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Complete convergence and complete moment convergence for a class of random variables

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Abstract

In this paper, we establish the complete convergence and complete moment convergence and obtain the equivalence of the complete convergence and complete moment convergence for the class of random variables satisfying a Rosenthal-type maximal inequality. Baum-Katz-type theorem and Hsu-Robbins-type theorem are extended to the case of this class of random variables.

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1 Introduction

The concept of complete convergence was introduced by Hsu and Robbins [1] as follows. A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant C if $\sum_{n=1}^{\infty} P(|U_n - C| > \varepsilon) < \infty$ for all $\varepsilon > 0$. In view of the Borel-Cantelli lemma, this implies that $U_n \rightarrow C$ almost surely (a.s.). The converse is true if the $\{U_n, n \geq 1\}$ are independent. Hsu and Robbins [1] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdős [2] proved the converse. The result of Hsu-Robbins-Erdős is a fundamental theorem in probability theory which has been generalized and extended in several directions by many authors. One of the most important generalizations is that by Baum and Katz [3] for the strong law of large numbers as follows.

Theorem A (Baum and Katz [3]) *Let $\alpha > 1/2$ and $\alpha p > 1$. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables. Assume further that $EX_1 = 0$ if $\alpha \leq 1$. Then the following statements are equivalent:*

- (i) $E|X_1|^p < \infty$;
- (ii) $\sum_{n=1}^{\infty} n^{\alpha p - 2} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_i \right| > \varepsilon n^{\alpha}\right) < \infty$, for all $\varepsilon > 0$.

Many authors studied the Baum-Katz-type theorem for dependent random variables; see, for example, Peligrad [4] for a strong stationary ρ -mixing sequence, Peligrad and Gut

[5] for a ρ^* -mixing sequence, Stoica [6, 7] for a martingale difference sequence, Stoica [8] for bounded subsequences, Wang and Hu [9] for φ -mixing random variables, and so forth.

One of the most interesting inequalities to probability theory is the Rosenthal-type maximal inequality. For a sequence $\{X_i, 1 \leq i \leq n\}$ of i.i.d. random variables with $E|X_1|^q < \infty$ for some $q \geq 2$, there exist positive constants C_q depending only on q such that

$$E\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (X_i - EX_i) \right|^q\right) \leq C_q \left\{ \sum_{i=1}^n E|X_i|^q + \left(\sum_{i=1}^n EX_i^2 \right)^{q/2} \right\}.$$

The inequality above has been obtained for dependent random variables by many authors. See, for example, Shao [10] for negatively associated random variables, Utev and Peligrad [11] for ρ^* -mixing random variables, Wang *et al.* [12] for φ -mixing random variables with the mixing coefficients satisfying certain conditions, and so forth.

The purpose of this work is to obtain complete convergence and complete moment convergence for a sequence of random variables satisfying a Rosenthal-type maximal inequality.

Throughout the paper, let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space (Ω, \mathcal{A}, P) . Let $I(A)$ be the indicator function of the set A . Denote $S_n = \sum_{i=1}^n X_i$, $S_0 = 0$, $\ln^+ x = \ln \max(x, e)$, $x^+ = xI(x \geq 0)$. The symbol C denotes a positive constant which may be different in various places.

The following definition will be used frequently in the paper.

Definition 1.1 A sequence of random variables $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| > x) \leq CP(|X| > x)$$

for all $x \geq 0$ and $n \geq 1$.

In this paper, the present investigation is to provide the complete convergence and complete moment convergence for a sequence of the class of random variables satisfying a Rosenthal-type maximal inequality and prove the equivalence of the complete convergence and complete moment convergence. Baum-Katz-type theorem and Hsu-Robbins-type theorem are extended to the case of the class of random variables satisfying a Rosenthal-type maximal inequality. As a result, the Marcinkiewicz-Zygmund strong law of large numbers for the class of random variables is obtained.

2 Main results

Theorem 2.1 Let $\alpha > 1/2$, $\alpha p \geq 1$ and $p > 1$. Suppose that $\{X_n, n \geq 1\}$ is a sequence of zero mean random variables which is stochastically dominated by a random variable X with $E|X|^p < \infty$. Assume that for any $q \geq 2$, there exists a positive constant C_q depending only on q such that

$$E\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ti} - EY_{ti}) \right|^q\right) \leq C_q \left\{ \sum_{i=1}^n E|Y_{ti}|^q + \left(\sum_{i=1}^n EY_{ti}^2 \right)^{q/2} \right\}, \tag{2.1}$$

where $Y_{ti} = -tI(X_i < -t) + X_iI(|X_i| \leq t) + tI(X_i > t)$ for all $t > 0$. Then

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\left(\max_{1 \leq j \leq n} |S_j| \geq \varepsilon n^\alpha\right) < \infty, \quad \text{for all } \varepsilon > 0 \tag{2.2}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} E\left(\max_{1 \leq j \leq n} |S_j| - \varepsilon n^\alpha\right)^+ < \infty, \quad \text{for all } \varepsilon > 0. \tag{2.3}$$

Furthermore, (2.2) is equivalent to (2.3).

Corollary 2.1 *Let $1 < p < 2$. Suppose that $\{X_n, n \geq 1\}$ is a sequence of zero mean random variables which is stochastically dominated by a random variable X with $E|X|^p < \infty$. Assume further that (2.1) holds, then*

$$\frac{S_n}{n^{1/p}} \rightarrow 0 \quad \text{a.s.}$$

For $p = 1$, we have the following theorem.

Theorem 2.2 *Let $\alpha > 0$. Suppose that $\{X_n, n \geq 1\}$ is a sequence of zero mean random variables which is stochastically dominated by a random variable X with $E|X| \ln^+ |X| < \infty$. Assume further that (2.1) holds, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{-2} E\left(\max_{1 \leq j \leq n} |S_j| - \varepsilon n^\alpha\right)^+ < \infty. \tag{2.4}$$

Theorem 2.3 *Let $\alpha > 1/2$, $p > 1$ and $\alpha p > 1$. Suppose that $\{X_n, n \geq 1\}$ is a sequence of zero mean random variables which is stochastically dominated by a random variable X with $E|X|^p < \infty$. Assume further that (2.1) holds, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\left(\sup_{j \geq n} \left|\frac{S_j}{j^\alpha}\right| \geq \varepsilon\right) < \infty. \tag{2.5}$$

Remark 2.1 In (2.1), $\{Y_{ti}, i \geq 1\}$ is a monotone transformation of $\{X_i, i \geq 1\}$. If $\{X_i, i \geq 1\}$ is a sequence of independent random variables, then (2.1) is clearly satisfied. There are many sequences of dependent random variables satisfying (2.1) for all $q \geq 2$. Examples include sequences of NA random variables (see Shao [10]), ρ^* -mixing random variables (see Utev and Peligrad [11]), φ -mixing random variables with the mixing coefficients satisfying certain conditions (see Wang *et al.* [12]), ρ^- -mixing random variables with the mixing coefficients satisfying certain conditions (see Wang and Lu [13]).

Remark 2.2 In Theorem 2.1, we not only generalize the Baum-Katz-type theorem for the class of random variables satisfying (2.1), but also consider the case $\alpha p = 1$. Furthermore, if we take $\alpha = 1$ and $p = 2$, then we can get the Hsu-Robbins-type theorem (see Hsu and Robbins [1]) for the class of random variables satisfying (2.1).

3 Lemmas

In this section, the following lemmas are very useful to prove the main results of the paper.

Lemma 3.1 (cf. Wu [14]) *Let $\{X_n, n \geq 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X . Then for any $a > 0$ and $b > 0$, the following two statements hold:*

$$E|X_n|^a I(|X_n| \leq b) \leq C_1 \{E|X|^a I(|X| \leq b) + b^a P(|X| > b)\}$$

and

$$E|X_n|^a I(|X_n| > b) \leq C_2 E|X|^a I(|X| > b),$$

where C_1 and C_2 are positive constants.

Lemma 3.2 *Under the conditions of Theorem 2.1,*

$$\sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} \int_{n^\alpha}^{\infty} P(\max_{1 \leq j \leq n} |S_j| > t) dt < \infty. \tag{3.1}$$

Proof For fixed $n \geq 1$, denote $Y'_{ii} = X_i - Y_{ii}$, $i \geq 1$. Then it follows that

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} \int_{n^\alpha}^{\infty} P(\max_{1 \leq j \leq n} |S_j| > t) dt \\ & \leq \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} \int_{n^\alpha}^{\infty} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ii} - EY_{ii}) \right| > t/2\right) dt \\ & \quad + \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} \int_{n^\alpha}^{\infty} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y'_{ii} - EY'_{ii}) \right| > t/2\right) dt \\ & := I + J. \end{aligned}$$

For J , noting that $|Y'_{ii}| \leq |X_i| I(|X_i| > t)$, we have by Markov's inequality, Lemma 3.1 and $E|X|^p < \infty$ that

$$\begin{aligned} J & \leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} \int_{n^\alpha}^{\infty} t^{-1} \sum_{i=1}^n E|Y'_{ii}| dt \\ & \leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} \int_{n^\alpha}^{\infty} t^{-1} \sum_{i=1}^n E|X_i| I(|X_i| > t) dt \\ & \leq C \sum_{n=1}^{\infty} n^{\alpha p - 1 - \alpha} \int_{n^\alpha}^{\infty} t^{-1} E|X| I(|X| > t) dt \\ & = C \sum_{n=1}^{\infty} n^{\alpha p - 1 - \alpha} \sum_{m=n}^{\infty} \int_{m^\alpha}^{(m+1)^\alpha} t^{-1} E|X| I(|X| > t) dt \\ & \leq C \sum_{n=1}^{\infty} n^{\alpha p - 1 - \alpha} \sum_{m=n}^{\infty} m^{-1} E|X| I(|X| > m^\alpha) \end{aligned}$$

$$\begin{aligned}
 &= C \sum_{m=1}^{\infty} m^{-1} E|X| I(|X| > m^\alpha) \sum_{n=1}^m n^{\alpha p-1-\alpha} \\
 &\leq C \sum_{m=1}^{\infty} m^{-1} E|X| I(|X| > m^\alpha) m^{\alpha p-\alpha} \\
 &= C \sum_{m=1}^{\infty} m^{\alpha p-1-\alpha} E|X| I(|X| > m^\alpha) \\
 &= C \sum_{m=1}^{\infty} m^{\alpha p-1-\alpha} \sum_{n=m}^{\infty} E|X| I(n < |X|^{1/\alpha} \leq n+1) \\
 &\leq C \sum_{n=1}^{\infty} E|X| I(n < |X|^{1/\alpha} \leq n+1) \sum_{m=1}^n m^{\alpha p-1-\alpha} \\
 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-\alpha} E|X| I(n < |X|^{1/\alpha} \leq n+1) \leq CE|X|^p < \infty.
 \end{aligned} \tag{3.2}$$

For I , by Markov's inequality and (2.1), we have that for $q \geq 2$,

$$\begin{aligned}
 I &\leq C_q \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^\alpha}^{\infty} t^{-q} E \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ti} - EY_{ti}) \right|^q \right) dt \\
 &\leq C_q \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^\alpha}^{\infty} t^{-q} \sum_{i=1}^n E|Y_{ti}|^q dt + C_q \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^\alpha}^{\infty} t^{-q} \left(\sum_{i=1}^n EY_{ti}^2 \right)^{q/2} dt \\
 &:= I_1 + I_2.
 \end{aligned} \tag{3.3}$$

We will consider the following three cases:

Case 1. $\alpha > 1/2$, $\alpha p > 1$ and $p \geq 2$.

Taking $q > \max(p, \frac{\alpha p-1}{\alpha-\frac{1}{2}})$, which implies that $\alpha p - 2 - \alpha q + q/2 < -1$. We have by Lemma 3.1 and the proof of (3.2) that

$$\begin{aligned}
 I_1 &\leq C_q \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^\alpha}^{\infty} t^{-q} \sum_{i=1}^n (E|X_i|^q I(|X_i| \leq t) + t^q P(|X_i| > t)) dt \\
 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} \int_{n^\alpha}^{\infty} t^{-q} E|X|^q I(|X| \leq t) dt + C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} \int_{n^\alpha}^{\infty} t^{-1} E|X| I(|X| > t) dt \\
 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} \sum_{m=n}^{\infty} \int_{m^\alpha}^{(m+1)^\alpha} t^{-q} E|X|^q I(|X| \leq t) dt + C \\
 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} \sum_{m=n}^{\infty} m^{\alpha-1-\alpha q} E|X|^q I(|X| \leq (m+1)^\alpha) + C \\
 &= C \sum_{m=1}^{\infty} m^{\alpha-1-\alpha q} E|X|^q I(|X| \leq (m+1)^\alpha) \sum_{n=1}^m n^{\alpha p-1-\alpha} + C \\
 &\leq C \sum_{m=1}^{\infty} m^{\alpha p-1-\alpha q} E|X|^q I(m^\alpha < |X| \leq (m+1)^\alpha)
 \end{aligned}$$

$$\begin{aligned}
 &+ C \sum_{m=1}^{\infty} m^{\alpha p-1-\alpha q} E|X|^q I(|X| \leq m^\alpha) + C \\
 \leq &C \sum_{m=1}^{\infty} m^{-1} E|X|^p I(m^\alpha < |X| \leq (m+1)^\alpha) \\
 &+ C \sum_{m=1}^{\infty} m^{\alpha(p-q)-1} \sum_{j=1}^m j^{\alpha q} P(j-1 < |X|^{1/\alpha} \leq j) + C \\
 \leq &CE|X|^p + C \sum_{j=1}^{\infty} j^{\alpha q} P(j-1 < |X|^{1/\alpha} \leq j) \sum_{m=j}^{\infty} m^{\alpha(p-q)-1} + C \\
 \leq &C \sum_{j=1}^{\infty} j^{\alpha p} P(j-1 < |X|^{1/\alpha} \leq j) + C \leq CE|X|^p + C < \infty.
 \end{aligned} \tag{3.4}$$

Note that $EX^2 < \infty$ if $E|X|^p < \infty$ for $p \geq 2$. We have that

$$\begin{aligned}
 I_2 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^\alpha}^{\infty} t^{-q} \left(\sum_{i=1}^n EX_i^2 \right)^{q/2} dt \\
 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^\alpha}^{\infty} t^{-q} \left(\sum_{i=1}^n EX^2 \right)^{q/2} dt \\
 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha+q/2} \int_{n^\alpha}^{\infty} t^{-q} dt \\
 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q+q/2} < \infty.
 \end{aligned}$$

Case 2. $\alpha > 1/2$, $\alpha p > 1$ and $1 < p < 2$.

Take $q = 2$. Similar to the proofs of (3.3) and (3.4), we have that

$$\begin{aligned}
 I &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^\alpha}^{\infty} t^{-2} \sum_{i=1}^n (EX_i^2 I(|X_i| \leq t) + t^2 P(|X_i| > t)) dt \\
 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} \sum_{m=n}^{\infty} m^{-\alpha-1} EX^2 I(|X| \leq (m+1)^\alpha) + C \\
 &= C \sum_{m=1}^{\infty} m^{-\alpha-1} EX^2 I(|X| \leq (m+1)^\alpha) \sum_{n=1}^m n^{\alpha p-1-\alpha} + C \\
 &\leq C \sum_{m=1}^{\infty} m^{\alpha p-1-2\alpha} EX^2 I(m^\alpha < |X| \leq (m+1)^\alpha) \\
 &\quad + C \sum_{m=1}^{\infty} m^{\alpha p-1-2\alpha} EX^2 I(|X| \leq m^\alpha) + C \\
 &\leq C \sum_{m=1}^{\infty} m^{-1} E|X|^p I(m^\alpha < |X| \leq (m+1)^\alpha) \\
 &\quad + C \sum_{m=1}^{\infty} m^{\alpha(p-2)-1} \sum_{j=1}^m j^{2\alpha} P(j-1 < |X|^{1/\alpha} \leq j) + C
 \end{aligned}$$

$$\begin{aligned}
 &\leq CE|X|^p + C \sum_{j=1}^{\infty} j^{2\alpha} P(j-1 < |X|^{1/\alpha} \leq j) \sum_{m=j}^{\infty} m^{\alpha(p-2)-1} + C \\
 &\leq C \sum_{j=1}^{\infty} j^{\alpha p} P(j-1 < |X|^{1/\alpha} \leq j) + C \\
 &\leq CE|X|^p + C < \infty.
 \end{aligned} \tag{3.5}$$

Case 3. $\alpha > 1/2$, $\alpha p = 1$ and $p > 1$.

Take $q = 2$. Note that $1/2 < \alpha < 1$ if $\alpha p = 1$. Similar to the proofs of (3.5), it follows that $I < \infty$. From the statements above, (3.1) is proved. The proof of the lemma is completed. \square

Lemma 3.3 (cf. Sung [15]) *Let $\{Y_n, n \geq 1\}$ and $\{Z_n, n \geq 1\}$ be sequences of random variables. Then for any $q > 1$, $\varepsilon > 0$ and $a > 0$,*

$$E \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_i + Z_i) \right| - \varepsilon a \right)^+ \leq \left(\frac{1}{\varepsilon^q} + \frac{1}{q-1} \right) \frac{1}{a^{q-1}} E \max_{1 \leq j \leq n} \left| \sum_{i=1}^j Y_i \right|^q + E \max_{1 \leq j \leq n} \left| \sum_{i=1}^j Z_i \right|.$$

4 The proofs of main results

Proof of Theorem 2.1 First, we prove (2.2). For fixed $n \geq 1$, let $X_{ni} = -n^\alpha I(X_i < -n^\alpha) + X_i I(|X_i| \leq n^\alpha) + n^\alpha I(X_i > n^\alpha)$ and $X'_{ni} = X_i - X_{ni}$, $i \geq 1$. Then it is easy to have that

$$\begin{aligned}
 &\sum_{n=1}^{\infty} n^{\alpha p-2} P \left(\max_{1 \leq j \leq n} |S_j| \geq \varepsilon n^\alpha \right) \\
 &\leq \sum_{n=1}^{\infty} n^{\alpha p-2} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (X_{ni} - EX_{ni}) \right| \geq \varepsilon n^\alpha / 2 \right) \\
 &\quad + \sum_{n=1}^{\infty} n^{\alpha p-2} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (X'_{ni} - EX'_{ni}) \right| \geq \varepsilon n^\alpha / 2 \right) \\
 &:= I^* + J^*.
 \end{aligned}$$

For J^* , noting that $|X'_{ni}| \leq |X_i| I(|X_i| > n^\alpha)$, we have by Markov's inequality, Lemma 3.1 and the proof of (3.2) that

$$\begin{aligned}
 J^* &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^n E|X'_{ni}| \\
 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^n E|X_i| I(|X_i| > n^\alpha) \\
 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E|X| I(|X| > n^\alpha) \\
 &\leq CE|X|^p < \infty.
 \end{aligned} \tag{4.1}$$

For I^* , by Markov's inequality and (2.1), we have that for any $q \geq 2$,

$$\begin{aligned} I^* &\leq C_q \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha q} E \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (X_{ni} - EX_{ni}) \right|^q \right) \\ &\leq C_q \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha q} \sum_{i=1}^n E |X_{ni}|^q + C_q \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha q} \left(\sum_{i=1}^n EX_{ni}^2 \right)^{q/2} \\ &:= I_1^* + I_2^*. \end{aligned} \tag{4.2}$$

We consider the following three cases:

Case 1. $\alpha > 1/2$, $\alpha p > 1$ and $p \geq 2$.

Take $q > \max(p, \frac{\alpha p - 1}{\alpha - \frac{1}{2}})$, which implies that $\alpha p - 2 - \alpha q + q/2 < -1$.

For I_1^* , we have by C_r 's inequality, the proofs of (3.2) and (3.4) that

$$\begin{aligned} I_1^* &\leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha q} \sum_{i=1}^n (E |X_i|^q I(|X_i| \leq n^\alpha) + n^{\alpha q} P(|X_i| > n^\alpha)) \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha q} \sum_{i=1}^n (E |X|^q I(|X| \leq n^\alpha) + n^{\alpha q} P(|X| > n^\alpha)) \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p - 1 - \alpha q} E |X|^q I(|X| \leq n^\alpha) + C \sum_{n=1}^{\infty} n^{\alpha p - 1 - \alpha} E |X| I(|X| > n^\alpha) \\ &< \infty. \end{aligned} \tag{4.3}$$

For I_2^* , note that $EX^2 < \infty$ if $E|X|^p < \infty$ for $p \geq 2$. We have that

$$\begin{aligned} I_2^* &\leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha q} \left(\sum_{i=1}^n EX_i^2 \right)^{q/2} \leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha q} \left(\sum_{i=1}^n EX^2 \right)^{q/2} \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha q + q/2} < \infty. \end{aligned}$$

Case 2. $\alpha > 1/2$, $\alpha p > 1$ and $1 < p < 2$.

Take $q = 2$. Similar to the proofs of (4.2), (4.3), (3.5) and (4.1), we have that

$$\begin{aligned} I^* &\leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - 2\alpha} \sum_{i=1}^n (EX_i^2 I(|X_i| \leq n^\alpha) + n^{2\alpha} P(|X_i| > n^\alpha)) \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p - 1 - 2\alpha} EX^2 I(|X| \leq n^\alpha) + C \sum_{n=1}^{\infty} n^{\alpha p - 1 - \alpha} E |X| I(|X| > n^\alpha) \\ &< \infty. \end{aligned} \tag{4.4}$$

Case 3. $\alpha > 1/2$, $\alpha p = 1$ and $p > 1$.

Take $q = 2$. Note that $1/2 < \alpha < 1$ if $\alpha p = 1$. Similar to the proof of (4.4), it follows that $I^* < \infty$. From all the statements above, we have proved (2.2).

Next, we prove (2.3). Since $S_j = \sum_{i=1}^j X_i$ and $X_i = (X_{ni} - EX_{ni}) + (X'_{ni} - EX'_{ni})$, $i = 1, 2, \dots, n$. By Lemma 3.3, the proofs of (4.1) and $I^* < \infty$, it follows that

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1 \leq j \leq n} |S_j| - \varepsilon n^\alpha\right)^+ \\ & \leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} E\left(\max_{1 \leq j \leq n} \left|\sum_{i=1}^j (X_{ni} - EX_{ni})\right|^q\right) \\ & \quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1 \leq j \leq n} \left|\sum_{i=1}^j (X'_{ni} - EX'_{ni})\right|\right) \\ & \leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} E\left(\max_{1 \leq j \leq n} \left|\sum_{i=1}^j (X_{ni} - EX_{ni})\right|^q\right) \\ & \quad + C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^n E|X_i|I(|X_i| > n^\alpha) \\ & < \infty. \end{aligned}$$

Hence (2.3) holds.

We will prove the equivalence of (2.2) and (2.3). First, we prove that (2.2) implies (2.3). In fact, for all $\varepsilon > 0$, we have by Lemma 3.2 that

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1 \leq j \leq n} |S_j| - \varepsilon n^\alpha\right)^+ \\ & = \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_0^\infty P\left(\left(\max_{1 \leq j \leq n} |S_j| - \varepsilon n^\alpha\right)^+ > t\right) dt \\ & = \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_0^\infty P\left(\max_{1 \leq j \leq n} |S_j| - \varepsilon n^\alpha > t\right) dt \\ & = \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_0^{n^\alpha} P\left(\max_{1 \leq j \leq n} |S_j| - \varepsilon n^\alpha > t\right) dt \\ & \quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^\alpha}^\infty P\left(\max_{1 \leq j \leq n} |S_j| - \varepsilon n^\alpha > t\right) dt \\ & \leq \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1 \leq j \leq n} |S_j| \geq \varepsilon n^\alpha\right) \\ & \quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^\alpha}^\infty P\left(\max_{1 \leq j \leq n} |S_j| > t\right) dt < \infty. \end{aligned}$$

Next, we prove that (2.3) implies (2.2). It is easy to see that

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1 \leq j \leq n} |S_j| - \varepsilon n^\alpha\right)^+ \\ & = \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_0^\infty P\left(\max_{1 \leq j \leq n} |S_j| - \varepsilon n^\alpha > t\right) dt \end{aligned}$$

$$\begin{aligned}
 &\geq \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_0^{\varepsilon n^\alpha} P\left(\max_{1 \leq j \leq n} |S_j| > \varepsilon n^\alpha + t\right) dt \\
 &\geq \varepsilon \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1 \leq j \leq n} |S_j| \geq 2\varepsilon n^\alpha\right). \tag{4.5}
 \end{aligned}$$

Hence, by (4.5), (2.3) implies (2.2). The proof of the theorem is completed. \square

Proof of Corollary 2.1 Taking $\alpha p = 1$ in Theorem 2.1, we have that for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} |S_j| \geq \varepsilon n^{1/p}\right) < \infty.$$

The rest of the proof is similar to that of Theorem 3.1 in Dung and Tien [16] and is omitted. \square

Proof of Theorem 2.2 We use the same notation as that in the proof of Theorem 2.1. Taking $q = 2$ and $a = n^\alpha$ in Lemma 3.3, by (2.1) and Lemma 3.1, it follows that

$$\begin{aligned}
 &\sum_{n=1}^{\infty} n^{-2} E\left(\max_{1 \leq j \leq n} |S_j| - \varepsilon n^\alpha\right)^+ \\
 &\leq C \sum_{n=1}^{\infty} n^{-\alpha-2} E\left(\max_{1 \leq j \leq n} \left(\sum_{i=1}^j (X_{ni} - EX_{ni})\right)^2\right) + \sum_{n=1}^{\infty} n^{-2} E\left(\max_{1 \leq j \leq n} \left|\sum_{i=1}^j (X'_{ni} - EX'_{ni})\right|\right) \\
 &\leq C \sum_{n=1}^{\infty} n^{-\alpha-2} \sum_{i=1}^n (EX_i^2 I(|X_i| \leq n^\alpha) + n^{2\alpha} P(|X_i| > n^\alpha)) \\
 &\quad + \sum_{n=1}^{\infty} n^{-2} \sum_{i=1}^n E|X_i| I(|X_i| > n^\alpha) \\
 &\leq C \sum_{n=1}^{\infty} n^{-\alpha-1} EX^2 I(|X| \leq n^\alpha) + C \sum_{n=1}^{\infty} n^{-1} E|X| I(|X| > n^\alpha) \\
 &\leq C \sum_{n=1}^{\infty} n^{-\alpha-1} \sum_{m=1}^n m^{2\alpha} P(m-1 < |X|^{1/\alpha} \leq m) \\
 &\quad + C \sum_{n=1}^{\infty} n^{-1} \sum_{m=n}^{\infty} E|X| I(m < |X|^{1/\alpha} \leq m+1) \\
 &= C \sum_{m=1}^{\infty} m^{2\alpha} P(m-1 < |X|^{1/\alpha} \leq m) \sum_{n=m}^{\infty} n^{-\alpha-1} \\
 &\quad + C \sum_{m=1}^{\infty} E|X| I(m < |X|^{1/\alpha} \leq m+1) \sum_{n=1}^m n^{-1} \\
 &\leq C \sum_{m=1}^{\infty} m^\alpha P(m-1 < |X|^{1/\alpha} \leq m) + C \sum_{m=1}^{\infty} E|X| I(m < |X|^{1/\alpha} \leq m+1) \ln^+ m \\
 &\leq CE|X| + CE|X| \ln^+ |X| < \infty.
 \end{aligned}$$

Hence, (2.4) holds. \square

Proof of Theorem 2.3 Inspired by the proof of Theorem 12.1 of Gut [17], we have that

$$\begin{aligned}
 \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\sup_{j \geq n} \left| \frac{S_j}{j^\alpha} \right| \geq \varepsilon\right) &= \sum_{m=1}^{\infty} \sum_{n=2^{m-1}}^{2^m-1} n^{\alpha p-2} P\left(\sup_{j \geq n} \left| \frac{S_j}{j^\alpha} \right| \geq \varepsilon\right) \\
 &\leq C \sum_{m=1}^{\infty} P\left(\sup_{j \geq 2^{m-1}} \left| \frac{S_j}{j^\alpha} \right| \geq \varepsilon\right) \sum_{n=2^{m-1}}^{2^m-1} 2^{m(\alpha p-2)} \\
 &\leq C \sum_{m=1}^{\infty} 2^{m(\alpha p-1)} P\left(\sup_{j \geq 2^{m-1}} \left| \frac{S_j}{j^\alpha} \right| \geq \varepsilon\right) \\
 &= C \sum_{m=1}^{\infty} 2^{m(\alpha p-1)} P\left(\sup_{k \geq m} \max_{2^{k-1} \leq j < 2^k} \left| \frac{S_j}{j^\alpha} \right| \geq \varepsilon\right) \\
 &\leq C \sum_{m=1}^{\infty} 2^{m(\alpha p-1)} \sum_{k=m}^{\infty} P\left(\max_{1 \leq j \leq 2^k} |S_j| \geq \varepsilon 2^{\alpha(k-1)}\right) \\
 &\leq C \sum_{k=1}^{\infty} P\left(\max_{1 \leq j \leq 2^k} |S_j| \geq \varepsilon 2^{\alpha(k-1)}\right) \sum_{m=1}^k 2^{m(\alpha p-1)} \\
 &\leq C \sum_{k=1}^{\infty} 2^{k(\alpha p-1)} P\left(\max_{1 \leq j \leq 2^k} |S_j| \geq \varepsilon 2^{\alpha(k-1)}\right) \\
 &\leq C \sum_{k=1}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} n^{\alpha p-2} P\left(\max_{1 \leq j \leq n} |S_j| \geq \left(\frac{\varepsilon}{4^\alpha}\right) n^\alpha\right) \\
 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1 \leq j \leq n} |S_j| \geq \left(\frac{\varepsilon}{4^\alpha}\right) n^\alpha\right). \tag{4.6}
 \end{aligned}$$

The desired result (2.5) follows from (2.2) and (4.6) immediately. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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