lipschitz continuous in the second argument, we get that

$$\left\|F(x_{n-1},y_n)-F(x_{n-1},y_{n-1})\right\|_1 \le \gamma_1\|y_n-y_{n-1}\|_2. \tag{4.8}$$

By virtue of the Lipschitz continuity in the first and second arguments of N_1, and $\hat{\mathcal{H}}$-Lipschitz continuity of A and B, we arrive at

$$\begin{aligned}
&\left\|N_1(u_n,v_n)-N_1(u_{n-1},v_{n-1})\right\|_1 \\
&\quad\le \left\|N_1(u_n,v_n)-N_1(u_{n-1},v_n)\right\|_1+\left\|N_1(u_{n-1},v_n)-N_1(u_{n-1},v_{n-1})\right\|_1 \\
&\quad\le b_1\|u_n-u_{n-1}\|_1+c_1\|v_n-v_{n-1}\|_2 \\
&\quad\le b_1\left(1+\frac{1}{n}\right)\hat{\mathcal{H}}_1\big(A(x_n),A(x_{n-1})\big)+c_1\left(1+\frac{1}{n}\right)\hat{\mathcal{H}}_2\big(B(y_n),B(y_{n-1})\big) \\
&\quad\le \left(1+\frac{1}{n}\right)\theta_4\|x_n-x_{n-1}\|_1+c_1\left(1+\frac{1}{n}\right)l_B\|y_n-y_{n-1}\|_2.
\end{aligned} \tag{4.9}$$

By (4.6)-(4.9), we obtain that

$$\begin{aligned}
\left\|a_n^*-a_{n-1}^*\right\|_1 \le{}& \sqrt{\theta_2+2\lambda\rho\alpha_1+\lambda^2\rho^2\beta_1^2}\,\|x_n-x_{n-1}\|_1+\lambda\rho\gamma_1\|y_n-y_{n-1}\|_2 \\
&+\lambda\left(1+\frac{1}{n}\right)\theta_4\|x_n-x_{n-1}\|_1+\lambda c_1\left(1+\frac{1}{n}\right)l_B\|y_n-y_{n-1}\|_2.
\end{aligned} \tag{4.10}$$

Keeping in mind (4.4), (4.5) and (4.10), we conclude directly that

$$\begin{aligned}
\|x_{n+1}-x_n\|_1 \le{}& \left[\theta_1+\frac{\tau_1}{\sigma_1}\left(\sqrt{\theta_2+2\lambda\rho\alpha_1+\lambda^2\rho^2\beta_1^2}+\lambda\left(1+\frac{1}{n}\right)\theta_4\right)\right]\|x_n-x_{n-1}\|_1 \\
&+\left[\frac{\tau_1}{\sigma_1}\lambda\left(\rho\gamma_1+c_1\left(1+\frac{1}{n}\right)l_B\right)+\lambda_1\right]\|y_n-y_{n-1}\|_2.
\end{aligned} \tag{4.11}$$

In a similar way, we see that

$$\begin{aligned}
\|y_{n+1}-y_n\|_2 \le{}& \left[\theta_1'+\frac{\tau_2}{\sigma_2}\left(\sqrt{\theta_2'+2\mu\nu\alpha_2+\mu^2\nu^2\beta_2^2}+\mu\left(1+\frac{1}{n}\right)\theta_4'\right)\right]\|y_n-y_{n-1}\|_2 \\
&+\left[\frac{\tau_2}{\sigma_2}\mu\left(\nu\gamma_2+c_2\left(1+\frac{1}{n}\right)l_D\right)+\lambda_2\right]\|x_n-x_{n-1}\|_1.
\end{aligned} \tag{4.12}$$

Adding the inequalities (4.11) and (4.12), we have

$$\|x_{n+1}-x_n\|_1+\|y_{n+1}-y_n\|_2 \le \zeta_n\big(\|x_n-x_{n-1}\|_1+\|y_n-y_{n-1}\|_2\big), \tag{4.13}$$

where

$$\begin{aligned}
\zeta_n ={}& \max\left\{\theta_1+\frac{\tau_1}{\sigma_1}\left(\sqrt{\theta_2+2\lambda\rho\alpha_1+\lambda^2\rho^2\beta_1^2}+\lambda\left(1+\frac{1}{n}\right)\theta_4\right)\right. \\
&+\frac{\tau_2}{\sigma_2}\mu\left(\nu\gamma_2+c_2\left(1+\frac{1}{n}\right)l_D\right)+\lambda_2, \\
&\theta_1'+\frac{\tau_2}{\sigma_2}\left(\sqrt{\theta_2'+2\mu\nu\alpha_2+\mu^2\nu^2\beta_2^2}+\mu\left(1+\frac{1}{n}\right)\theta_4'\right)
\end{aligned}$$

$$+\frac{\tau_1}{\sigma_1}\lambda\left(\rho\gamma_1+c_1\left(1+\frac{1}{n}\right)l_B\right)+\lambda_1\bigg\}$$

$$\begin{aligned}\to \zeta = \max\bigg\{&\theta_1+\frac{\tau_1}{\sigma_1}\left(\sqrt{\theta_2+2\lambda\rho\alpha_1+\lambda^2\rho^2\beta_1^2}+\lambda\theta_4\right)+\mu\theta_3+\lambda_2,\\ &\theta_1'+\frac{\tau_2}{\sigma_2}\left(\sqrt{\theta_2'+2\mu\nu\alpha_2+\mu^2\nu^2\beta_2^2}+\mu\theta_4'\right)+\lambda\theta_3'+\lambda_1\bigg\}\quad \text{as } n\to\infty. \end{aligned}\tag{4.14}$$

In view of (4.2) and (4.14), we know that $0<\zeta<1$ and so (4.13) implies that $\{x_n\}_{n\in\omega}$, $\{y_n\}_{n\in\omega}$ are both Cauchy sequences. Thus, there exist $x\in\mathcal{H}_1$ and $y\in\mathcal{H}_2$ satisfying $x_n\to x$ and $y_n\to y$ as $n\to\infty$.

It follows from the Lipschitz continuity of A and (3.5) that

$$\|u_n-u_{n-1}\|_1\le\left(1+\frac{1}{n}\right)l_A\|x_n-x_{n-1}\|_1,$$

which together with (4.2), (4.13) and (4.14) yields that $\{u_n\}_{n\in\omega}$ is a Cauchy sequence. Similarly, we infer that $\{v_n\}_{n\in\omega}$, $\{w_n\}_{n\in\omega}$, $\{z_n\}_{n\in\omega}$ are Cauchy sequences as well. Further, there exist $u,w\in\mathcal{H}_1, v,z\in\mathcal{H}_2$ such that $u_n\to u$, $v_n\to v$, $w_n\to w$, $z_n\to z$ as $n\to\infty$. Moreover,

$$\begin{aligned}d_1\big(u,A(x)\big)&\le\|u-u_n\|_1+\hat{\mathcal{H}}_1\big(A(x_n),A(x)\big)\\&\le\|u-u_n\|_1+l_A\|x_n-x\|_1\to 0\quad\text{as } n\to\infty.\end{aligned}$$

Since $A(x)$ is closed, we have $u\in A(x)$. In a similar way, we obtain that $v\in B(y)$, $w\in C(x)$, $z\in D(y)$. On account of the continuity of g_i, m_i, H_i, F, G, N_i, $J^{H_1,\eta_1}_{M_1(\cdot,y),\lambda\rho}$, $J^{H_2,\eta_2}_{M_2(x,\cdot),\mu\nu}$ and Algorithm 3.1, we find that (x,y,u,v,w,z) satisfy the following relations:

$$\begin{aligned}(g_1-m_1)(x)&=J^{H_1,\eta_1}_{M_1(\cdot,y),\lambda\rho}\big[H_1(g_1-m_1)(x)+\lambda f-\lambda N_1(u,v)-\lambda\rho F(x,y)\big],\\(g_2-m_2)(y)&=J^{H_2,\eta_2}_{M_2(x,\cdot),\mu\nu}\big[H_2(g_2-m_2)(y)+\mu g-\mu N_2(w,z)-\mu\nu G(x,y)\big].\end{aligned}$$

It follows that (x,y,u,v,w,z) is a solution of the problem (3.1) from Lemma 3.1. This completes the proof. □

Remark 4.1 Theorem 4.1 generalizes and unifies those results in [2–6, 13, 15–18] from the following aspects:

(d1) The problem (3.1) includes the variational inequalities and variational inclusions in [2, 15–17] as special cases;

(d2) For every $i\in\{1,2\}$, the (H_i,η_i)-monotone operators and $H_i(g_i-m_i)$-relaxed monotone operators we utilized here are much more universal than those used in [3, 4, 6, 13, 18];

(d3) Algorithm 3.1 is very different from those in [2–6, 13, 15–18] for finding approximate solutions.

Theorem 4.2 *Let ρ, ν, η_i, H_i, g_i, m_i, A, B, C, D, F, G, N_i, M_i, $i\in\{1,2\}$, be the same as in Theorem 4.1. For $n\in\omega$, let $M_1^n:\mathcal{H}_1\times\mathcal{H}_2\to 2^{\mathcal{H}_1}$, $M_2^n:\mathcal{H}_1\times\mathcal{H}_2\to 2^{\mathcal{H}_2}$ be set-valued operators such that $M_1^n(\cdot,y):\mathcal{H}_1\to 2^{\mathcal{H}_1}$ is (H_1,η_1)-monotone for each fixed $y\in\mathcal{H}_2$,*

$M_2^n(x,\cdot):\mathcal{H}_2\to 2^{\mathcal{H}_2}$ *is* (H_2,η_2)*-monotone for each fixed* $x\in\mathcal{H}_1$. *Assume that there exist positive constants* λ_1, λ_2, λ *and* μ *satisfying* (4.2)-(4.3),

$$\begin{aligned}
&\bigl\|J^{H_1,\eta_1}_{M_1^n(\cdot,y_1),\lambda\rho}(k)-J^{H_1,\eta_1}_{M_1^n(\cdot,y_2),\lambda\rho}(k)\bigr\|_1\\
&\quad\le\lambda_1\|y_1-y_2\|_2,\quad\forall(y_1,y_2,k)\in\mathcal{H}_2\times\mathcal{H}_2\times\mathcal{H}_1,\\
&\bigl\|J^{H_2,\eta_2}_{M_2^n(x_1,\cdot),\mu\nu}(m)-J^{H_2,\eta_2}_{M_2^n(x_2,\cdot),\mu\nu}(m)\bigr\|_2\\
&\quad\le\lambda_2\|x_1-x_2\|_1,\quad\forall(x_1,x_2,m)\in\mathcal{H}_1\times\mathcal{H}_1\times\mathcal{H}_2
\end{aligned}\tag{4.15}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}J^{H_1,\eta_1}_{M_1^n(\cdot,y_1),\lambda\rho}(k)=J^{H_1,\eta_1}_{M_1(\cdot,y_1),\lambda\rho}(k),\quad\forall(y_1,k)\in\mathcal{H}_2\times\mathcal{H}_1,\\
&\lim_{n\to\infty}J^{H_2,\eta_2}_{M_2^n(x_1,\cdot),\mu\nu}(m)=J^{H_2,\eta_2}_{M_2(x_1,\cdot),\mu\nu}(m),\quad\forall(x_1,m)\in\mathcal{H}_1\times\mathcal{H}_2.
\end{aligned}\tag{4.16}$$

Then the problem (3.1) *admits a solution* $(x,y,u,v,w,z)\in\mathcal{H}_1\times\mathcal{H}_2\times A(x)\times B(y)\times C(x)\times D(y)$ *and the sequence* $\{(x_n,y_n,u_n,v_n,w_n,z_n)\}_{n\in\omega}$ *generated by Algorithm* 3.2 *converges to* (x,y,u,v,w,z) *as* $n\to\infty$.

Proof It follows from Theorem 4.1 that the problem (3.1) possesses a solution $(x,y,u,v,w,z)\in\mathcal{H}_1\times\mathcal{H}_2\times A(x)\times B(y)\times C(x)\times D(y)$. Further, we get that by Lemma 3.1

$$\begin{aligned}
x&=\bigl(1-\alpha_n^*\bigr)x+\alpha_n^*\bigl[x-(g_1-m_1)(x)+J^{H_1,\eta_1}_{M_1(\cdot,y),\lambda\rho}\bigl(a^*\bigr)\bigr],\\
y&=\bigl(1-\alpha_n^*\bigr)y+\alpha_n^*\bigl[y-(g_2-m_2)(y)+J^{H_2,\eta_2}_{M_2(x,\cdot),\mu\nu}\bigl(b^*\bigr)\bigr],
\end{aligned}\tag{4.17}$$

where

$$\begin{aligned}
a^*&=H_1(g_1-m_1)(x)+\lambda f-\lambda N_1(u,v)-\lambda\rho F(x,y),\\
b^*&=H_2(g_2-m_2)(y)+\mu g-\mu N_2(w,z)-\mu\nu G(x,y).
\end{aligned}$$

As in the proof of Theorem 4.1, we conclude that by (3.6) and (4.17)

$$\begin{aligned}
&\|x_{n+1}-x\|_1\\
&\quad\le\bigl(1-\alpha_n^*\bigr)\|x_n-x\|_1+\alpha_n^*\bigl[\bigl\|x_n-x-\bigl((g_1-m_1)(x_n)-(g_1-m_1)(x)\bigr)\bigr\|_1\\
&\qquad+\bigl\|J^{H_1,\eta_1}_{M_1^n(\cdot,y_n),\lambda\rho}\bigl(a_n^*\bigr)-J^{H_1,\eta_1}_{M_1^n(\cdot,y),\lambda\rho}\bigl(a_n^*\bigr)\bigr\|_1\\
&\qquad+\bigl\|J^{H_1,\eta_1}_{M_1^n(\cdot,y),\lambda\rho}\bigl(a_n^*\bigr)-J^{H_1,\eta_1}_{M_1^n(\cdot,y),\lambda\rho}\bigl(a^*\bigr)\bigr\|_1+P_n\bigr]+\|e_n\|_1\\
&\quad\le\bigl(1-\alpha_n^*\bigr)\|x_n-x\|_1+\alpha_n^*\left[\theta_1\|x_n-x\|_1+\lambda_1\|y_n-y\|_2+\frac{\tau_1}{\sigma_1}\bigl\|a_n^*-a^*\bigr\|_1+P_n\right]+\|e_n\|_1\\
&\quad=\left[1-\alpha_n^*+\alpha_n^*\left(\theta_1+\frac{\tau_1}{\sigma_1}\left(\sqrt{\theta_2+2\lambda\rho\alpha_1+\lambda^2\rho^2\beta_1^2}+\lambda\left(1+\frac{1}{n}\right)\theta_4\right)\right)\right]\|x_n-x\|_1\\
&\qquad+\alpha_n^*\left[\frac{\tau_1}{\sigma_1}\lambda\left(\rho\gamma_1+c_1\left(1+\frac{1}{n}\right)l_B\right)+\lambda_1\right]\|y_n-y\|_2+\alpha_n^*P_n+\|e_n\|_1,
\end{aligned}\tag{4.18}$$

where $P_n = \|J^{H_1,\eta_1}_{M_1^n(\cdot,y),\lambda\rho}(a^*) - J^{H_1,\eta_1}_{M_1(\cdot,y),\lambda\rho}(a^*)\|_1$. Similarly, we get directly that

$$\begin{aligned}
&\|y_{n+1} - y\|_2 \\
&\quad \le \left[1 - \alpha_n^* + \alpha_n^*\left(\theta_1' + \frac{\tau_2}{\sigma_2}\left(\sqrt{\theta_2' + 2\mu\nu\alpha_2 + \mu^2\nu^2\beta_2^2} + \mu\left(1 + \frac{1}{n}\right)\theta_4'\right)\right)\right]\|y_n - y\|_2 \\
&\qquad + \alpha_n^*\left[\frac{\tau_2}{\sigma_2}\mu\left(\nu\gamma_2 + c_2\left(1 + \frac{1}{n}\right)l_D\right) + \lambda_2\right]\|x_n - x\|_1 + \alpha_n^* Q_n + \|f_n\|_2,
\end{aligned} \tag{4.19}$$

where $Q_n = \|J^{H_2,\eta_2}_{M_2^n(x,\cdot),\mu\nu}(b^*) - J^{H_2,\eta_2}_{M_2^n(x,\cdot),\mu\nu}(b^*)\|_2$. Adding (4.18) and (4.19), we conclude that

$$\begin{aligned}
&\|x_{n+1} - x\|_1 + \|y_{n+1} - y\|_2 \\
&\quad \le \left[1 - \alpha_n^*(1 - \zeta_n)\right]\left(\|x_n - x\|_1 + \|y_n - y\|_2\right) + \alpha_n^* P_n + \alpha_n^* Q_n + \|e_n\|_1 + \|f_n\|_2,
\end{aligned} \tag{4.20}$$

here, ζ_n and ζ are defined by (4.14). Thus, $\zeta_n \to \zeta$ as $n \to \infty$ and $0 < \zeta < 1$. Hence, for $\theta = \frac{1+\zeta}{2}$, there exist $n_0 \in \mathbb{N}$ such that $\zeta_n < \theta$ for every $n \ge n_0$. By (b) and (c) in Algorithm 3.2 and (4.20), we gain that

$$\begin{aligned}
&\|x_{n+1} - x\|_1 + \|y_{n+1} - y\|_2 \\
&\quad \le \left[1 - \alpha_n^*(1 - \theta)\right]\left(\|x_n - x\|_1 + \|y_n - y\|_2\right) + \alpha_n^* P_n \\
&\qquad + \alpha_n^* Q_n + \xi_n^* \alpha_n^* + \varsigma_n^* \alpha_n^* + \left\|e_n'\right\|_1 + \left\|f_n'\right\|_2.
\end{aligned} \tag{4.21}$$

On the grounds of (4.16), (a)-(c) in Algorithm 3.2 and Lemma 2.2, we obtain that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. The remainder of the proof is the same as that in Theorem 4.1 and is omitted. This completes the proof. □

Remark 4.2 Theorem 4.2 improves and extends those corresponding results in [1, 2, 4, 5, 15–18, 21].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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