A viscosity hybrid steepest-descent method for a system of equilibrium and fixed point problems for an infinite family of strictly pseudo-contractive mappings

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Abstract

Based on a viscosity hybrid steepest-descent method, in this paper, we introduce an iterative scheme for finding a common element of a system of equilibrium and fixed point problems of an infinite family of strictly pseudo-contractive mappings which solves the variational inequality $\langle (\gamma f - \mu F)q, p - q \rangle \le 0$ for $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Furthermore, we also prove the strong convergence theorems for the proposed iterative scheme and give a numerical example to support and illustrate our main theorem.

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1 Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. A self-mapping $f : C \to C$ is said to be a contraction on C if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \le \alpha \|x - y\|$, $\forall x, y \in C$. We denote by Π_C the collection of mappings f verifying the above inequality and note that each $f \in \Pi_C$ has a unique fixed point in C.

A mapping $T : C \to C$ is said to be λ-strictly pseudo-contractive if there exists a constant $\lambda \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \lambda \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C, \tag{1.1}$$

and we denote by $F(T)$ the set of fixed points of the mapping T; that is, $F(T) = \{x \in C : Tx = x\}$.

Note that T is the class of λ-strictly pseudo-contractive mappings including the class of nonexpansive mappings T on C (that is, $\|Tx - Ty\| \le \|x - y\|$, $x, y \in C$) as a subclass. That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive.

A mapping $F : C \to C$ is called k-Lipschitzian if there exists a positive constant k such that

$$\|Fx - Fy\| \le k\|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

F is said to be η-strongly monotone if there exists a positive constant η such that

$$\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

Definition 1.1 A bounded linear operator A is said to be *strongly positive*, if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H.$$

In 2006, Marino and Xu [1] introduced the following iterative scheme: for $x_1 = x \in C$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n, \quad n \geq 1. \tag{1.4}$$

They proved that under appropriate conditions of the sequence $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution of the variational inequality $\langle (\gamma f - A) q, p - q \rangle \leq 0$, $p \in F(T)$, which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In 2010, Jung [2] extended the result of Marino and Xu [1] to the class of k-strictly pseudo-contractive mappings $T : C \to H$ with $F(T) \neq \emptyset$ and introduced the following iterative scheme: for $x_1 = x \in C$,

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) P_C S x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \quad n \geq 1, \end{cases} \tag{1.5}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1 - k) Tx$. He proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to a fixed point q of T, which is the unique solution of the variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad p \in F(T).$$

Later, Tian [3] considered the following iterative method for a nonexpansive mapping $T : H \to H$ with $F(T) \neq \emptyset$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n F) T x_n, \quad n \geq 1, \tag{1.6}$$

where F is a k-Lipschitzian and η-strongly monotone operator. He proved that the sequence $\{x_n\}$ generated by (1.6) converges to a fixed point q in $F(T)$, which is the unique solution of the variational inequality

$$\langle (\gamma f - \mu F) q, p - q \rangle \leq 0, \quad p \in F(T).$$

In 2010, Saeidi [4] introduced the following modified hybrid steepest-descent iterative algorithm for finding a common element of the set of solutions of a system of equilibrium

problems for a family $\mathcal{F} = \{F_j : C \times C \to \mathbb{R}, j = 1, 2, \ldots, M\}$ and the set of common fixed points for a family of infinitely nonexpansive mappings $\mathcal{S} = \{S_i : C \to C\}$ with respect to W-mappings (see [5]):

$$\begin{cases} y_n = W_n J_{r_M,n}^{F_M} \cdots J_{r_2,n}^{F_2} J_{r_1,n}^{F_1} x_n, \\ x_{n+1} = \beta x_n + (1 - \beta)(I - \lambda_n B) y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where B is a relaxed (γ, r)-cocoercive, k-Lipschitzian mapping such that $r > \gamma k^2$. Then, under weaker hypotheses on coefficients, he proved the strongly convergence of the proposed iterative algorithm to the unique solution of the variational inequality.

Recently, Wang [6] extended and improved all the above results. He introduced a new iterative scheme: for $x_1 = x \in C$,

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) W_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n F) y_n, \quad n \geq 1, \end{cases} \tag{1.8}$$

where W_n is a mapping defined by (2.3), and F is a k-Lipschitzian and η-strongly monotone operator with $0 < \mu < 2\eta/k^2$. He proved that the sequence $\{x_n\}$ generated by (1.7) converges strongly to a common fixed point of an infinite family of λ_i-strictly pseudo-contractive mappings, which is a unique solution of the variational inequality

$$\langle (\gamma f - \mu F) q, p - q \rangle \leq 0, \quad p \in \bigcap_{i=1}^{\infty} F(T_i).$$

Very recently, He, Liu and Cho [7] introduced an explicit scheme which was defined by the following suitable sequence:

$$z_{n+1} = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A) W_n S_{r_1,n}^1 S_{r_2,n}^2 \cdots S_{r_K,n}^K z_n, \quad \forall n \in N.$$

They generated W_n-mapping by $\{T_i\}$ and $\{\lambda_n\}$ where $\{T_i\}$ is a family of nonexpansive mappings from H into itself. They found that if $\{r_{k,n}\}_{k=1}^{K}$, $\{\epsilon_n\}$ and $\{\lambda_n\}$ satisfy appropriate conditions and $F := (\bigcap_{k=1}^{K} SEP(G_K)) \cap (\bigcap_{n \in N} F(T_n)) \neq \emptyset$, then $\{z_n\}$ converges strongly to $x^* \in F$, which satisfies the variational inequality $\langle (A - \gamma f) x^*, x - x^* \rangle \geq 0$ for all $x \in F$.

In this paper, we introduce a new iterative scheme in a Hilbert space H which is a mixed iterative scheme of (1.7) and (1.8). We prove that the sequence converges strongly to a common element of the set of solutions of the system of equilibrium problems and the set of common fixed points of an infinite family of strictly pseudo-contractive mappings by using a viscosity hybrid steepest-descent method. The results obtained in this paper improved and extended the above mentioned results and many others. Finally, we give a simple numerical example to support and illustrate our main theorem in the last part.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We have

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle, \quad \forall x, y \in H. \tag{2.1}$$

Recall that the nearest projection P_C from H to C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

We recall some lemmas which will be needed in the rest of this paper.

Lemma 2.1 *In a Hilbert space H, the following inequality holds*:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle, \quad x, y \in H.$$

Lemma 2.2 *Let B be a k-Lipschitzian and η-strongly monotone operator on a Hilbert space H with $k > 0$, $\eta > 0$, $0 < \mu < 2\eta/k^2$ and $0 < t < 1$. Then $S = (I - t\mu B) : H \to H$ is a contraction with a contractive coefficient $1 - t\tau$ and $\tau = \frac{1}{2}\mu(2\eta - \mu k^2)$.*

Proof From (1.2), (1.3) and (2.1), we have

$$
\begin{aligned}
\|Sx - Sy\|^2 &= \left\|(x - y) - t\mu(Bx - By)\right\|^2 \\
&= \|x - y\|^2 + t^2 \mu^2 \|Bx - By\|^2 - 2\mu t \langle Bx - By, x - y \rangle \\
&\le \|x - y\|^2 + t^2 \mu^2 k^2 \|x - y\|^2 - 2\mu t \eta \|x - y\|^2 \\
&= \left[1 - t\mu(2\eta - \mu k^2)\right]\|x - y\|^2 \\
&\le (1 - t\tau)^2 \|x - y\|^2,
\end{aligned}
$$

where $\tau = \frac{1}{2}\mu(2\eta - \mu k^2)$, and so, $\|Sx - Sy\| \le (1 - t\tau)\|x - y\|$.

Hence, S is a contraction with a contractive coefficient $1 - t\tau$. $\qquad\square$

Lemma 2.3 *Let H be a Hilbert space. For a given $z \in H$ and $u \in C$,*

$$u = P_C z \quad \Leftrightarrow \quad \langle u - z, v - u \rangle \ge 0, \quad \forall v \in C.$$

Lemma 2.4 *Let H be a real Hilbert space. For q which solves the variational inequality $\langle(\gamma f - \mu B)q, p - q\rangle \le 0, f \in \Pi_H, p \in F(T)$, the following statement is true*:

$$\langle(\gamma f - \mu B)q, p - q\rangle \le 0 \quad \Leftrightarrow \quad P_\Theta(I - \mu B + \gamma f)q = q, \tag{2.2}$$

where $\Theta := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^{M} SEP(F_j))$.

Proof From Lemma (2.3), it follows that

$$
\begin{aligned}
q = P_\Theta(I - \mu B + \gamma f)q \quad &\Leftrightarrow \quad \langle q - (I - \mu B + \gamma f)q, p - q \rangle \ge 0, \quad p \in \Theta, \\
&\Leftrightarrow \quad \langle(\mu B - \gamma f)q, p - q\rangle \ge 0 \\
&\Leftrightarrow \quad \langle(\gamma f - \mu B)q, p - q\rangle \le 0. \qquad\square
\end{aligned}
$$

Lemma 2.5 [8] *Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$; if the sequence $\{x_n\}$ weakly converges to x and $(I - T)x_n$ converges strongly to y, then $(I - T)x = y$.*

Lemma 2.6 [9] *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and $\{\gamma_n\}$ be a sequence in $[0, 1]$ which satisfies the following condition:*

$$0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1.$$

Suppose that $x_{n+1} = \gamma_n x_n + (1 - \gamma_n)z_n$, $n \geq 0$ and $\limsup_{n \to \infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n \to \infty} \|z_n - x_n\| = 0$.

Lemma 2.7 [10, 11] *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n \delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:
  (i) *$\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=0}^{\infty} = \infty$;*
  (ii) *$\limsup_{n \to \infty} \delta_n \leq 0$ or $\sum_{n=0}^{\infty} \lambda_n \delta_n < \infty$;*
  (iii) *$\gamma_n \geq 0 \ (n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.*
Then $\lim_{n \to \infty} s_n = 0$.

Lemma 2.8 [12] *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a λ-strictly pseudo-contractive mapping. Define a mapping $S : C \to C$ by $Sx = \alpha x + (1 - \alpha)Tx$ for all $x \in C$ and $\alpha \in [\lambda, 1)$. Then S is a nonexpansive mapping such that $F(S) = F(T)$.*

In this work, we defined the mapping W_n by

$$
\begin{aligned}
&U_{n,n+1} = I, \\
&U_{n,n} = \gamma_n T'_n U_{n,n+1} + (1 - \gamma_n)I, \\
&U_{n,n-1} = \gamma_{n-1} T'_{n-1} U_{n,n} + (1 - \gamma_{n-1})I, \\
&\vdots \\
&U_{n,k} = \gamma_k T'_k U_{n,k+1} + (1 - \gamma_k)I, \\
&U_{n,k-1} = \gamma_{k-1} T'_{k-1} U_{n,k} + (1 - \gamma_{k-1})I, \\
&\vdots \\
&U_{n,2} = \gamma_2 T'_2 U_{n,3} + (1 - \gamma_2)I, \\
&W_n = U_{n,1} = \gamma_1 T'_1 U_{n,2} + (1 - \gamma_1)I,
\end{aligned}
\tag{2.3}
$$

where $\gamma_1, \gamma_2, \ldots$ are real numbers such that $0 \leq \gamma_n \leq 1$, $T'_i = \theta_i I + (1 - \theta_i)T_i$ where T_i is a λ_i-strictly pseudo-contractive mapping of C into itself and $\theta_i \in [\lambda_i, 1)$. By Lemma 2.8, we know that T'_i is a nonexpansive mapping and $F(T_i) = F(T'_i)$. As a result, it can be easily seen that W_n is also a nonexpansive mapping.

Lemma 2.9 [5] *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T'_1, T'_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T'_i) \neq \emptyset$ and $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_i \leq b < 1$ for each $i = 1, 2, \ldots$. Then for any $x \in C$ and $k \in N$, the limit $\lim_{n \to \infty} U_{n,k} x$ exists.*

By using Lemma 2.8, one can define the mapping W of C into itself as follows:

$$Wx := \lim_{n \to \infty} W_n x = \lim_{n \to \infty} U_{n,1} x, \quad x \in C. \tag{2.4}$$

Such a mapping W is called the modified W-mapping generated by $T_1, T_2, \ldots, \gamma_1, \gamma_2, \ldots$ and $\theta_1, \theta_2, \ldots$.

Lemma 2.10 [5] *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T'_1, T'_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T'_i) \neq \emptyset$ and $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_i \leq b < 1$ for each $i = 1, 2, \ldots$. Then $F(W) = \bigcap_{i=1}^{\infty} F(T'_i)$.*

Combining Lemmas 2.7-2.9, one can get that $F(W) = \bigcap_{i=1}^{\infty} F(T'_i) = \bigcap_{i=1}^{\infty} F(T_i)$.

Lemma 2.11 [13] *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T'_i : C \to C\}$ be a family of infinite nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T'_i) \neq \emptyset$, $\{\gamma_i\}$ be a real sequence such that $0 < \gamma_i \leq b < 1$, for each $i \geq 1$. If K is any bounded subset of C, then*

$$\lim_{n \to \infty} \sup_{x \in K} \| Wx - W_n x \| = 0. \tag{2.5}$$

For solving the equilibrium problem, let us give the following assumptions on a bifunction $F : C \times C \to \mathbb{R}$, which were imposed in [14]:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.12 [14] *Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Then for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0. \tag{2.6}$$

Lemma 2.13 [15] *Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$, define a mapping $J_r^F : H \to C$ as follows*:

$$J_r^F(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\} \tag{2.7}$$

for all $x \in H$. Then the following conclusions hold:

(1) J_r^F *is single-valued*;
(2) J_r^F *is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\left\| J_r^F(x) - J_r^F(y) \right\|^2 \leq \langle J_r^F(x) - J_r^F(y), x - y \rangle;$$

(3) $F(J_r^F) = EP(F)$;

(4) $EP(F)$ *is closed and convex.*

Lemma 2.14 [5] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let*
$\mathcal{T} = \{T_i\}_{i=1}^N$ *be an infinite family of nonexpanxive mappings with $F(\mathcal{T}) = \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ and*
$\{\gamma_i\}$ *be a real sequence such that $0 < \gamma_i \leq b < 1$ for each $i \geq 1$. Then:*

(1) W_n *is nonexpansive and $F(W_n) = \bigcap_{i=1}^n F(T_i)$ for each $n \geq 1$;*

(2) *for each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} U_{n,k}x$ exists;*

(3) *the mapping $W : C \to C$ defined by $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$ is a*
nonexpansive mapping satisfying $F(W) = F(\mathcal{T})$ and it is called the W-mapping
generated by $T_1, T_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$;

(4) *if K is any bounded subset of C, then $\lim_{n\to\infty} \sup_{x\in K} \| Wx - W_n x \| = 0$.*

3 Main results

In this section, we will introduce an iterative scheme by using a viscosity hybrid steepest-descent method for finding a common element of the set of variational inequalities, fixed points for an infinite family of strictly pseudo-contractive mappings and the set of solutions of a system of equilibrium problems in a real Hilbert space.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H, let*
$T_i : H \to H$ *be a λ_i-strictly pseudo-contractive mapping with $\bigcap_{i=1}^\infty F(T_i) \neq \emptyset$, $\mathcal{F} = \{F_j : j = 1, 2, 3, \ldots, M\}$ be a finite family of bifunctions $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4) *and γ_i*
be a real sequence such that $0 \leq \gamma_i \leq b \leq 1$ for each $i \geq 1$. Let B be a k-Lipschitzian and η-strongly monotone operator on C with $0 < \mu < \eta/k^2$ and $f \in \Pi_H$ with $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$ and $\tau < 1$. Assume that $\Theta := (\bigcap_{i=1}^\infty F(T_i)) \cap (\bigcap_{j=1}^M SEP(F_j)) \neq \emptyset$. Let the mapping W_n be
defined by (2.3). *Let $\{x_n\}$ be the sequence generated by $x_1 \in H$ and*

$$\begin{cases} u_n = J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} \cdots J_{r_{2,n}}^{F_2} J_{r_{1,N}}^{F_1} x_n, \\ y_n = \beta_n x_n + (1 - \beta_n) W_n u_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu B) y_n, \quad \forall n \in \mathbb{N} \text{ and } n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ which satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and $\Sigma_{n=1}^\infty \alpha_n = \infty$;*

(C2) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n \leq a < 1$ *for some constant $a \in (0, 1)$;*

(C3) $\liminf_{n\to\infty} r_{j,n} > 0$, *for each $j = 1, 2, \ldots, M$.*

Then the sequence $\{x_n\}$ converges strongly to $q \in \Theta$, where $q = P_\Theta(I - \mu B + \gamma f)q$, which is
the unique solution of the variational inequality

$$\langle (\gamma f - \mu B)q, p - q \rangle \leq 0, \quad \forall p \in \Theta, \tag{3.2}$$

or equivalently, q is the unique solution of the minimization problem

$$\min_{x\in\Theta} \frac{1}{2} \langle Ax, x \rangle + h(x),$$

where h is a potential function for γf.

Proof We will divide the proof of Theorem 3.1 into several steps.

Step 1. We show that $\{x_n\}$ is bounded. Let $p \in \Theta$. Since for each $k = 1, 2, 3, \ldots, M$, $J_{r_{k,n}}^{F_k}$ is nonexpansive. Given $\Im_n^k = J_{r_{k,n}}^{F_k} J_{r_{k-1,n}}^{F_{k-1}} J_{r_{k-2,n}}^{F_{k-2}} \cdots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1}$ for $k \in \{1, 2, 3, \ldots, M\}$ and $\Im_n^0 = I$, for each $n \in \mathbb{N}$, we have

$$\left\| \Im_n^k x_n - p \right\| = \left\| \Im_n^k x_n - \Im_n^k p \right\| \leq \|x_n - p\|.$$

Consider,

$$\begin{aligned}
\|y_n - p\| &= \left\| \beta_n x_n + (1 - \beta_n) W_n u_n - p \right\| \\
&= \left\| \beta_n (x_n - p) + (1 - \beta_n)(W_n u_n - p) \right\| \\
&\leq \beta_n \|x_n - p\| + (1 - \beta_n) \left\| (W_n u_n - p) \right\| \\
&= \beta_n \|x_n - p\| + (1 - \beta_n) \left\| W_n \Im_n^M x_n - p \right\| \\
&\leq \beta_n \|x_n - p\| + (1 - \beta_n) \left\| \Im_n^M x_n - p \right\| \\
&\leq \beta_n \|x_n - p\| + (1 - \beta_n) \|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned} \tag{3.3}$$

From Lemma 2.2, (3.1) and (3.3), it follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\| \alpha_n \gamma f(x_n) + (I - \alpha_n \mu B) y_n - p \right\| \\
&= \left\| \alpha_n \big(\gamma f(x_n) - \mu B p \big) + (I - \alpha_n \mu B) y_n - (I - \alpha_n \mu B) p \right\| \\
&= \left\| (I - \alpha_n \mu B)(y_n - p) + \alpha_n \big(\gamma f(x_n) - \mu B p + \gamma f(p) - \gamma f(p) \big) \right\| \\
&\leq \left\| (I - \alpha_n \mu B)(y_n - p) \right\| + \alpha_n \gamma \left\| f(x_n) - f(p) \right\| + \alpha_n \left\| \gamma f(p) - \mu B p \right\| \\
&\leq (1 - \alpha_n \tau) \|y_n - p\| + \alpha_n \gamma \alpha \|x_n - p\| + \alpha_n \left\| \gamma f(p) - \mu B p \right\| \\
&\leq (1 - \alpha_n \tau) \|x_n - p\| + \alpha_n \gamma \alpha \|x_n - p\| + \alpha_n \left\| \gamma f(p) - \mu B p \right\| \\
&= \big[1 - \alpha_n (\tau - \gamma \alpha) \big] \|x_n - p\| + \alpha_n \frac{\tau - \gamma \alpha}{\tau - \gamma \alpha} \left\| \gamma f(p) - \mu B p \right\| \\
&\leq \max \left\{ \|x_n - p\|, \frac{\|\gamma f(p) - \mu B p\|}{\tau - \gamma \alpha} \right\}; \quad n \geq 1.
\end{aligned} \tag{3.4}$$

By mathematical induction, we have

$$\|x_n - p\| \leq \max \left\{ \|x_1 - p\|, \frac{\|\gamma f(p) - \mu B p\|}{\tau - \gamma \alpha} \right\}, \quad \forall n \geq 1, \tag{3.5}$$

and we obtain $\{x_n\}$ is bounded. So are $\{y_n\}$, $\{W_n \Im_n^k(x_n)\}$ and $\{f(x_n)\}$.

Step 2. We claim that if $\{x_n\}$ is a bounded sequence in C, then

$$\lim_{n \to \infty} \left\| \Im_n^k x_n - \Im_{n+1}^k x_n \right\| = 0 \tag{3.6}$$

for every $k \in \{1, 2, 3, \ldots, M\}$. From Step 2 of the proof in [16, Theorem 3.1], we have for $k \in \{1, 2, 3, \ldots, M\}$,

$$\lim_{n \to \infty} \left\| J_{r_{k,n+1}}^{F_k} x_n - J_{r_{k,n}}^{F_k} x_n \right\| = 0. \tag{3.7}$$

Note that for every $k \in \{1, 2, 3, \ldots, M\}$, we have

$$\Im_n^k = J_{r_{k,n}}^{F_k} J_{r_{k-1,n}}^{F_{k-1}} J_{r_{k-2,n}}^{F_{k-2}} \cdots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} = J_{r_{k,n}}^{F_k} \Im_n^{k-1}.$$

So, we note that

$$
\begin{aligned}
\left\| \Im_n^k x_n - \Im_{n+1}^k x_n \right\| &= \left\| J_{r_{k,n}}^{F_k} \Im_n^{k-1} x_n - J_{r_{k,n+1}}^{F_k} \Im_{n+1}^{k-1} x_n \right\| \\
&\leq \left\| J_{r_{k,n}}^{F_k} \Im_n^{k-1} x_n - J_{r_{k,n+1}}^{F_k} \Im_n^{k-1} x_n \right\| + \left\| J_{r_{k,n+1}}^{F_k} \Im_n^{k-1} x_n - J_{r_{k,n+1}}^{F_k} \Im_{n+1}^{k-1} x_n \right\| \\
&\leq \left\| J_{r_{k,n}}^{F_k} \Im_n^{k-1} x_n - J_{r_{k,n+1}}^{F_k} \Im_n^{k-1} x_n \right\| + \left\| \Im_n^{k-1} x_n - \Im_{n+1}^{k-1} x_n \right\| \\
&\leq \left\| J_{r_{k,n}}^{F_k} \Im_n^{k-1} x_n - J_{r_{k,n+1}}^{F_k} \Im_n^{k-1} x_n \right\| + \left\| J_{r_{k-1,n}}^{F_{k-1}} \Im_n^{k-2} x_n - J_{r_{k-1,n+1}}^{F_{k-1}} \Im_n^{k-2} x_n \right\| \\
&\quad + \left\| \Im_n^{k-2} x_n - \Im_{n+1}^{k-2} x_n \right\| \\
&\leq \left\| J_{r_{k,n}}^{F_k} \Im_n^{k-1} x_n - J_{r_{k,n+1}}^{F_k} \Im_n^{k-1} x_n \right\| + \left\| J_{r_{k-1,n}}^{F_{k-1}} \Im_n^{k-2} x_n - J_{r_{k-1,n+1}}^{F_{k-1}} \Im_n^{k-2} x_n \right\| \\
&\quad + \cdots + \left\| J_{r_{2,n}}^{F_2} \Im_n^1 x_n - J_{r_{2,n+1}}^{F_2} \Im_n^1 x_n \right\| + \left\| J_{r_{1,n}}^{F_1} x_n - J_{r_{1,n+1}}^{F_1} x_n \right\|. \tag{3.8}
\end{aligned}
$$

Now, applying (3.7) to (3.8), we conclude (3.6).

Step 3. We show that $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$.

We define a sequence $\{z_n\}$ by $z_n = (x_{n+1} - \beta_n x_n)/(1 - \beta_n)$, so that $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$. We now observe that

$$
\begin{aligned}
z_{n+1} - z_n &= \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1} \gamma f(x_{n+1}) + (I - \mu \alpha_{n+1} B) y_{n+1} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} \\
&\quad - \frac{\alpha_n \gamma f(x_n) + (I - \mu \alpha_n B) y_n - \beta_n x_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1}(\gamma f(x_{n+1}) - \mu B y_{n+1})}{1 - \beta_{n+1}} - \frac{\alpha_n(\gamma f(x_n) - \mu B y_n)}{1 - \beta_n} + W_{n+1} u_{n+1} - W_n u_n. \tag{3.9}
\end{aligned}
$$

It follows from (3.9) that

$$
\begin{aligned}
\|z_{n+1} - z_n\| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \left(\left\| \gamma f(x_{n+1}) \right\| + \|\mu B y_{n+1}\| \right) + \frac{\alpha_n}{1 - \beta_n} \left(\left\| \gamma f(x_n) \right\| + \|\mu B y_n\| \right) \\
&\quad + \| W_{n+1} u_{n+1} - W_n u_n \|. \tag{3.10}
\end{aligned}
$$

We observe that

$$
\begin{aligned}
&\| W_{n+1} u_{n+1} - W_n u_n \| \\
&= \left\| W_{n+1} \Im_{n+1}^k x_{n+1} - W_n \Im_n^k x_n \right\| \\
&\leq \left\| W_{n+1} \Im_{n+1}^k x_{n+1} - W_{n+1} \Im_n^k x_n \right\| + \left\| W_{n+1} \Im_n^k x_{n+1} - W_n \Im_n^k x_n \right\|, \tag{3.11}
\end{aligned}
$$

and compute

$$
\begin{aligned}
\left\| W_{n+1} \Im_n^k x_{n+1} - W_n \Im_n^k x_n \right\| &\leq \left\| W_{n+1} \Im_n^k x_{n+1} - W_{n+1} \Im_n^k x_n \right\| + \left\| W_{n+1} \Im_n^k x_n - W_n \Im_n^k x_n \right\| \\
&\leq \|x_{n+1} - x_n\| + \left\| W_{n+1} \Im_n^k x_n - W_n \Im_n^k x_n \right\|. \tag{3.12}
\end{aligned}
$$

Consider,

$$
\begin{aligned}
\left\| W_{n+1}\mathfrak{J}_n^k x_n - W_n\mathfrak{J}_n^k x_n \right\| &= \left\| \gamma_1 T_1' U_{n+1,2}\mathfrak{J}_n^k x_n - \gamma_1 T_1' U_{n,2}\mathfrak{J}_n^k x_n \right\| \\
&\leq \gamma_1 \left\| U_{n+1,2}\mathfrak{J}_n^k x_n - U_{n,2}\mathfrak{J}_n^k x_n \right\| \\
&= \gamma_1 \left\| \gamma_2 T_2' U_{n+1,3}\mathfrak{J}_n^k x_n - \gamma_2 T_2' U_{n,3}\mathfrak{J}_n^k x_n \right\| \\
&\leq \cdots \\
&\leq \gamma_1\gamma_2\cdots\gamma_n \left\| U_{n+1,n+1}\mathfrak{J}_n^k x_n - U_{n,n+1}\mathfrak{J}_n^k x_n \right\| \\
&\leq M_1 \prod_{i=1}^n \gamma_i,
\end{aligned}
\tag{3.13}
$$

where $M_1 \geq 0$ is a constant such that $\|U_{n+1,n+1}u_n - U_{n,n+1}u_n\| \leq M_1$ for all $n \geq 1$.

Substituting (3.11) and (3.13) into (3.10), we can obtain

$$
\begin{aligned}
\|z_{n+1} - z_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\left\| \gamma f(x_{n+1}) \right\| + \|\mu B y_{n+1}\| \right) + \frac{\alpha_n}{1-\beta_n}\left(\left\| \gamma f(x_n) \right\| + \|\mu B y_n\| \right) \\
&\quad + \left\| W_{n+1}\mathfrak{J}_{n+1}^k x_{n+1} - W_{n+1}\mathfrak{J}_n^k x_{n+1} \right\| + \|x_{n+1} - x_n\| + M_1 \prod_{i=1}^n \gamma_i \\
&\leq M_2 \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n} \right) + \left\| W_{n+1}\mathfrak{J}_{n+1}^k x_{n+1} - W_{n+1}\mathfrak{J}_n^k x_{n+1} \right\| \\
&\quad + \|x_{n+1} - x_n\| + M_1 \prod_{i=1}^n \gamma_i
\end{aligned}
\tag{3.14}
$$

where $M_2 = \sup\{\|\gamma f(x_n)\| + \|\mu B y_n\|, n \geq 1\}$.

It follows from (3.14) that

$$
\begin{aligned}
\|z_{n+1} - z_n\| &\leq M_2 \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n} \right) + \left\| \mathfrak{J}_{n+1}^k x_{n+1} - \mathfrak{J}_n^k x_{n+1} \right\| \\
&\quad + \|x_{n+1} - x_n\| + M_1 \prod_{i=1}^n \gamma_i.
\end{aligned}
$$

Hence, we have

$$
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq M_2 \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n} \right) + \left\| \mathfrak{J}_{n+1}^k x_{n+1} - \mathfrak{J}_n^k x_{n+1} \right\| + M_1 \prod_{i=1}^n \gamma_i.
$$

From $\lim_{n\to\infty} \|\mathfrak{J}_n^k x_n - \mathfrak{J}_{n+1}^k x_n\| = 0$ and the condition $\lim_{n\to\infty} \alpha_n = 0$ and $0 < \lim_{n\to\infty} \inf \beta_n < \lim_{n\to\infty} \sup \beta_n \leq a < 1$ for some $a \in (0,1)$, it follows that

$$
\lim_{n\to\infty} \sup \left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \right) \leq 0.
\tag{3.15}
$$

By Lemma 2.5, we obtain

$$
\lim_{n\to\infty} \|z_n - x_n\| = 0.
\tag{3.16}
$$

From $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$ and by (3.16), we get

$$\|x_{n+1} - x_n\| = (1 - \beta_n)\|z_n - x_n\|. \tag{3.17}$$

Hence,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \beta_n)\|z_n - x_n\| = 0.$$

Step 4. We claim that $\lim_{n\to\infty} \|x_n - W_n u_n\| = 0$.

$$
\begin{aligned}
\|x_n - W_n u_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - W_n u_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n \gamma f(x_n) + (I - \alpha_n \mu B)y_n - W_n u_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n \gamma f(x_n) + y_n - \alpha_n \mu B y_n - W_n u_n\| \\
&\leq \|x_{n+1} - x_n\| + \alpha_n \|\gamma f(x_n) - \mu B y_n\| + \|y_n - W_n u_n\| \\
&\leq \|x_{n+1} - x_n\| + \alpha_n \|\gamma f(x_n) - \mu B y_n\| + \|\beta_n x_n + (1 - \beta_n)W_n u_n - W_n u_n\| \\
&\leq \|x_{n+1} - x_n\| + \alpha_n \|\gamma f(x_n) - \mu B y_n\| + \beta_n \|x_n - W_n u_n\|.
\end{aligned}
$$

It follows that

$$\|x_n - W_n u_n\| \leq \frac{1}{1 - \beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1 - \beta_n}\|\gamma f(x_n) - \mu B y_n\|.$$

By the conditions (C1) and (C2), we obtain

$$\lim_{n\to\infty} \|x_n - W_n u_n\| = 0. \tag{3.18}$$

Step 5. We show that

$$\lim_{n\to\infty} \left\|\Im_n^k x_n - \Im_n^{k+1} x_n\right\| = 0, \quad \forall k = 1, 2, \ldots, M - 1 \tag{3.19}$$

for any $p \in \Theta$ and $\forall k = 1, 2, \ldots, M - 1$. We note that $J_{r_{k+1,n}}^{F_{k+1}}$ is firmly nonexpansive by Lemma 2.12, then we observe that

$$
\begin{aligned}
\left\|\Im_n^{k+1} x_n - p\right\| &= \left\|J_{r_{k+1,n}}^{F_{k+1}} \Im_n^k x_n - J_{r_{k+1,n}}^{F_{k+1}} p\right\| \\
&\leq \left\langle J_{r_{k+1,n}}^{F_{k+1}} \Im_n^k x_n - J_{r_{k+1,n}}^{F_{k+1}} p, \Im_n^k x_n - p\right\rangle \\
&= \left\langle \Im_n^{k+1} x_n - p, \Im_n^k x_n - p\right\rangle \\
&= \frac{1}{2}\left(\left\|\Im_n^{k+1} x_n - p\right\|^2 + \left\|\Im_n^k x_n - p\right\|^2 - \left\|\Im_n^{k+1} x_n - \Im_n^k x_n\right\|^2\right),
\end{aligned}
$$

and hence

$$
\begin{aligned}
\left\|\Im_n^{k+1} x_n - p\right\|^2 &\leq \left\|\Im_n^k x_n - p\right\|^2 - \left\|\Im_n^{k+1} x_n - \Im_n^k x_n\right\|^2 \\
&\leq \|x_n - p\|^2 - \left\|\Im_n^{k+1} x_n - \Im_n^k x_n\right\|^2. \tag{3.20}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\| \alpha_n \big(\gamma f(x_n) - \mu Bp \big) + (I - \alpha_n \mu B)(y_n - p) \right\|^2 \\
&= \alpha_n^2 \left\| \big(\gamma f(x_n) - \mu Bp \big) \right\|^2 + (1 - \alpha_n \tau)^2 \|y_n - p\|^2 \\
&\quad + 2\alpha_n \big\langle \big(\gamma f(x_n) - \mu Bp \big), (I - \alpha_n \mu B)(y_n - p) \big\rangle \\
&\leq \alpha_n^2 \left\| \big(\gamma f(x_n) - \mu Bp \big) \right\|^2 + (1 - \alpha_n \tau)^2 \big[\beta_n \|x_n - p\|^2 + (1 - \beta_n) \big\| \Im_n^M x_n - p \big\|^2 \big] \\
&\quad + 2\alpha_n \big\langle \big(\gamma f(x_n) - \mu Bp \big), (I - \alpha_n \mu B)(y_n - p) \big\rangle \\
&\leq (1 - \alpha_n \tau)^2 \beta_n \|x_n - p\|^2 + (1 - \alpha_n \tau)^2 (1 - \beta_n) \big\| \Im_n^M x_n - p \big\|^2 + c_n \\
&= (1 - \alpha_n \tau)^2 \beta_n \|x_n - p\|^2 + \big(1 - 2\alpha_n(\tau - \gamma) \big)(1 - \beta_n) \big\| \Im_n^M x_n - p \big\|^2 \\
&\quad + \alpha_n^2 \tau^2 (1 - \beta_n) \big\| \Im_n^M x_n - p \big\|^2 \\
&\quad - 2\alpha_n \gamma (1 - \beta_n) \big\| \Im_n^M x_n - p \big\|^2 + c_n \\
&\leq (1 - \alpha_n \tau)^2 \beta_n \|x_n - p\|^2 + \big(1 - 2\alpha_n(\tau - \gamma) \big)(1 - \beta_n) \big\| \Im_n^M x_n - p \big\|^2 \\
&\quad + \alpha_n^2 \tau^2 (1 - \beta_n) \big\| \Im_n^M x_n - p \big\|^2 + c_n \\
&= \big(1 - 2\alpha_n(\tau - \gamma) \big) \beta_n \|x_n - p\|^2 + \alpha_n^2 \tau^2 \beta_n \|x_n - p\|^2 - 2\alpha_n \gamma \beta_n \|x_n - p\|^2 \\
&\quad + \big(1 - 2\alpha_n(\tau - \gamma) \big)(1 - \beta_n) \big\| \Im_n^{k+1} x_n - p \big\|^2 \\
&\quad + \alpha_n^2 \tau^2 (1 - \beta_n) \big\| \Im_n^M x_n - p \big\|^2 + c_n \\
&= \big(1 - 2\alpha_n(\tau - \gamma) \big) \beta_n \|x_n - p\|^2 + \alpha_n^2 \tau^2 \beta_n \|x_n - p\|^2 - 2\alpha_n \gamma \beta_n \|x_n - p\|^2 \\
&\quad + \big(1 - 2\alpha_n(\tau - \gamma) \big)(1 - \beta_n) \big[\|x_n - p\|^2 - \big\| \Im_n^{k+1} x_n - \Im_n^k x_n \big\|^2 \big] \\
&\quad + \alpha_n^2 \tau^2 (1 - \beta_n) \big\| \Im_n^M x_n - p \big\|^2 + c_n \\
&\leq \big(1 - 2\alpha_n(\tau - \gamma) \big) \|x_n - p\|^2 + \alpha_n^2 \tau^2 \|x_n - p\|^2 \\
&\quad - \big(1 - 2\alpha_n(\tau - \gamma) \big)(1 - \beta_n) \big\| \Im_n^{k+1} x_n - \Im_n^k x_n \big\|^2 + c_n,
\end{aligned}
$$

where

$$
c_n = 2\alpha_n \big\langle \big(\gamma f(x_n) - \mu Bp \big), (I - \alpha_n \mu B)(y_n - p) \big\rangle. \tag{3.21}
$$

It follows from the condition (C1) that

$$
\lim_{n \to \infty} c_n = 0. \tag{3.22}
$$

So, we obtain

$$
\begin{aligned}
\big(1 - 2\alpha_n(\tau - \gamma) \big)&(1 - \beta_n) \big\| \Im_n^{k+1} x_n - \Im_n^k x_n \big\|^2 \\
&\leq \big(1 - 2\alpha_n(\tau - \gamma) \big) \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + \alpha_n^2 \tau^2 \|x_n - p\|^2 + c_n \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^2 \tau^2 \|x_n - p\|^2 + c_n \\
&= \|x_n - x_{n+1}\| \big(\|x_n - p\| - \|x_{n+1} - p\| \big) + \alpha_n^2 \tau^2 \|x_n - p\|^2 + c_n.
\end{aligned}
$$

Using the condition (C1), (3.17) and (3.22), we obtain

$$\lim_{n \to \infty} \left\| \Im_n^{k+1} x_n - \Im_n^k x_n \right\| = 0, \quad \forall k = 1, 2, \ldots, M - 1. \tag{3.23}$$

Step 6. We show that $\limsup_{n \to \infty} \langle (\gamma f - \mu B)q, x_n - q \rangle \le 0$, where $q = P_\Theta(I - \mu B + \gamma f)q$.

The Banach contraction principle guarantees that $P_\Theta(I - \mu B + \gamma f)$ has a unique fixed point q which is the unique solution of (3.1). Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k \to \infty} \langle (\gamma f - \mu B)q, x_{n_m} - q \rangle = \limsup_{n \to \infty} \langle (\gamma f - \mu B)q, x_n - q \rangle.$$

Since $\{x_{n_k}\}$ is bounded, then there exists a subsequence $\{x_{n_{k_i}}\}$ which converges weakly to $z \in H$. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup z$. We claim that $z \in \Theta$.

Next, we need to show that $z \in \bigcap_{j=1}^{M} SEP(F_j)$. First, by (A2) and given $y \in C$ and $k \in \{1, 2, \ldots, M - 1\}$, we have

$$\frac{1}{r_{k+1,n}} \langle y - \Im_n^{k+1} x_n, \Im_n^{k+1} x_n - \Im_n^k x_n \rangle \ge F_{k+1}\left(y, \Im_n^{k+1} x_n\right).$$

Thus,

$$\left\langle y - \Im_{n_m}^{k+1} x_{n_m}, \frac{\Im_{n_m}^{k+1} x_{n_m} - \Im_{n_m}^k x_{n_m}}{r_{k+1,n_m}} \right\rangle \ge F_{k+1}\left(y, \Im_{n_m}^{k+1} x_{n_m}\right). \tag{3.24}$$

From (A4), $F(y, \cdot)$ is a lower semicontinuous and convex, and thus weakly semicontinuous. The condition (C3) and (3.23) imply that

$$\frac{\Im_{n_m}^{k+1} x_{n_m} - \Im_{n_m}^k x_{n_m}}{r_{k+1,n_m}} \to 0, \tag{3.25}$$

in norm. Therefore, letting $m \to \infty$ in (3.24) yields

$$F_{k+1}(y, z) \le \lim_{m \to \infty} F_{k+1}\left(y, \Im_{n_m}^{k+1} x_{n_m}\right) \le 0,$$

for all $y \in H$ and $k \in \{1, 2, \ldots, M - 1\}$. Replacing y with $y_t = ty + (1 - t)z$ with $t \in (0, 1)$ and using (A1) and (A4), we obtain

$$0 = F_{k+1}(y_t, y_t) \le tF_{k+1}(y_t, y) + (1 - t)F_{k+1}(y_t, z) \le tF_{k+1}(y_t, y).$$

Hence, $F_{k+1}(ty + (1 - t)z, y) \ge 0$, for all $t \in (0, 1)$ and $y \in H$. Letting $t \to 0^+$ and using (A3), we conclude that $F_{k+1}(z, y) \ge 0$ for all $y \in H$ and $k \in \{1, 2, \ldots, M\}$. Therefore,

$$z \in EP(F_j), \quad \forall j = 1, 2, \ldots, M, \tag{3.26}$$

that is,

$$z \in \bigcap_{j=1}^{M} SEP(F_j). \tag{3.27}$$

Next, we show that $z \in \bigcap_{i=1}^{\infty} F(T_i)$. By Lemma 2.6, we have

$$W_{n_m} z \to W z, \quad \forall z \in C, \tag{3.28}$$

and $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$. Assume that $z \notin F(W)$, then $z \neq Wz$. Therefore, from the Opial property of a Hilbert space, (3.27), (3.28) and Step 4, we have

$$
\begin{aligned}
\liminf_{m \to \infty} \|x_{n_m} - z\| &< \liminf_{m \to \infty} \|x_{n_m} - Wz\| \\
&\leq \liminf_{m \to \infty} \big\{ \big\| x_{n_m} - W_{n_m} \Im_{n_m}^M x_{n_m} \big\| \\
&\quad + \big\| W_{n_m} \Im_{n_m}^M x_{n_m} - W_{n_m} \Im_{n_m}^M z \big\| + \big\| W_{n_m} \Im_{n_m}^M z - Wz \big\| \big\} \\
&\leq \liminf_{m \to \infty} \big\{ \|x_{n_m} - z\| + \|W_{n_m} z - Wz\| \big\} \\
&\leq \liminf_{m \to \infty} \|x_{n_m} - z\|.
\end{aligned}
$$

It is a contradiction. Thus z belongs to $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$. Hence, $z \in \Theta$.

Hence, by Lemma 2.4, we obtain

$$
\begin{aligned}
\lim_{n \to \infty} \sup \langle (\gamma f - \mu B)q, x_n - q \rangle &= \lim_{m \to \infty} \sup \langle (\gamma f - \mu B)q, x_{n_m} - q \rangle \\
&= \langle (\gamma f - \mu B)q, z - q \rangle \leq 0.
\end{aligned}
$$

Step 7. We claim that x_n converges strongly to $q = P_\Theta (I - \mu B + \gamma f)q$. We observe that

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &= \big\| \alpha_n \gamma f(x_n) + (I - \mu \alpha_n B)y_n - q \big\|^2 \\
&= \big\| \alpha_n \gamma f(x_n) + (I - \mu \alpha_n B)y_n - q + \mu \alpha_n B q - \mu \alpha_n F q \big\|^2 \\
&\leq \big\| (I - \mu \alpha_n B)y_n - (I - \mu \alpha_n B)q \big\|^2 + 2\alpha_n \langle \gamma f(x_n) - \mu B q, x_{n+1} - q \rangle \\
&\leq (1 - \alpha_n \tau)^2 \|y_n - q\|^2 \\
&\quad + 2\alpha_n \langle \gamma f(x_n) - \gamma f(q), x_{n+1} - q \rangle + 2\alpha_n \langle \gamma f(q) - \mu B q, x_{n+1} - q \rangle \\
&\leq (1 - \alpha_n \tau)^2 \|x_n - q\|^2 + \alpha_n \gamma \alpha \big(\|x_n - q\|^2 + \|x_{n+1} - q\|^2 \big) \\
&\quad + 2\alpha_n \langle \gamma f(q) - \mu B q, x_{n+1} - q \rangle \\
&\leq \frac{(1 - \alpha_n \tau)^2}{1 - \alpha_n \gamma \alpha} \|x_n - q\|^2 + \frac{\alpha_n \gamma \alpha}{1 - \alpha_n \gamma \alpha} \|x_n - q\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n \gamma \alpha} \langle \gamma f(q) - \mu B q, x_{n+1} - q \rangle \\
&= \left(1 - \frac{2\alpha_n (\tau - \gamma \alpha) + \alpha_n^2 \tau^2}{1 - \alpha_n \gamma \alpha} \right) \|x_n - q\|^2 + \frac{2\alpha_n}{1 - \alpha_n \gamma \alpha} \langle \gamma f(q) - \mu B q, x_{n+1} - q \rangle \\
&= \left(1 - \frac{2\alpha_n (\tau - \gamma \alpha)}{1 - \alpha_n \gamma \alpha} \right) \|x_n - q\|^2 \\
&\quad + \frac{2\alpha_n (\tau - \gamma \alpha)}{1 - \alpha_n \gamma \alpha} \left(\frac{1}{(\tau - \gamma \alpha)} \langle \gamma f(q) - \mu B q, x_{n+1} - q \rangle \right. \\
&\quad \left. + \frac{\alpha_n \tau^2}{2(\tau - \gamma \alpha)} \|x_n - q\|^2 \right)
\end{aligned}
$$

$$= \left(1 - \frac{2\alpha_n(\tau - \gamma\alpha)}{1 - \alpha_n\gamma\alpha}\right)\|x_n - q\|^2$$

$$+ \frac{2\alpha_n(\tau - \gamma\alpha)}{1 - \alpha_n\gamma\alpha}\left(\frac{1}{(\tau - \gamma\alpha)}\langle\gamma f(q) - \mu Bq, x_{n+1} - q\rangle + \frac{\alpha_n\tau^2}{2(\tau - \gamma\alpha)}M_3\right),$$

where $M_3 = \sup_{n\geq 1}\|x_n - q\|^2$. Put $\lambda_n = \frac{2\alpha_n(\tau-\gamma\alpha)}{1-\alpha_n\gamma\alpha}$ and $\delta_n = \frac{1}{\tau-\gamma\alpha}\langle\gamma f(q) - \mu Bq, x_{n+1} - q\rangle$. It follows that

$$\|x_{n+1} - q\|^2 \leq (1 - \lambda_n)\|x_n - q\|^2 + \lambda_n\delta_n + \gamma_n.$$

From (C1), (C2) and Step 5, it follows that $\sum_{n=0}^{\infty}\lambda_n = \infty$ and $\limsup_{n\to\infty}\delta_n \leq 0$. Hence, by Lemma 2.7, the sequence $\{x_n\}$ converges strongly to q. □

Using Theorem 3.1, we obtain the following corollaries.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}$ be an infinite family of nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, $\mathcal{F} = \{F_j : j = 1, 2, 3, \ldots, M\}$ be a finite family of bifunctions $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4) and γ_i be a real sequence such that $0 \leq \gamma_i \leq b \leq 1$ for each $i \geq 1$. Let B be a k-Lipschitzian and η-strongly monotone operator on C with $0 < \mu < \eta/k^2$ and $f \in \Pi_H$ with $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$ and $\tau < 1$. Assume that $\Theta := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^{M} SEP(F_j)) \neq \emptyset$. Let the mapping W_n be defined by (2.3). Let $\{x_n\}$ be the sequence generated by $x_1 \in H$ and*

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)W_n J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} \cdots J_{r_{2,n}}^{F_2} J_{r_{1,N}}^{F_1} x_n, \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n\mu B)y_n, \quad \forall n \in \mathbb{N} \text{ and } n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0, 1)$ which satisfy the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty}\beta_n < \limsup_{n\to\infty}\beta_n \leq a < 1$ *for some constant $a \in (0, 1)$;*

(C3) $\liminf_{n\to\infty} r_{j,n} > 0$, *for each $j = 1, 2, \ldots, M$.*

Then the sequence $\{x_n\}$ converges strongly to $q \in \Theta$ where $q = P_{\Theta}(I - \mu B + \gamma f)q$, which is the unique solution of the variational inequality

$$\langle(\gamma f - \mu B)q, p - q\rangle \leq 0, \quad \forall p \in \Theta.$$

Remark 3.3 Corollary 3.2 extends and improves Theorem 3.1 from f an infinite family of nonexpansive mappings to a family of strictly pseudo contractive mappings.

If $M = 1$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}$ be an infinite family of nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, $\mathcal{F} = \{F_j : j = 1, 2, 3, \ldots, M\}$ be a finite family of bifunctions $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4) and γ_i be a real sequence such that $0 \leq \gamma_i \leq b \leq 1$ for each $i \geq 1$. Let B be a k-Lipschitzian and η-strongly monotone operator on C with $0 < \mu < \eta/k^2$ and $f \in \Pi_H$ with $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$ and $\tau < 1$. Assume that $\Theta := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^{M} SEP(F_j)) \neq \emptyset$. Let the mapping W_n be*

defined by (2.3). *Let* $\{x_n\}$ *be the sequence generated by* $x_1 \in H$ *and*

$$
\begin{cases}
F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\
y_n = \beta_n x_n + (1 - \beta_n) W_n u_n, \\
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu B) y_n, & \forall n \in \mathbb{N} \text{ and } n \geq 1,
\end{cases}
$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are the sequences in* $(0, 1)$ *which satisfy the following conditions*:

(C1) $\lim_{n \to \infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < \liminf_{n \to \infty} \beta_n < \limsup_{n \to \infty} \beta_n \leq a < 1$ *for some constant* $a \in (0, 1)$;

(C3) $\liminf_{n \to \infty} r_{j,n} > 0$, *for each* $j = 1, 2, \ldots, M$.

Then the sequence $\{x_n\}$ *converges strongly to* $q \in \Theta$, *where* $q = P_\Theta(I - \mu B + \gamma f)q$, *which is the unique solution of the variational inequality*

$$
\langle (\gamma f - \mu B)q, p - q \rangle \leq 0, \quad \forall p \in \Theta.
$$

If $M = 1$, $W_n = W$, $\gamma = 1$, $A = I$ and $\mu = 1$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.5 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\{T_i\}$ *be an infinite family of nonexpansive mappings with* $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, $\mathcal{F} = \{F_j : j = 1, 2, 3, \ldots, M\}$ *be a finite family of bifunctions* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4) *and* γ_i *be a real sequence such that* $0 \leq \gamma_i \leq b \leq 1$ *for each* $i \geq 1$. *Let* B *be a* k-*Lipschitzian and* η-*strongly monotone operator on* C *with* $0 < \mu < \eta/k^2$ *and* $f \in \Pi_H$ *with* $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$ *and* $\tau < 1$. *Assume that* $\Theta := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^{M} SEP(F_j)) \neq \emptyset$. *Let the mapping* W_n *be defined by* (2.3). *Let* $\{x_n\}$ *be the sequence generated by* $x_1 \in H$ *and*

$$
\begin{cases}
y_n = \beta_n x_n + (1 - \beta_n) W J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} \cdots J_{r_{2,n}}^{F_2} J_{r_{1,N}}^{F_1} x_n, \\
x_{n+1} = \alpha_n f(x_n) + (I - \alpha_n \mu B) y_n, & \forall n \in \mathbb{N} \text{ and } n \geq 1,
\end{cases}
$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are the sequences in* $(0, 1)$ *which satisfy the following conditions*:

(C1) $\lim_{n \to \infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < \liminf_{n \to \infty} \beta_n < \limsup_{n \to \infty} \beta_n \leq a < 1$ *for some constant* $a \in (0, 1)$;

(C3) $\liminf_{n \to \infty} r_{j,n} > 0$, *for each* $j = 1, 2, \ldots, M$.

Then the sequence $\{x_n\}$ *converges strongly to* $q \in \Theta$ *where* $q = P_\Theta(I - \mu B + \gamma f)q$, *which is the unique solution of the variational inequality*

$$
\langle (\gamma f - \mu B)q, p - q \rangle \leq 0, \quad \forall p \in \Theta.
$$

4 Numerical example

In this section, we give a real numerical example of Theorem 3.1 as follows.

Example 4.1 Let $H = \mathbb{R}$, $C = [0, \frac{1}{4}]$, $T_n = I$. $F_k(x, y) = 0, \forall x, y \in H$, $r_{k,n} = 1$, $k \in \{1, 2, 3, \ldots, K\}$, $B = I$, $f(x) = x^2$, $\beta_n = \frac{1}{2}$, $\alpha_n = \frac{1}{n}$ for every $n \in \mathbb{N}$ and $\mu = 1$. Then $\{x_n\}$ is the sequence generated by

$$
x_{n+1} = \frac{x_n^2}{n} + \left(1 - \frac{1}{n}\right) x_n, \tag{4.1}
$$

and $z \to 0$ as $n \to \infty$, where 0 is the unique solution of the minimization problem

$$\min_{x \in C} \frac{x^3}{3} + \frac{x^2}{2} + C_1, \tag{4.2}$$

where C_1 is a constant.

Proof We divide the proof into four steps.

Step 1. Using the idea in [7], we can show that

$$J_{r_{k,n}}^k x = P_C x, \quad \forall x \in H, k \in \{1, 2, \dots, K\}, \tag{4.3}$$

where

$$P_C x = \begin{cases} \frac{x}{|x|}, & x \in H - C, \\ x, & x \in C. \end{cases} \tag{4.4}$$

Since $F_k(x, y) = 0$, $\forall x, y \in C$, $k \in \{1, 2, \dots, K\}$, with the definition of $J_r(x)$, $\forall x \in H$ in Lemma 2.13, we have

$$J_r^F(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}. \tag{4.5}$$

By the equivalent property of the nearest projection P_C from H to C, we can conclude that if we take $x \in C$, $J_{r_{k,n}}^k x = P_C x = Ix$. By (3) in Lemma 2.13, we have

$$\bigcap_{k=1}^K SEP(F_k) = C. \tag{4.6}$$

Step 2. We show that

$$W_n = I. \tag{4.7}$$

Since $T_i' = \theta_i I + (1 - \theta_i) T_i$, where T_i is a λ_i-strictly pseudo-contractive mapping and $\theta_i \in [\lambda_i, 1)$, it can be easily seen that T_i' is a nonexpansive mapping. By (2.3), we have

$$W_1 = U_{1,1} = \gamma_1 T_1' U_{1,2} + (1 - \gamma_1)I,$$

$$W_2 = U_{2,1} = \gamma_1 T_1' U_{2,2} + (1 - \gamma_1)I = \gamma_1 T_1' \big(\gamma_2 T_2' U_{2,3} + (1 - \gamma_2)I\big) + (1 - \gamma_1)I$$

$$= \gamma_1 \gamma_2 T_1' T_2' + \gamma_1(1 - \gamma_2)T_1' + (1 - \gamma_1)I,$$

$$W_3 = U_{3,1} = \gamma_1 T_1' U_{3,2} + (1 - \gamma_1)I = \gamma_1 T_1' \big(\gamma_2 T_2' U_{3,3} + (1 - \gamma_2)I\big) + (1 - \gamma_1)I$$

$$= \gamma_1 \gamma_2 T_1' T_2' U_{3,3} + \gamma_1(1 - \gamma_2)T_1' + (1 - \gamma_1)I$$

$$= \gamma_1 \gamma_2 T_1' T_2' \big(\gamma_3 T_3' U_{3,4} + (1 - \gamma_3)I\big) + \gamma_1(1 - \gamma_2)T_1' + (1 - \gamma_1)I$$

$$= \gamma_1 \gamma_2 \gamma_3 T_1' T_2' T_3' + \gamma_1 \gamma_2(1 - \gamma_3)T_1' T_2' + \gamma_1(1 - \gamma_2)T_1' + (1 - \gamma_1)I,$$

and we compute (2.3) in the same way as above, so we obtain

$$W_n = U_{n,1} = \gamma_1\gamma_2\cdots\gamma_n T_1' T_2' \cdots T_n' + \gamma_1\gamma_2\cdots\gamma_{n-1}(1-\gamma_n)T_1' T_2' \cdots T_{n-1}'$$
$$+ \gamma_1\gamma_2\cdots\gamma_{n-2}(1-\gamma_{n-1})T_1' T_2' \cdots T_{n-2}' + \cdots + \gamma_1(1-\gamma_2)T_1' + (1-\gamma_1)I.$$

Since $T_n' = I$, $\gamma_n = \beta$, $n \in \mathbb{N}$, hence,

$$W_n = \left[\beta^n + \beta^{n-1}(1-\beta) + \cdots + \beta(1-\beta) + (1-\beta)\right]I = I.$$

Step 3. We prove

$$x_{n+1} = \frac{x_n^2}{n} + \left(1 - \frac{1}{n}\right)x_n \quad \text{and} \quad x_n \to 0 \quad \text{as } n \to \infty,$$

where 0 is the unique solution of the minimization problem

$$\min_{x \in C} \frac{x^3}{3} + \frac{x^2}{2} + C_1.$$

Since we let $B = I$, γ is a real number, so we choose $\gamma = 1$. From (4.3), (4.4) and (4.7), we can obtain a special sequence $\{x_n\}$ of Theorem 3.1 as follows:

$$x_{n+1} = \frac{x_n^2}{n} + \left(1 - \frac{1}{n}\right)x_n.$$

Since $T_n = I$, $n \in \mathbb{N}$, we have

$$\bigcap_{n \in \mathbb{N}} F(T_n) = H.$$

Combining it with (4.6), we obtain

$$\Theta := \left(\bigcap_{k=1}^{K} SEP(F_k)\right) \cap \left(\bigcap_{n \in \mathbb{N} F(T_n)}\right) = C = \left[0, \frac{1}{4}\right].$$

It is obvious that $x_n \to 0$, 0 is the unique solution of the minimization problem $\min_{x \in C} \frac{x^3}{3} + \frac{x^2}{2} + C_1$, where C_1 is a constant number.

Step 4. In this step, we give the numerical results that support our main theorem as shown by plotting graphs using Matlab 7.11.0. We choose two different initial values as $x_1 = 0.1$ and $x_1 = 0.15$ in Table 1, Figure 1, and Figure 2, respectively. From the example, we can see that $\{x_n\}$ converges to 0. □

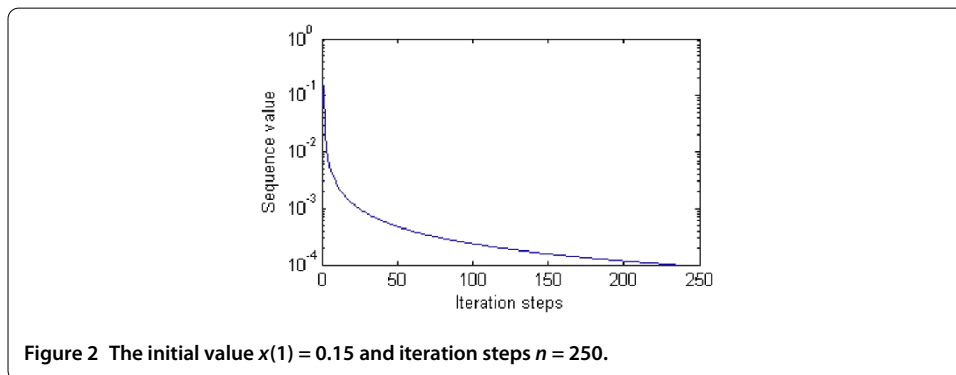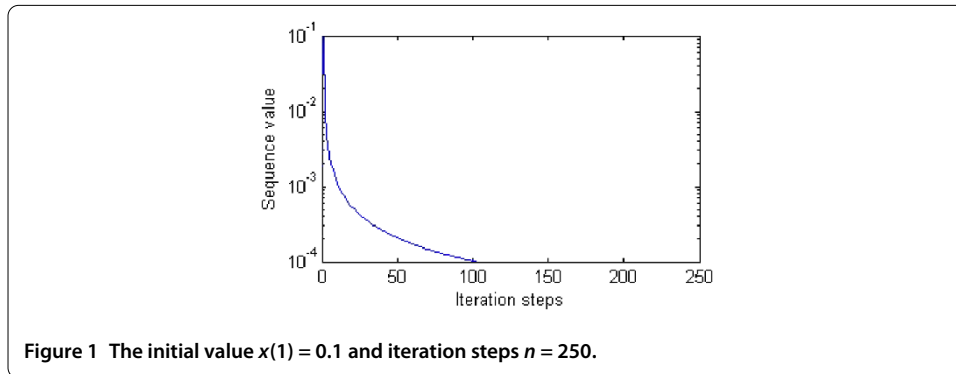**Table 1 The sequence values on each different iteration step**

Iteration step (n)	$x(1) = 0.1$	$x(1) = 0.15$	Iteration step (n)	$x(1) = 0.1$	$x(1) = 0.15$
0	0.1000	0.1500	7	0.0015	0.0033
1	0.0100	0.0225	8	0.0013	0.0029
2	0.0051	0.0115	9	0.0011	0.0026
3	0.0034	0.0077	203	0.0001	0.0001
4	0.0025	0.0058	204	0.0000	0.0001
5	0.0020	0.0046	205	0.0000	0.0001
6	0.0017	0.0039	250	0.0000	0.0000

Figure 1 The initial value $x(1) = 0.1$ and iteration steps $n = 250$.

Figure 2 The initial value $x(1) = 0.15$ and iteration steps $n = 250$.

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