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An application of matrix inequalities to certain functional inequalities involving fractional powers

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Abstract
We will show certain functional inequalities involving fractional powers, making use of the Furuta inequality and Tanahashi's argument.
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1 Introduction

Let x be an arbitrary positive real number. One can easily see the inequality

$$\bigl(x^{\frac{3}{2}}-1\bigr)\bigl(x^2-1\bigr) \le \frac{6}{5}\bigl(x^{\frac{5}{2}}-1\bigr)(x-1),$$

for instance, is reduced to a simple polynomial inequality by putting $t = x^{\frac{1}{2}}$. However, at least to the author, it seems not easy to give an elementary proof of the inequality

$$x^{\frac{2-\sqrt{2}+\sqrt{3}}{4}}\bigl(x^{\sqrt{2}}-1\bigr)\bigl(x^{\frac{\sqrt{2}+\sqrt{3}}{2}}-1\bigr) \le \frac{1}{\sqrt{2}}\bigl(x^{\sqrt{2}+\sqrt{3}}-1\bigr)(x-1),$$

which has a very similar form to the preceding one although their corresponding numerical parts are different.

The purpose of this article is to show the following theorem.

Theorem 1.1 *Let $0 \le p$, $1 \le q$ and $0 \le r$ with $p+r \le (1+r)q$. If $0 < x$, then*

$$x^{\frac{1+r-\frac{p+r}{q}}{2}}\bigl(x^p-1\bigr)\bigl(x^{\frac{p+r}{q}}-1\bigr) \le \frac{p}{q}\bigl(x^{p+r}-1\bigr)(x-1). \tag{1}$$

An elementary approach to proving the inequality (1) might be to consider the power series expansion.

Put $t = x-1$, $c = \frac{1+r-\frac{p+r}{q}}{2}$ and

$$f(t) = \frac{p}{q}\bigl((1+t)^{p+r}-1\bigr)t - (1+t)^c\bigl((1+t)^p-1\bigr)\bigl((1+t)^{\frac{p+r}{q}}-1\bigr).$$

Then we can expand $f(t)$ around $t = 0$ as

$$\begin{aligned} f(t) &= \frac{p}{q}\left\{p + r + \binom{p+r}{2}t + \binom{p+r}{3}t^2 + \binom{p+r}{4}t^3 + \binom{p+r}{5}t^4 + \cdots\right\} \cdot t^2 \\ &\quad - \left\{1 + ct + \binom{c}{2}t^2 + \binom{c}{3}t^3 + \binom{c}{4}t^4 + \cdots\right\} \\ &\quad \cdot \left\{p + \binom{p}{2}t + \binom{p}{3}t^2 + \binom{p}{4}t^3 + \binom{p}{5}t^4 + \cdots\right\} \\ &\quad \cdot \left\{\frac{p+r}{q} + \binom{\frac{p+r}{q}}{2}t + \binom{\frac{p+r}{q}}{3}t^2 + \binom{\frac{p+r}{q}}{4}t^3 + \binom{\frac{p+r}{q}}{5}t^4 + \cdots\right\} \cdot t^2 \\ &= a_4t^4 + a_5t^5 + a_6t^6 + \cdots. \end{aligned}$$

Thus, the constant term and the coefficients of t, t^2 and t^3 are 0. Further, one can obtain

$$\begin{aligned} a_4 &= \frac{p(p+r)}{24q}\left(r^2 + 2pr + 1 - \left(\frac{p+r}{q}\right)^2\right), \\ a_5 &= \frac{p(p+r)(p+r-3)}{48q}\left(r^2 + 2pr + 1 - \left(\frac{p+r}{q}\right)^2\right) \end{aligned}$$

and

$$\begin{aligned} a_6 = -\frac{p(p+r)}{5760q}\bigg\{&3\left(\frac{p+r}{q}\right)^4 + 10\left(\frac{p+r}{q}\right)^2\left\{3(p+r)(p+r-8) + p^2 + 41\right\} \\ &- 33(p+r)^4 + 240(p+r)^3 + 30(p+r)^2\left(p^2 - 15\right) - 240(p+r)\left(p^2 - 1\right) \\ &+ \left(3p^2 + 413\right)\left(p^2 - 1\right)\bigg\}. \end{aligned}$$

Thus, if the assumption for the parameters p, q and r in Theorem 1.1 is satisfied, then we have $0 < a_4$. However, the signature of a_5 and a_6 depends on parameters, and one cannot see any signs of a simple rule among the coefficients of higher order terms. Although $f(t)$ is non-negative on a sufficiently small neighborhood of $t = 0$, it seems difficult to show that $f(t)$ is non-negative entirely on $-1 < t < \infty$ by such an argument as above.

Let us recall some fundamental concepts on related matrix inequalities. A capital letter means a matrix whose entries are complex numbers. A square matrix T is said to be positive semidefinite (denoted by $0 \leq T$) if $0 \leq (Tx, x)$ for all vectors x. We write $0 < T$ if T is positive semidefinite and invertible. For two selfadjoint matrices T_1 and T_2 of the same size, a matrix inequality $T_1 \leq T_2$ is defined by $0 \leq T_2 - T_1$.

The celebrated Löwner-Heinz theorem includes:

Theorem 1.2 [1, 2] *Let $0 \leq p \leq 1$. If $0 \leq B \leq A$, then $B^p \leq A^p$.*

For $1 < p$, $0 \leq B \leq A$ does not always ensure $B^p \leq A^p$. Furuta obtained an epoch-making extension of the Löwner-Heinz inequality by using the Löwner-Heinz inequality itself.

Theorem 1.3 [3] *Let $0 \leq p$, $1 \leq q$ and $0 \leq r$ with $p + r \leq (1 + r)q$. If $0 \leq B \leq A$, then*

$$\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}} \leq A^{\frac{p+r}{q}}. \tag{2}$$

The following result by Tanahashi is a full description of the best possibility of the range

$$p + r \le (1 + r)q \quad \text{and} \quad 1 \le q$$

as far as all parameters are positive.

Theorem 1.4 [4] *Let p, q, r be positive real numbers. If* $(1 + r)q < p + r$ *or* $0 < q < 1$, *then there exist* 2×2 *matrices* A, B *with* $0 < B \le A$ *that do not satisfy the inequality*

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \le A^{\frac{p+r}{q}}.$$

One notices the coincidence between the assumption on parameters in Theorem 1.1 and Theorem 1.3. As a matter of fact, the inequality (1) is a particular conclusion of the Furuta inequality. We should point out that Tanahashi's argument in [4] is almost sufficient to deduce the former from the latter. In the next section, we will prove Theorem 1.1 using Theorem 1.3 and Tanahashi's argument.

2 Proof of Theorem 1.1

As we mentioned above, our proof of Theorem 1.1 has a major part which is parallel to [4]. Our matrix A is a little different from that in [4], we use a variable y instead of ε and δ. It simplifies the argument to an extent, though the improvement is not essential.

Throughout this paper, we assume that $1 < a < b$ and $0 < y$. We will consider matrices

$$A = \begin{pmatrix} a & \sqrt{(a-1)y} \\ \sqrt{(a-1)y} & b + y \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}.$$

Then we have $0 < B \le A$. The eigenvalues of A are $\frac{a+b+y\pm\sqrt{d}}{2}$, where $d = a^2 + b^2 + y^2 - 2ab + 2(a + b - 2)y$.

Lemma 2.1 $0 < d < (a + b + y)^2$ *and* $a - b - y - \sqrt{d} \neq 0$.

Proof Obviously,

$$d = (a - b)^2 + y\bigl(y + 2(a + b - 2)\bigr) > 0,$$
$$d = (a + b + y)^2 - 4(ab + y) < (a + b + y)^2.$$

If $a - b - y - \sqrt{d} = 0$, then we would have $a = 1$ or $y = 0$, which is contrary to the assumption. □

Let

$$c = \frac{-2\sqrt{(a-1)y}}{a - b - y - \sqrt{d}}$$

and

$$U = \frac{1}{\sqrt{c^2+1}} \begin{pmatrix} c & 1 \\ 1 & -c \end{pmatrix}.$$

Then U is unitary and

$$U^* A U = \frac{1}{2} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

where

$$d_1 = a + b + y + \sqrt{d}, \qquad d_2 = a + b + y - \sqrt{d}.$$

By the assumption and Theorem 1.3, A and B satisfy the inequality (2). Then

$$\left(U^* A^{\frac{r}{2}} U U^* B^p U U^* A^{\frac{r}{2}} U\right)^{\frac{1}{q}} \le U^* A^{\frac{p+r}{q}} U,$$

hence we have

$$\left\{ \begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix} U^* \begin{pmatrix} 1 & 0 \\ 0 & b^p \end{pmatrix} U \begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix} \right\}^{\frac{1}{q}} \le 2^{-\frac{p}{q}} \begin{pmatrix} d_1^{\frac{p+r}{q}} & 0 \\ 0 & d_2^{\frac{p+r}{q}} \end{pmatrix}. \tag{3}$$

Denote

$$\begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix} U^* \begin{pmatrix} 1 & 0 \\ 0 & b^p \end{pmatrix} U \begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix} = \frac{1}{c^2+1} \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix},$$

where

$$\begin{aligned} A_1 &= d_1^r\left(c^2 + b^p\right), \\ A_2 &= d_2^r\left(1 + c^2 b^p\right), \\ A_3 &= d_1^{\frac{r}{2}} d_2^{\frac{r}{2}} c\left(1 - b^p\right) = \left((a+b+y)^2 - d\right)^{\frac{r}{2}} c\left(1 - b^p\right) = (4ab + 4y)^{\frac{r}{2}} c\left(1 - b^p\right). \end{aligned}$$

Lemma 2.2 *Let p, q, r be positive real numbers. Then $A_2 < A_1$ and $A_3 < 0$.*

Proof Since $d_2 < d_1$ and $0 < r$, we have $d_2^r < d_1^r$. Moreover,

$$\left(c^2 + b^p\right) - \left(1 + c^2 b^p\right) = \left(c^2 - 1\right)\left(1 - b^p\right), \quad 1 - b^p < 0$$

and

$$c^2 - 1 = -\frac{2(a-b)^2 + 2y^2 + 4(b-a)y + 2(b-a+y)\sqrt{d}}{(a-b-y-\sqrt{d})^2} < 0,$$

hence we have $1 + c^2 b^p < c^2 + b^p$. Thus $A_2 < A_1$.

It is obvious that $1 - b^p < 0$ and $0 < c$, and hence $A_3 < 0$. □

Let

$$V = \frac{1}{\sqrt{A_1 - A_2 + 2\varepsilon_1}} \begin{pmatrix} \sqrt{A_1 - A_2 + \varepsilon_1} & -\sqrt{\varepsilon_1} \\ -\sqrt{\varepsilon_1} & -\sqrt{A_1 - A_2 + \varepsilon_1} \end{pmatrix},$$

where

$$2\varepsilon_1 = -A_1 + A_2 + \sqrt{(A_1 - A_2)^2 + 4A_3^2}.$$

Then it is easy to see that $A_3 = -\sqrt{(A_1 - A_2 + \varepsilon_1)\varepsilon_1}$, V is unitary and

$$V^* \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix} V = \begin{pmatrix} A_1 + \varepsilon_1 & 0 \\ 0 & A_2 - \varepsilon_1 \end{pmatrix}.$$

The following lemma is one of the most important points in Tanahashi's argument. Although the substance is presented in the whole proof of [4, Theorem], we should restate and prove it in our context for the readers' convenience.

Lemma 2.3

$$\begin{aligned} &\varepsilon_1 \bigl\{\gamma d_1^{\frac{p+r}{q}} - (A_2 - \varepsilon_1)^{\frac{1}{q}}\bigr\}\bigl\{(A_1 + \varepsilon_1)^{\frac{1}{q}} - \gamma d_2^{\frac{p+r}{q}}\bigr\} \\ &\quad \le (A_1 - A_2 + \varepsilon_1)\bigl\{\gamma d_1^{\frac{p+r}{q}} - (A_1 + \varepsilon_1)^{\frac{1}{q}}\bigr\}\bigl\{\gamma d_2^{\frac{p+r}{q}} - (A_2 - \varepsilon_1)^{\frac{1}{q}}\bigr\}, \end{aligned} \tag{4}$$

where $\gamma = (\frac{c^2+1}{2^p})^{\frac{1}{q}}$.

Proof The formula (3) implies

$$\bigl(c^2 + 1\bigr)^{-\frac{1}{q}} V \begin{pmatrix} (A_1 + \varepsilon_1)^{\frac{1}{q}} & 0 \\ 0 & (A_2 - \varepsilon_1)^{\frac{1}{q}} \end{pmatrix} V^* \le 2^{-\frac{p}{q}} \begin{pmatrix} d_1^{\frac{p+r}{q}} & 0 \\ 0 & d_2^{\frac{p+r}{q}} \end{pmatrix}. \tag{5}$$

Write the left-hand matrix as

$$\bigl(c^2 + 1\bigr)^{-\frac{1}{q}} (A_1 - A_2 + 2\varepsilon_1)^{-1} \begin{pmatrix} B_1 & B_3 \\ B_3 & B_2 \end{pmatrix},$$

where

$$B_1 = (A_1 - A_2 + \varepsilon_1)(A_1 + \varepsilon_1)^{\frac{1}{q}} + \varepsilon_1 (A_2 - \varepsilon_1)^{\frac{1}{q}},$$

$$B_2 = \varepsilon_1 (A_1 + \varepsilon_1)^{\frac{1}{q}} + (A_1 - A_2 + \varepsilon_1)(A_2 - \varepsilon_1)^{\frac{1}{q}},$$

$$B_3 = -\sqrt{A_1 - A_2 + \varepsilon_1}\sqrt{\varepsilon_1}\bigl\{(A_1 + \varepsilon_1)^{\frac{1}{q}} - (A_2 - \varepsilon_1)^{\frac{1}{q}}\bigr\}.$$

Then, by the formula (5), we have

$$0 \le \begin{pmatrix} \gamma (A_1 - A_2 + 2\varepsilon_1) d_1^{\frac{p+r}{q}} - B_1 & -B_3 \\ -B_3 & \gamma (A_1 - A_2 + 2\varepsilon_1) d_2^{\frac{p+r}{q}} - B_2 \end{pmatrix}.$$

So, its determinant is also non-negative. We expand it to obtain

$$\begin{aligned}0 &\le \gamma^2(A_1 - A_2 + 2\varepsilon_1)^2 d_1^{\frac{p+r}{q}} d_2^{\frac{p+r}{q}} - \gamma(A_1 - A_2 + 2\varepsilon_1) d_1^{\frac{p+r}{q}} B_2 \\ &\quad - \gamma(A_1 - A_2 + 2\varepsilon_1) d_2^{\frac{p+r}{q}} B_1 + B_1 B_2 - B_3^2. \end{aligned} \tag{6}$$

Now,

$$\begin{aligned} &B_1B_2 - B_3^2 \\ &\quad = \bigl\{(A_1 - A_2 + \varepsilon_1)(A_1 + \varepsilon_1)^{\frac{1}{q}} + \varepsilon_1(A_2 - \varepsilon_1)^{\frac{1}{q}}\bigr\}\bigl\{\varepsilon_1(A_1 + \varepsilon_1)^{\frac{1}{q}} + (A_1 - A_2 + \varepsilon_1)(A_2 - \varepsilon_1)^{\frac{1}{q}}\bigr\} \\ &\qquad - (A_1 - A_2 + \varepsilon_1)\varepsilon_1\bigl\{(A_1 + \varepsilon_1)^{\frac{1}{q}} - (A_2 - \varepsilon_1)^{\frac{1}{q}}\bigr\}^2 \\ &\quad = (A_1 - A_2 + 2\varepsilon_1)^2 (A_1 + \varepsilon_1)^{\frac{1}{q}} (A_2 - \varepsilon_1)^{\frac{1}{q}}. \end{aligned}$$

Hence, the formula (6) implies

$$\begin{aligned} 0 &\le (A_1 - A_2 + 2\varepsilon_1)\bigl\{\gamma^2(A_1 - A_2 + 2\varepsilon_1) d_1^{\frac{p+r}{q}} d_2^{\frac{p+r}{q}} - \gamma d_1^{\frac{p+r}{q}} B_2 - \gamma d_2^{\frac{p+r}{q}} B_1\bigr\} \\ &\quad + (A_1 - A_2 + 2\varepsilon_1)^2 (A_1 + \varepsilon_1)^{\frac{1}{q}} (A_2 - \varepsilon_1)^{\frac{1}{q}}. \end{aligned}$$

Cancel the common positive factor $A_1 - A_2 + 2\varepsilon_1$ and substitute the definitions for B_1 and B_2. Then a simple calculation shows that

$$\begin{aligned} &-\varepsilon_1\bigl\{\gamma^2 d_1^{\frac{p+r}{q}} d_2^{\frac{p+r}{q}} - \gamma d_1^{\frac{p+r}{q}} (A_1 + \varepsilon_1)^{\frac{1}{q}} - \gamma d_2^{\frac{p+r}{q}} (A_2 - \varepsilon_1)^{\frac{1}{q}} + (A_1 + \varepsilon_1)^{\frac{1}{q}} (A_2 - \varepsilon_1)^{\frac{1}{q}}\bigr\} \\ &\quad \le (A_1 - A_2 + \varepsilon_1) \\ &\qquad \cdot \bigl\{\gamma^2 d_1^{\frac{p+r}{q}} d_2^{\frac{p+r}{q}} - \gamma d_1^{\frac{p+r}{q}} (A_2 - \varepsilon_1)^{\frac{1}{q}} - \gamma d_2^{\frac{p+r}{q}} (A_1 + \varepsilon_1)^{\frac{1}{q}} + (A_1 + \varepsilon_1)^{\frac{1}{q}} (A_2 - \varepsilon_1)^{\frac{1}{q}}\bigr\}. \end{aligned}$$

By factorizing, we have

$$\begin{aligned} &-\varepsilon_1\bigl\{\gamma d_1^{\frac{p+r}{q}} - (A_2 - \varepsilon_1)^{\frac{1}{q}}\bigr\}\bigl\{\gamma d_2^{\frac{p+r}{q}} - (A_1 + \varepsilon_1)^{\frac{1}{q}}\bigr\} \\ &\quad \le (A_1 - A_2 + \varepsilon_1)\bigl\{\gamma d_1^{\frac{p+r}{q}} - (A_1 + \varepsilon_1)^{\frac{1}{q}}\bigr\}\bigl\{\gamma d_2^{\frac{p+r}{q}} - (A_2 - \varepsilon_1)^{\frac{1}{q}}\bigr\}. \end{aligned}$$

This completes the proof of Lemma 2.3. □

Now, we estimate each term of the inequality (4) with respect to $y \to +0$. A key point in making use of the inequality (4) is that both estimations of the factor ε_1 on the left-hand side and the factor $\gamma d_1^{\frac{p+r}{q}} - (A_1 + \varepsilon_1)^{\frac{1}{q}}$ on the right-hand side contain a common subfactor y. After the cancellation of this y, we will derive the desired functional inequality by letting $y \to +0$, $a \to 1 + 0$ and applying l'Hopital's rule. Terms in other factors can be roughly estimated.

In the following, o means $o(y)$, that is

$$\frac{o}{y} \to 0 \quad (y \to +0),$$

and $o(1)$ denotes a term such that $o(1) \to 0$ $(y \to +0)$.

One can establish the following formulae:

$$\sqrt{d} = (b-a)\left\{1 + \frac{a+b-2}{(b-a)^2}y + o(y)\right\},$$

$$d_1^{\frac{p+r}{q}} = (2b)^{\frac{p+r}{q}}\left\{1 + \frac{p+r}{q}\cdot\frac{b-1}{b(b-a)}y + o(y)\right\},$$

$$d_2^{\frac{p+r}{q}} = (2a)^{\frac{p+r}{q}}\left\{1 + \frac{p+r}{q}\cdot\frac{-a+1}{a(b-a)}y + o(y)\right\},$$

$$c = \frac{-2\sqrt{(a-1)y}}{a-b-y-(b-a+\frac{a+b-2}{b-a}y + o(y))} = \sqrt{y}\cdot\frac{\sqrt{a-1}}{b-a}\left\{1 - \frac{b-1}{(b-a)^2}y + o(y)\right\},$$

$$c^2 + 1 = 1 + \frac{a-1}{(b-a)^2}y + o(y),$$

$$\begin{aligned}
&\left(c^2+1\right)^{\frac{1}{q}} d_1^{\frac{p+r}{q}} \\
&\quad = \left\{1 + \frac{a-1}{q(b-a)^2}y + o(y)\right\}(2b)^{\frac{p+r}{q}}\left\{1 + \frac{p+r}{q}\cdot\frac{b-1}{b(b-a)}y + o(y)\right\} \\
&\quad = (2b)^{\frac{p+r}{q}}\left\{1 + \frac{1}{qb(b-a)^2}\big((a-1)b + (p+r)(b-1)(b-a)\big)y + o(y)\right\},
\end{aligned}$$

$$\left(c^2+1\right)^{\frac{1}{q}} d_2^{\frac{p+r}{q}} = (2a)^{\frac{p+r}{q}}\big(1 + o(1)\big),$$

$$\begin{aligned}
A_1 &= (2b)^r\left\{1 + \frac{r(b-1)}{b(b-a)}y + o(y)\right\}\left\{b^p + \frac{a-1}{(b-a)^2}y + o(y)\right\} \\
&= 2^r b^{p+r}\left\{1 + \frac{1}{b(b-a)^2}\big(r(b-1)(b-a) + b^{1-p}(a-1)\big)y + o(y)\right\},
\end{aligned}$$

$$A_2 = (2a)^r\big(1 + o(1)\big),$$

$$A_3^2 = (4ab + 4y)^r y\frac{a-1}{(b-a)^2}\big(1 + o(1)\big)\big(1 - b^p\big)^2 = y4^r a^r b^r\frac{a-1}{(b-a)^2}\big(1 - b^p\big)^2\big(1 + o(1)\big),$$

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{2}(A_1 - A_2)\left(-1 + \sqrt{1 + \frac{4A_3^2}{(A_1 - A_2)^2}}\right) = \frac{A_3^2}{A_1 - A_2} + o \\
&= \frac{y4^r a^r b^r (a-1)(b-a)^{-2}(1-b^p)^2(1+o(1))}{2^r b^{p+r}(1+o(1)) - (2a)^r(1+o(1))} + o \\
&= \frac{y2^r a^r b^r (a-1)(1-b^p)^2}{(b-a)^2(b^{p+r} - a^r)}\big(1 + o(1)\big),
\end{aligned}$$

$$\begin{aligned}
&(A_1 + \varepsilon_1)^{\frac{1}{q}} \\
&\quad = \Bigg(2^r b^{p+r}\left\{1 + \frac{1}{b(b-a)^2}\big(r(b-1)(b-a) + b^{1-p}(a-1)\big)y + o(y)\right\} \\
&\qquad + \frac{y2^r a^r b^r (a-1)(1-b^p)^2}{(b-a)^2(b^{p+r} - a^r)}\big(1 + o(1)\big)\Bigg)^{\frac{1}{q}} \\
&\quad = 2^{\frac{r}{q}} b^{\frac{p+r}{q}}\Bigg\{1 + \frac{1}{qb(b-a)^2}\left(r(b-1)(b-a) + b^{1-p}(a-1) + \frac{a^r b^{1-p}(a-1)(1-b^p)^2}{b^{p+r} - a^r}\right)y \\
&\qquad + o(y)\Bigg\},
\end{aligned}$$

$$
\begin{aligned}
&(A_2-\varepsilon_1)^{\frac{1}{q}} = 2^{\frac{r}{q}} a^{\frac{r}{q}}\bigl(1+o(1)\bigr),\\
&A_1 - A_2 + \varepsilon_1 = 2^r\bigl(b^{p+r}-a^r\bigr)\bigl(1+o(1)\bigr),\\
&\gamma d_1^{\frac{p+r}{q}} - (A_2-\varepsilon_1)^{\frac{1}{q}} = 2^{\frac{r}{q}}\bigl(b^{\frac{p+r}{q}} - a^{\frac{r}{q}}\bigr)\bigl(1+o(1)\bigr),\\
&\gamma d_2^{\frac{p+r}{q}} - (A_1+\varepsilon_1)^{\frac{1}{q}} = 2^{\frac{r}{q}}\bigl(a^{\frac{p+r}{q}} - b^{\frac{p+r}{q}}\bigr)\bigl(1+o(1)\bigr),\\
&\gamma d_2^{\frac{p+r}{q}} - (A_2-\varepsilon_1)^{\frac{1}{q}} = 2^{\frac{r}{q}}\bigl(a^{\frac{p+r}{q}} - a^{\frac{r}{q}}\bigr)\bigl(1+o(1)\bigr).
\end{aligned}
$$

Now, we have the estimation of the most delicate factor in the formula (4), whose constant term is canceled by subtraction.

$$
\begin{aligned}
&\gamma d_1^{\frac{p+r}{q}} - (A_1+\varepsilon_1)^{\frac{1}{q}}\\
&\quad = 2^{-\frac{p}{q}}\cdot(2b)^{\frac{p+r}{q}}\left\{1+\frac{1}{qb(b-a)^2}\bigl((a-1)b+(p+r)(b-1)(b-a)\bigr)y+o(y)\right\}\\
&\qquad - 2^{\frac{r}{q}} b^{\frac{p+r}{q}}\left\{1+\frac{1}{qb(b-a)^2}\left(r(b-1)(b-a)+b^{1-p}(a-1)+\frac{a^r b^{1-p}(a-1)(1-b^p)^2}{b^{p+r}-a^r}\right)y\right.\\
&\qquad \left. + o(y)\right\}\\
&\quad = 2^{\frac{r}{q}}\frac{b^{\frac{p+r}{q}-1}}{q(b-a)^2}\left\{(a-1)b+p(b-1)(b-a)-b^{1-p}(a-1)-\frac{a^r b^{1-p}(a-1)(1-b^p)^2}{b^{p+r}-a^r}\right\}y\\
&\qquad \cdot\bigl(1+o(1)\bigr)
\end{aligned}
$$

Substitute these estimations for the inequality (4), cancel the positive factor y, and let $y \to +0$, then we have

$$
\begin{aligned}
&\frac{2^r a^r b^r (a-1)(1-b^p)^2}{(b-a)^2(b^{p+r}-a^r)}\cdot 2^{\frac{r}{q}}\bigl(b^{\frac{p+r}{q}}-a^{\frac{r}{q}}\bigr)\cdot 2^{\frac{r}{q}}\bigl(b^{\frac{p+r}{q}}-a^{\frac{p+r}{q}}\bigr)\\
&\quad \le 2^r\bigl(b^{p+r}-a^r\bigr)\\
&\qquad \cdot 2^{\frac{r}{q}}\frac{b^{\frac{p+r}{q}-1}}{q(b-a)^2}\left\{(a-1)b+p(b-1)(b-a)-b^{1-p}(a-1)-\frac{a^r b^{1-p}(a-1)(1-b^p)^2}{b^{p+r}-a^r}\right\}\\
&\qquad \cdot 2^{\frac{r}{q}}\bigl(a^{\frac{p+r}{q}}-a^{\frac{r}{q}}\bigr),
\end{aligned}
$$

and hence

$$
\begin{aligned}
&a^r b^r\bigl(1-b^p\bigr)^2\cdot\bigl(b^{\frac{p+r}{q}}-a^{\frac{r}{q}}\bigr)\cdot\bigl(b^{\frac{p+r}{q}}-a^{\frac{p+r}{q}}\bigr)\\
&\quad \le \bigl(b^{p+r}-a^r\bigr)^2\\
&\qquad \cdot\frac{b^{\frac{p+r}{q}-1}}{q}\left\{(a-1)b+p(b-1)(b-a)-b^{1-p}(a-1)-\frac{a^r b^{1-p}(a-1)(1-b^p)^2}{b^{p+r}-a^r}\right\}\\
&\qquad \cdot\frac{a^{\frac{p+r}{q}}-a^{\frac{r}{q}}}{a-1}.
\end{aligned}
$$

Letting $a \to 1 + 0$ and applying l'Hopital's rule, we have

$$b^r\left(1 - b^p\right)^2\left(b^{\frac{p+r}{q}} - 1\right)^2 \le \left(b^{p+r} - 1\right)^2 b^{\frac{p+r}{q}-1}\frac{p^2}{q^2}(b-1)^2.$$

This implies that, for arbitrary $1 < b$,

$$b^{\frac{1+r-\frac{p+r}{q}}{2}}\left(b^p - 1\right)\left(b^{\frac{p+r}{q}} - 1\right) \le \frac{p}{q}\left(b^{p+r} - 1\right)(b-1). \tag{7}$$

For arbitrary $0 < x < 1$, substitute $\frac{1}{x}$ for b in (7) and multiply by $x, x^p, x^{p+r}, x^{\frac{p+r}{q}}$ both sides. It is easy to see that x itself satisfies (7). This completes the proof of Theorem 1.1.

Competing interests
The author declares that he has no competing interests.

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