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Boundary regularity for elliptic systems with supquadratic growth condition

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Abstract

We consider boundary regularity for weak solutions of quasilinear elliptic systems with the supquadratic growth condition and obtain a general criterion for a weak solution to be regular in the neighborhood of a given boundary point. By an element covering argument combined with existing interior partial regularity results, we establish the boundary regularity result.

Keywords: quasilinear elliptic systems; *A*-harmonic approximation technique; boundary partial regularity

1 Introduction

In this paper, we consider the partial regularity for weak solutions of elliptic systems

$$-D_\alpha\big(A_{ij}^{\alpha\beta}(x,u)D_\beta u^j\big) = B_i(x,u,Du), \quad x \in \Omega, \tag{1.1}$$

where Ω is a bounded domain in R^n with a boundary of class C^1, $n \geq 2$, u and B_i take values in R^N, $N > 1$. Each $A_{ij}^{\alpha\beta}$ maps $\Omega \times R^N$ into R, and each B_i maps $\Omega \times R^N \times R^{nN}$ into R. In theory partial regularity to (1.1) needs *a priori* existence of weak solutions. In order to define weak solutions to (1.1), one needs to impose certain structural and regularity conditions on $A_{ij}^{\alpha\beta}$ and the inhomogeneity B_i as follows. For $m > 2$,

(H1) There exists $L > 0$ such that

$$A_{ij}^{\alpha\beta}(x,\xi)(\nu,\tilde{\nu}) \leq L\big(1 + |\xi|^2\big)^{\frac{m-2}{2}} |\nu||\tilde{\nu}| \quad \text{for all } (x,\xi,) \in \overline{\Omega} \times R^N, \nu, \tilde{\nu} \in R^{nN}.$$

(H2) $A_{ij}^{\alpha\beta}(x,\xi)$ is uniformly strongly elliptic, that is, for some $\lambda > 0$, we have

$$A_{ij}^{\alpha\beta}(x,\xi)(\nu,\nu) \geq \lambda\big(1 + |\xi|^2\big)^{\frac{m-2}{2}} |\nu|^2 \quad \text{for all } (x,\xi) \in \overline{\Omega} \times R^N, \nu \in R^{nN}.$$

(H3) There exists a monotone nondecreasing concave function $\omega(s) : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$, continuous at 0, such that for all $x,y \in \Omega$, $u,v \in R^N$,

$$\big|A_{ij}^{\alpha\beta}(x,u) - A_{ij}^{\alpha\beta}(y,v)\big| \leq \omega\big(|x-y|^m + |u-v|^m\big).$$

(H4) The inhomogeneity satisfies the following growth condition:

$$\big|B_i(x,\xi,v)\big| \leq \big(1 + |\xi|^2\big)^{\frac{m-2}{2}} Q(x),$$

for all $x \in \overline{\Omega}$, $\xi \in R^N$, $v \in R^{nN}$ and where $Q(x) \in L^{m,n-2m+m\sigma}$ for some $\sigma > 0$.

(H5) There exists s with $s > n$ and a function $g \in H^{1,s}(\Omega, R^N)$ such that

$$u|_{\partial \Omega} = g|_{\partial \Omega}.$$

Note that we trivially have $g \in H^{1,m}(\Omega, R^N)$. Further, by the Sobolev embedding theorem, we have $g \in C^{0,\kappa}(\Omega, R^N)$ for any $\kappa \in [0, 1 - \frac{n}{s}]$. If $g|_{\partial \Omega} \equiv 0$, we will take $g \equiv 0$ on Ω.

If the domain in this case is a half unit ball, then the boundary condition is as follows:

(H5)′ There exists s with $s > n$ and a function $g \in H^{1,s}(B^+, R^N)$ such that

$$u|_D = g|_D.$$

Here we write $B_\rho(x_0) = \{x \in R^n : |x - x_0| < \rho\}$, and further $B_\rho = B_\rho(0)$, $B = B_1$. Similarly, we denote upper half balls as follows: for $x_0 \in R^{n-1} \times \{0\}$, we write $B_\rho^+(x_0)$ for $\{x \in R^n : x_n > 0, |x - x_0| < \rho\}$ and set $B_\rho^+ = B_\rho^+(0)$, $B^+ = B_1^+$. For $x_0 \in R^{n-1} \times \{0\}$, we further write $D_\rho(x_0)$ for $\{x \in R^n : x_n = 0, |x - x_0| < \rho\}$ and set $D_\rho = D_\rho(0)$, $D = D_1$.

Definition 1.1 By a weak solution of (1.1), we mean for all test-functions $\varphi \in C_0^1(\overline{\Omega}, R^N)$, a function $u \in W^{1,m}(\overline{\Omega}, R^N)$ satisfying

$$\int_\Omega A_{ij}^{\alpha\beta}(x,u)\big(D_\beta u^j, D_\alpha \varphi^i\big)\, dx = \int_\Omega B_i(x,u,Du) \cdot \varphi^i\, dx. \tag{1.2}$$

It is easily seen that by closure (1.2) holds for every φ in $W^{1,m}(\overline{\Omega}, R^N)$.

Under such assumptions on the structural conditions, full regularity of (1.1) at the boundary cannot, in general, be expected [1] even if the boundary data is smooth. Then our goal is to establish a partial boundary regularity for weak solutions of systems (1.1).

There are some previous partial regularity results for quasilinear systems. For example, Arkhipova has studied regularity up to the boundary for nonlinear and quasilinear systems [2–4]. For systems in a diagonal form, boundary regularity was first established by Wiegner [5], and the proof was generalized and extended by Hildebrandt-Widman [6]. In the case of minima of functionals of the form $\int_\Omega A(x,u)|Du|^2\, dx$, Jost-Meier [7] established full regularity in a neighborhood of the boundary.

The result which is most closely related to that given here was shown by Grotowski [8]. Grotowski obtained boundary partial regularity results for more general systems: $-D_\alpha A_i^\alpha(\cdot, u, Du) = f^i(\cdot, u, Du)$, $i = 1, \ldots, N$. In this paper, we extend the results in [9]. However the results in the current paper do not follow from those in [8]; in the current situation, we need only to impose weaker structure conditions and, at the same time, we can obtain stronger conclusions.

The technique of A-harmonic approximation is a natural extension of the technique of harmonic approximation. This technique has its origins in Simon's [10] proof of the regularity theorem of Allard [11]. The technique of A-harmonic approximation then refers to the direct analog of the above situation. The interior version of this technique has previously been applied by Duzaar and Grotowski in [12].

In this context, as the argument for combining the boundary and the interior estimates is relatively standard, we omit it and obtain the following results.

Theorem 1.1 *Consider a bounded domain Ω in R^N, with a boundary of class C^1. Let u be a bounded weak solution of* (1.1) *satisfying the boundary condition* (H5), *where the structure conditions* (H1)-(H3) *hold for $A_{ij}^{\alpha\beta}$, and* (H4) *holds for B_i. Consider a fixed $\gamma \in (0, \sigma]$. Then there exist positive R_1 and ε_0 (depending only on n, N, λ, L, b, M, $a(M)$, $\omega(\cdot)$ and γ) with the property that*

$$
\fint_{B_R(x_0) \cap \Omega} \left[\left| u - u_{x_0,R}' \right|^2 + \frac{|u - u_{x_0,R}'|^m}{R^{m-2}} \right] dx + \left[\|g\|_{H^{1,s}}^2 + \|g\|_{H^{1,s}}^m \right] R^{2(1-\frac{n}{s})} + R^{2\sigma} \leq \varepsilon_0^2
$$

for some $R \in (0, R_1]$ for a given $x_0 \in \partial\Omega$ implies $u \in C^{0,\gamma}(\overline{B}_{\frac{R}{2}}(x_0) \cap \overline{\Omega}, R^N)$.

Note in particular that the boundary condition (H5) means that $u_{x_0,R}'$ makes sense; in fact, we have $u_{x_0,R}' = g_{x_0,R}'$.

Combining this result with the analogous interior [13] and a standard covering argument allows us to obtain the following bound on the size of the singular set.

Corollary 1.1 *Under the assumptions of Theorem* 1.1, *the singular set of the weak solution u has $(n - m)$-dimensional Hausdorff measure zero in $\overline{\Omega}$.*

If the domain of the main step in proving Theorem 1.1 is a half ball, the result is then:

Theorem 1.2 *Consider a bounded weak solution of* (1.1) *on the upper half unit ball B^+ which satisfies the boundary condition* (H5)$'$, *where the structure conditions* (H1)-(H3) *hold for $A_{ij}^{\alpha\beta}$, and* (H4) *holds for B_i. Then there exist positive R_0 and ε_0 (depending only on n, N, λ, L, b, M, $a(M)$, M, $\omega(\cdot)$ and γ) with the property that*

$$
\fint_{B_R^+(x_0)} \left[\left| u - u_{x_0,R}' \right|^2 + \frac{|u - u_{x_0,R}'|^m}{R^{m-2}} \right] dx + \left[\|g\|_{H^{1,s}}^2 + \|g\|_{H^{1,s}}^m \right] R^{2(1-\frac{n}{s})} + R^{2\sigma} \leq \varepsilon_0^2
$$

for some $R \in (0, R_0]$ for a given $x_0 \in D$ implies that $u \in C^{0,\sigma}(\overline{B}_{\frac{R}{2}}^+(x_0), R^N)$ holds.

Note that analogously to above, the boundary condition (H5)$'$ ensures that $u_{x_0,R}'$ exists, and we have indeed $u_{x_0,R}' = g_{x_0,R}'$.

2 The *A*-harmonic approximation technique

In this section, we present the *A*-harmonic approximation lemma [9] and some standard results due to Campanato [14, 15].

Lemma 2.1 *Consider fixed positive λ and L, and $n, N \in N$ with $n \geq 2$. Then for any given $\varepsilon > 0$, there exists $\delta = \delta(n, N, \lambda, L, \varepsilon) \in (0, 1]$ with the following property: for any $A \in \mathrm{Bil}(R^{nN})$ satisfying*

$$
A(v, v) \geq \lambda |v|^2 \quad \text{for all } v \in R^{nN} \tag{2.1}
$$

and

$$
\left| A(v, \bar{v}) \right| \leq L |v| |\bar{v}| \quad \text{for all } v, \bar{v} \in R^{nN}, \tag{2.2}
$$

for any $w \in H^{1,2}(B_\rho^+(x_0), R^N)$ (for some $\rho > 0$, $x_0 \in R^n$) satisfying

$$\rho^{2-n} \int_{B_\rho^+(x_0)} |Dw|^2 \, dx \leq 1, \tag{2.3}$$

and

$$\left| \rho^{2-n} \int_{B_\rho^+(x_0)} A(Dg, D\varphi) \, dx \right| \leq \delta\rho \sup_{B_\rho^+(x_0)} |D\varphi| \tag{2.4}$$

and

$$w|_{D_\rho(x_0)} = 0 \tag{2.5}$$

for all $\varphi \in C_0^1(B_\rho^+(x_0), R^N)$, there exists an A-harmonic function

$$v \in \tilde{H} = \left\{ \tilde{w} \in H^{1,2}\big(B_\rho^+(x_0), R^N\big) \,\bigg|\, \rho^{2-n} \int_{B_\rho^+(x_0)} |D\tilde{w}|^2 \, dx \leq 1, \tilde{w}|_{D_\rho(x_0)} \equiv 0 \right\}$$

with

$$\rho^{-n} \int_{B_\rho^+(x_0)} |v - w|^2 \, dx \leq \varepsilon. \tag{2.6}$$

Next, we recall a slight modification of a characterization of Hölder continuous functions originally due to Campanato [14].

Lemma 2.2 *Consider $n \in N$, $n \geq 2$ and $x_0 \in R^{n-1} \times \{0\}$. Suppose that there are positive constants κ and α, with $\alpha \in (0, 1]$, such that for some $v \in L^2(B_{6R}^+(x_0))$, the following inequalities hold*:

$$\inf_{\mu \in R} \left\{ \rho^{-n} \int_{B_\rho^+(y)} |v - \mu|^2 \, dx \right\} \leq \kappa^2 \left(\frac{\rho}{R} \right)^{2\alpha}, \tag{2.7}$$

for all $y \in D_{2R}(x_0)$ and $\rho \leq 4R$; and

$$\inf_{\mu \in R} \rho^{-n} \left\{ \int_{B_\rho(y)} |v - \mu|^2 \, dx \right\} \leq \kappa^2 \left(\frac{\rho}{R} \right)^{2\alpha}, \tag{2.8}$$

for all $y \in B_{2R}^+(x_0)$ and $B_\rho(y) \subset B_{2R}^+(x_0)$.

Then there exists a Hölder continuous representative of the L^2-class of v on $\overline{B}_R^+(x_0)$, and for this representative $\overline{v}$,

$$\big|\overline{v}(x) - \overline{v}(z)\big| \leq C_\kappa \left(\frac{|x - z|}{R} \right)^\alpha \tag{2.9}$$

holds for all $x, z \in \overline{B}_R^+(x_0)$, for a constant C_κ depending only on n and α.

We close this section by a standard estimate for the solutions to homogeneous second-order elliptic systems with constant coefficients due originally to Campanato [15].

Lemma 2.3 *Consider fixed positive λ and L, and $n, N \in N$ with $n \geq 2$. Then there exists C_0 depending only on n, N, λ and L (without loss of generality, we take $C_0 \geq 1$) such that for $A \in \mathrm{Bil}(R^{nN})$ satisfying (2.1) and (2.2), any A-harmonic function h on $B_\rho^+(x_0)$ with $h|_{D_\rho(x_0)} \equiv 0$ satisfies*

$$\rho^2 \sup_{B_{\frac{\rho}{2}}^+(x_0)} |Dh|^2 \leq C_0 \rho^{2-n} \int_{B_\rho^+(x_0)} |Dh|^2 \, dx.$$

3 Caccioppoli inequality

In this section, we prove the Caccioppoli inequality. First of all, we recall two useful inequalities. The Sobolev embedding theorem yields the existence of a constant C_s depending only on s, n and N such that

$$\sup_{B_\rho^+(x_0)} \left| g - g'_{x_0,\rho} \right| \leq C_s \rho^{1-\frac{n}{s}} \|g\|_{H^{1,s}(B_\rho^+(x_0), R^N)} \tag{3.1}$$

holds for $x_0 \in D$, $\rho \leq 1 - |x_0|$. Obviously, the inequality remains true if we replace $\|g\|_{H^{1,s}(B_\rho^+(x_0), R^N)}$ by $\|g\|_{H^{1,s}(B^+, R^N)}$, which we will henceforth abbreviate simply as $\|g\|_{H^{1,s}}$.

Now, we note that the Poincare inequality in this setting for $x_0 \in D$, $\rho \leq 1 - |x_0|$ yields

$$\int_{B_\rho^+(x_0)} |g - g_{x_0,\rho}|^m \, dx \leq C_p \rho^m \int_{B_\rho^+(x_0)} |Dg|^m \, dx, \tag{3.2}$$

for a constant C_p which depends only on n.

Finally, we fix an exponent $\sigma \in (0, 1)$ as follows: if $g \equiv 0$, σ can be chosen arbitrarily (but henceforth fixed); otherwise, we take σ fixed in $(0, 1 - \frac{n}{s})$.

Then we prove an appropriate inequality for Caccioppoli.

Theorem 3.1 (Caccioppoli inequality) *Let u be a weak solution of systems* (1.1) *under* (H1)-(H5). *Then there exists $\rho_0 > 0$, ρ_0, depending only on L, M, s and $\|g\|_{H^{1,s}}$, such that for all $B_\rho^+(x_0) \subset B^+$, with $x_0 \in D^+$, $\rho < R < \rho_0$ there holds*

$$\int_{B_{\rho/2}^+(x_0)} |Du|^m \, dx$$

$$\leq C(\varepsilon, L) \int_{B_\rho^+(x_0)} \left[\frac{|u - u'_{x_0,\rho}|^2}{\rho^2} + \frac{|u - u'_{x_0,\rho}|^m}{\rho^m} \right] dx$$

$$+ C(\varepsilon) \alpha_n \left[\rho^{n-2+2\sigma} + \rho^{n-m+m\sigma} \right]$$

$$+ C(\varepsilon, L, C_s, C_p) \left[\|g\|_{H^{1,s}}^2 \left(\alpha_n \rho^n / 2 \right)^{1-\frac{2}{s}} + \|g\|_{H^{1,s}}^m \left(\alpha_n \rho^n / 2 \right)^{1-\frac{m}{s}} \right], \tag{3.3}$$

where C_1, C_2 depending only on λ and L, m and C_3 depending on these quantities and, in addition, on C_s, C_p and s.

Proof We consider a cut-off function $\eta \in C_0^\infty(B_\rho^+(x_0))$ satisfying $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_{\rho/2}^+(x_0)$ and $|\nabla \eta| < \frac{4}{\rho}$. Then the function $(u - g)\eta^2$ is in $W_0^{1,m}(B_\rho^+(x_0), R^N))$, and thus can

be taken as a test-function in (1.2). By (H2) and (H4), we have

$$\int_{B_\rho^+(x_0)} A_{ij}^{\alpha\beta}(\cdot, u)\big(D_\beta u^j, D_\alpha u^i\big)\eta^2\,dx$$

$$\leq L \int_{B_\rho^+(x_0)} \big(1 + |u|^2\big)^{\frac{m-2}{2}} |Du||Dg|\eta^2\,dx + 2 \int_{B_\rho^+(x_0)} \big(1 + |u|^2\big)^{\frac{m-2}{2}} |Du||D\eta|\eta|u - g|\,dx$$

$$+ \int_{B_\rho^+(x_0)} \big(1 + |u|^2\big)^{\frac{m-2}{2}} |Q(x)||u - g|\eta^2\,dx.$$

Using (H5) and Young's inequality, we have

$$\int_{B_\rho^+(x_0)} A_{ij}^{\alpha\beta}(\cdot, u)\big(D_\beta u^j, D_\alpha u^i\big)\eta^2\,dx$$

$$\leq \frac{3\varepsilon}{2} \int_{B_\rho^+(x_0)} \big(1 + |u|^2\big)^{\frac{m-2}{2}} |Du|^2\eta^2\,dx + \frac{L^2}{2\varepsilon} \int_{B_\rho^+(x_0)} \big(1 + |u|^2\big)^{\frac{m-2}{2}} |Dg|^2\eta^2\,dx$$

$$+ \frac{64L^2 + 1}{2\varepsilon\rho^2} \int_{B_\rho^+(x_0)} \big(1 + |u|^2\big)^{\frac{m-2}{2}} \big|u - u_{x_0,\rho}'\big|^2\,dx$$

$$+ \frac{64L^2 + 1}{2\varepsilon\rho^2} \int_{B_\rho^+(x_0)} \big(1 + |u|^2\big)^{\frac{m-2}{2}} \big|g - g_{x_0,\rho}'\big|^2\,dx$$

$$+ \varepsilon \int_{B_\rho^+(x_0)} \big(1 + |u|^2\big)^{\frac{m-2}{2}} \rho^2\eta^2 |Q(x)|^2\,dx$$

$$\leq \frac{3\varepsilon}{2} \int_{B_\rho^+(x_0)} \big(1 + |u|^2\big)^{\frac{m-2}{2}} |Du|^2\eta^2\,dx$$

$$+ C(\varepsilon, L) \int_{B_\rho^+(x_0)} \left[\frac{|u - u_{x_0,\rho}'|^2}{\rho^2} + \frac{|u - u_{x_0,\rho}'|^m}{\rho^m} \right] dx$$

$$+ C(\varepsilon, L) \int_{B_\rho^+(x_0)} \big[|Dg|^2\eta^2 + |Dg|^m\eta^m \big]\,dx$$

$$+ C(\varepsilon, L) \int_{B_\rho^+(x_0)} \left[\frac{|g - g_{x_0,\rho}'|^2}{\rho^2} + \frac{|g - g_{x_0,\rho}'|^m}{\rho^m} \right] dx$$

$$+ C(\varepsilon) \int_{B_\rho^+(x_0)} \big[\rho^2\eta^2 |Q(x)|^2 + \rho^m\eta^m |Q(x)|^m \big]\,dx.$$

Recalling the definition of $Q(x)$ and by (3.1) and (3.2), we have

$$\int_{B_\rho^+(x_0)} A_{ij}^{\alpha\beta}(\cdot, u)\big(D_\beta u^j, D_\alpha u^i\big)\eta^2\,dx$$

$$\leq \frac{3\varepsilon}{2} \int_{B_\rho^+(x_0)} \big(1 + |u|^2\big)^{\frac{m-2}{2}} |Du|^2\eta^2\,dx$$

$$+ C(\varepsilon, L) \int_{B_\rho^+(x_0)} \left[\frac{|u - u_{x_0,\rho}'|^2}{\rho^2} + \frac{|u - u_{x_0,\rho}'|^m}{\rho^m} \right] dx$$

$$+ C(\varepsilon, L, C_s, C_p)\Big[\|g\|_{H^{1,s}}^2 \big(\alpha_n\rho^n/2\big)^{1 - \frac{2}{s}} + \|g\|_{H^{1,s}}^m \big(\alpha_n\rho^n/2\big)^{1 - \frac{m}{s}} \Big]$$

$$+ C(\varepsilon)\alpha_n\big[\rho^{n-2+2\sigma} + \rho^{n-m+m\sigma} \big].$$

Using (H2), we can get with ε positive but arbitrary (to be fixed later)

$$\left(\lambda - \frac{3\varepsilon}{2}\right) \int_{B_R^+(x_0)} \left(1 + |u|^2\right)^{\frac{m-2}{2}} |Du|^2 \eta^2 \, dx$$

$$\leq C(\varepsilon, L) \int_{B_\rho^+(x_0)} \left[\frac{|u - u_{x_0,\rho}'|^2}{\rho^2} + \frac{|u - u_{x_0,\rho}'|^m}{\rho^m} \right] dx + C(\varepsilon) \alpha_n \left[\rho^{n-2+2\sigma} + \rho^{n-m+m\sigma} \right]$$

$$+ C(\varepsilon, L, C_s, C_p) \left[\|g\|_{H^{1,s}}^2 \left(\alpha_n \rho^n/2 \right)^{1-\frac{2}{s}} + \|g\|_{H^{1,s}}^m \left(\alpha_n \rho^n/2 \right)^{1-\frac{m}{s}} \right].$$

We can find a constant $p = \min\{1, (\frac{1+|u|^2}{|Du|^2})^{\frac{m-2}{2}}\}$ yields

$$\left(\lambda - \frac{3\varepsilon}{2}\right) p \int_{B_{\rho/2}^+(x_0)} |Du|^m \, dx$$

$$\leq C(\varepsilon, L) \int_{B_\rho^+(x_0)} \left[\frac{|u - u_{x_0,\rho}'|^2}{\rho^2} + \frac{|u - u_{x_0,\rho}'|^m}{\rho^m} \right] dx + C(\varepsilon) \alpha_n \left[\rho^{n-2+2\sigma} + \rho^{n-m+m\sigma} \right]$$

$$+ C(\varepsilon, L, C_s, C_p) \left[\|g\|_{H^{1,s}}^2 \left(\alpha_n \rho^n/2 \right)^{1-\frac{2}{s}} + \|g\|_{H^{1,s}}^m \left(\alpha_n \rho^n/2 \right)^{1-\frac{m}{s}} \right].$$

Choosing ε small enough, we can get the desired result immediately. $\qquad\square$

4 The proof of the main theorem

In this section, we proceed with the proof of the partial regularity result, and hence consider u to be a weak solution of (1.1). We consider $x_0 \in D$ and $y \in D_R(x_0)$, $D_\rho(y) \subset\subset D_R(x_0)$, for $R < 1 - |x_0|$, and $\varphi \in C_0^\infty(B_{\frac{\rho}{2}}^+(y), R^N)$ with $\sup_{B_\rho^+(y)} |D\varphi| \leq 1$. First of all, we have

Lemma 4.1

$$\int_{B_{\frac{\rho}{2}}^+(y)} A_{ij}^{\alpha\beta}\left(y, u_{y,\rho}'\right)\left(D_\beta u^j, D_\alpha \varphi^i\right) dx \leq C_4 \alpha_n \rho^{n-1} I^{\frac{1}{m}} \left(I^{\frac{m-1}{m}} + \omega(I) \right) \sup_{B_{\frac{\rho}{2}}^+(x_0)} |D\varphi|, \qquad (4.1)$$

here and hereafter, for $z \in D$, $r \in (0, 1 - |z|)$, we define

$$I(z, r) = \fint_{B_r^+(z)} \left[|u - u_{z,r}'|^2 + \frac{|u - u_{z,r}'|^m}{r^{m-2}} \right] dx + r^{2(1-\frac{s}{n})} \left[\|g\|_{H^{1,s}}^2 + \|g\|_{H^{1,s}}^m \right] + r^{2\sigma}.$$

Proof Consider $x_0 \in D$ and $y \in D_R(x_0)$, $D_\rho(y) \subset\subset D_R(x_0)$ for $R < 1 - |x_0|$, and $\varphi \in C_0^\infty(B_{\frac{\rho}{2}}^+(y), R^N)$ with $\sup_{B_\rho^+(y)} |D\varphi| \leq 1$. Using (1.2), we have

$$\int_{B_{\frac{\rho}{2}}^+(y)} A_{ij}^{\alpha\beta}\left(y, u_{y,\rho}'\right)\left(D_\beta u^j, D_\alpha \varphi^i\right) dx$$

$$\leq \int_{B_{\frac{\rho}{2}}^+(y)} \left(1 + |u|^2\right)^{\frac{m-2}{2}} |Q(x)| \rho \, dx \sup_{B_{\frac{\rho}{2}}^+(y)} |D\varphi|$$

$$+ \int_{B_{\frac{\rho}{2}}^+(y)} \left| A_{ij}^{\alpha\beta}(x, u) - A_{ij}^{\alpha\beta}\left(y, u_{y,\rho}'\right) \right| |Du| \, dx \sup_{B_{\frac{\rho}{2}}^+(y)} |D\varphi|. \qquad (4.2)$$

Henceforth, we restrict ρ sufficiently small and keeping the definition of $Q(x)$ in mind,

$$\int_{B^+_{\frac{\rho}{2}}(y)} \left(1+|u|^2\right)^{\frac{m-2}{2}} |Q(x)| \, dx$$

$$\leq C(m,\varepsilon)\left[\int_{B^+_{\frac{\rho}{2}}(y)} |Q(x)| \, dx + \int_{B^+_{\frac{\rho}{2}}(y)} \frac{|u-u'_{y,\rho}|^m}{\rho^m} \, dx + \int_{B^+_{\frac{\rho}{2}}(y)} |Q(x)|^{\frac{m}{2}} \rho^{\frac{m(m-2)}{2}}\right] dx$$

$$\leq C(m,\varepsilon)\left[\alpha_n \rho^n \left(\fint_{B^+_{\frac{\rho}{2}}(y)} |Q(x)|^m \, dx\right)^{\frac{1}{m}} \left(\fint_{B^+_{\frac{\rho}{2}}(y)} dx\right)^{1-\frac{1}{m}} + \int_{B^+_{\frac{\rho}{2}}(y)} \frac{|u-u'_{y,\rho}|^m}{\rho^m} \, dx\right.$$

$$\left. + \rho^{\frac{m(m-2)}{2}} \left(\int_{B^+_{\frac{\rho}{2}}(y)} |Q(x)|^m \, dx\right)^{\frac{1}{2}} \left(\int_{B^+_{\frac{\rho}{2}}(y)} dx\right)^{\frac{1}{2}}\right]$$

$$\leq C(m,\varepsilon)\rho^{n-2}\left[C\big(\|Q(x)\|_{L^m}\big)\alpha_n\rho^2 + \fint_{B^+_{\frac{\rho}{2}}(y)} \frac{|u-u'_{y,\rho}|^m}{\rho^{m-2}} \, dx + \alpha_n^{\frac{1}{2}} \rho^{\frac{(m-2)^2+m+m\sigma}{2}}\right]$$

$$\leq C\big(m,\varepsilon,\|Q(x)\|_{L^m},\alpha_n\big)\rho^{n-2}I(y,\rho), \tag{4.3}$$

where we define that for $z \in D$, $r \in (0, 1-|z|)$,

$$I(z,r) = \fint_{B^+_r(z)}\left[|u-u'_{z,r}|^2 + \frac{|u-u'_{z,r}|^m}{r^{m-2}}\right] dx + r^{2(1-\frac{s}{n})}\big[\|g\|^2_{H^{1,s}} + \|g\|^m_{H^{1,s}}\big] + r^{2\sigma}.$$

Applying (H3), Theorem 3.1 and Jensen's inequality, we can calculate from (4.2)

$$\fint_{B^+_{\frac{\rho}{2}}(y)} \left|A^{\alpha\beta}_{ij}(x,u) - A^{\alpha\beta}_{ij}\big(y,u'_{y,\rho}\big)\right||Du| \, dx$$

$$\leq \left(\fint_{B^+_{\frac{\rho}{2}}(y)} |Du|^m \, dx\right)^{\frac{1}{m}}\left(\fint_{B^+_{\frac{\rho}{2}}(y)} \omega^{\frac{m}{m-1}}\big(\rho^m + |u-u'_{y,\rho}|^m\big) \, dx\right)^{\frac{m-1}{m}}$$

$$\leq C\alpha_n\rho^{n-1}\left(\rho^m \fint_{B^+_\rho(y)}\left[\frac{|u-u'_{y,\rho}|^2}{\rho^2} + \frac{|u-u'_{y,\rho}|^m}{\rho^m}\right] dx + \rho^{2\sigma}\right.$$

$$\left. + \big[\|g\|^2_{H^{1,s}} + \|g\|^m_{H^{1,s}}\big]\rho^{2(1-\frac{n}{s})}\right)^{\frac{1}{m}} \cdot \omega\left(\rho^m + \fint_{B^+_{\frac{\rho}{2}}(y)} |u-u'_{y,\rho}|^m \, dx\right)$$

$$\leq C\alpha_n\rho^{n-1}I^{\frac{1}{m}}(y,\rho)\omega\big(I(y,\rho)\big). \tag{4.4}$$

Take (4.3) and (4.4) into (4.2), for arbitrary $\varphi \in C^\infty_0(\Omega, R^N)$, we thus have

$$\int_{B^+_{\frac{\rho}{2}}(y)} A^{\alpha\beta}_{ij}\big(y,u'_{y,\rho}\big)\big(D_\beta u^j, D_\alpha \varphi^i\big) \, dx \leq C_4\alpha_n\rho^{n-1}\big[I + I^{\frac{1}{m}}\omega(I)\big] \sup_{B^+_{\frac{\rho}{2}}(x_0)} |D\varphi|.$$

Multiplying through by $(\frac{\rho}{2})^{2-n}$, this yields

$$\left|\left(\frac{\rho}{2}\right)^{2-n} \int_{B^+_{\frac{\rho}{2}}(y)} A^{\alpha\beta}_{ij}\big(y,u'_{y,\rho}\big)\big(D_\beta u^j, D_\alpha \varphi^i\big) \, dx\right| \leq C_4 I^{\frac{1}{m}}\big[I^{\frac{m-1}{m}} + \omega(I)\big]\rho \sup_{B^+_{\frac{\rho}{2}}(x_0)} |D\varphi|, \tag{4.5}$$

for C_4 defined by $C_4 = C(m, \varepsilon, C_1, C_2, \alpha_n)$. $\qquad\square$

Lemma 4.2 *Consider u satisfying the conditions of Theorem* 1.1 *and* σ *fixed, then we can find* δ *and* s_0 *together with positive constants* C_9 *such that the smallness conditions*

$$0 < \omega(s_0) \leq \frac{\delta}{2} \quad and \quad I(x_0, R) \leq C_9^{-1} \min\left\{ \left(\frac{\delta}{2}\right)^{\frac{m}{m-1}}, s_0 \right\}$$

together imply the growth condition

$$I(y, \theta\rho) \leq \theta^{2\sigma} I(y, \rho).$$

Proof We now set $w = u - g$, using in turn (H1), Young's inequality and Hölder's inequality. We have from (4.5)

$$\left| \left(\frac{\rho}{2}\right)^{2-n} \int_{B_{\frac{\rho}{2}}^+(y)} A_{ij}^{\alpha\beta}(y, u'_{y,\rho}) \left(D_\beta w^j, D_\alpha \varphi^i\right) dx \right| \leq C_5 I^{\frac{1}{m}} \left[I^{\frac{m-1}{m}} + \omega(I) \right] \rho \sup_{B_{\frac{\rho}{2}}^+(x_0)} |D\varphi|, \quad (4.6)$$

for $c_5 = \max\{C_4, 2(\frac{\alpha_n}{2})^{1-\frac{1}{s}}\}$.

We now define C_6 by $C_6 = \max\{C_5, [2^{2n-1}(C_1 + \alpha_n^{-\frac{2}{s}} 2^{-(n+1)(1-\frac{2}{s})})]^{\frac{1}{2}}\}$ and set $v = \frac{w}{\gamma}$, for $\gamma = C_6 I^{\frac{1}{m}}$. From (4.5), we then have

$$\left| \left(\frac{\rho}{2}\right)^{2-n} \int_{B_{\frac{\rho}{2}}^+(y)} A_{ij}^{\alpha\beta}(y, u'_{y,\rho}) \left(D_\beta v^j, D_\alpha \varphi^i\right) dx \right| \leq \left(I^{\frac{m-1}{m}} + \omega(I) \right) \rho \sup_{B_{\frac{\rho}{2}}^+(x_0)} |D\varphi|, \quad (4.7)$$

and from (4.6), we observe from the definition of C_6 (recalling also the definition of γ)

$$\left(\frac{\rho}{2}\right)^{2-n} \int_{B_{\frac{\rho}{2}}^+(y)} |Dv|^2 dx < 1. \quad (4.8)$$

Further, we note

$$v|_{D_\rho(y)} = \frac{1}{\gamma} w|_{D_\rho(y)} = \frac{1}{\gamma} (u - g)|_{D_\rho(y)} \equiv 0. \quad (4.9)$$

For $\varepsilon > 0$, we take $\delta = \delta(n, N, \lambda, L, \varepsilon)$ to be the corresponding δ from A-harmonic approximation. Suppose that we could ensure that the smallness condition

$$I^{\frac{m-1}{m}} + \omega(I) \leq \delta \quad (4.10)$$

holds. Then in view of (4.7), (4.8) and (4.9), we would be able to apply Lemma 2.1 to conclude the existence of a function $h \in H^{1,2}(B_{\frac{\rho}{2}}^+(y), R^N)$ which is $A_{ij}^{\alpha\beta}(y, u'_{y,\rho})$-harmonic, with $h|_{D_{\frac{\rho}{2}}(y)} \equiv 0$ such that

$$\left(\frac{\rho}{2}\right)^{2-n} \int_{B_{\frac{\rho}{2}}^+(y)} |Dh|^2 dx \leq 1, \quad (4.11)$$

$$\left(\frac{\rho}{2}\right)^{-n} \int_{B_{\frac{\rho}{2}}^+(y)} |v - h|^2 dx \leq \varepsilon. \quad (4.12)$$

For $\theta \in (0, \frac{1}{4}]$ arbitrary (to be fixed later), we have from the Campanato theorem, noting (4.11) and recalling also that $h(y) = 0$,

$$\sup_{B^+_{\theta\rho}(y)} |h|^2 \leq \theta^2 \rho^2 \sup_{B^+_{\frac{\rho}{4}}(y)} |Dh|^2 \leq 4C_0 \theta^2. \tag{4.13}$$

Using (4.12) and (4.13) we observe

$$(\theta\rho)^{-n} \int_{B^+_{\theta\rho}(y)} |v|^2 \, dx \leq 2(\theta\rho)^{-n} \left[\left(\frac{\rho}{2}\right)^n \varepsilon + \frac{1}{2}\alpha_n(\theta\rho)^n \sup_{B^+_{\theta\rho}(y)} |h|^2 \right] \leq 2^{1-n}\theta^{-n}\varepsilon + 4\alpha_n C_0 \theta^2,$$

and hence, on multiplying this through by γ^2, we obtain the estimate

$$(\theta\rho)^{-n} \int_{B^+_{\theta\rho}(y)} |w|^2 \, dx \leq C_6^2 \left(2^{1-n}\theta^{-n}\varepsilon + 4\alpha_n C_0 \theta^2 \right) I^{\frac{2}{m}}. \tag{4.14}$$

For the time being, we restrict to the case that g does not vanish identically. Recalling that $w = u - g$, using in turn Poincare's, Sobolev's and then Hölder's inequalities, noting also that $u'_{y,\theta\rho} = g'_{y,\theta\rho}$, thus from (4.14), we can get

$$(\theta\rho)^{-n} \int_{B^+_{\theta\rho}(y)} \left| u - u'_{y,\theta\rho} \right|^2 \, dx$$

$$\leq 2(\theta\rho)^{-n} \left[\int_{B^+_{\theta\rho}(y)} |u - g|^2 \, dx + \int_{B^+_{\theta\rho}(y)} \left| g - g'_{y,\theta\rho} \right|^2 \, dx \right]$$

$$\leq 2C_6^2 \left(2^{1-n}\theta^{-n}\varepsilon + 4\alpha_n C_0 \theta^2 \right) I^{\frac{2}{m}} + 2C_p(\theta\rho)^{2-n} \left[\frac{1}{2}\alpha_n(\theta\rho)^n \right]^{1-\frac{2}{s}} \|g\|^2_{H^{1,s}}$$

$$\leq C_7 \left(\theta^{-n}\varepsilon + \theta^2 \right) I^{\frac{2}{m}} + C_7 \theta^{2(1-\frac{n}{s})} I \tag{4.15}$$

for $C_7 = \max\{8\alpha_n C_0 C_6^2, 2^{\frac{2}{s}} C_p \alpha_n^{1-\frac{2}{s}}\}$, and provided $\varepsilon = \theta^{n+2}$ and $\theta^2 I^{\frac{2}{m}} \leq \theta^{2(1-\frac{n}{s})} I$, we have

$$(\theta\rho)^{-n} \int_{B^+_{\theta\rho}(y)} \left| u - u'_{y,\theta\rho} \right|^2 \, dx \leq 3C_7 \theta^{2(1-\frac{n}{s})} I. \tag{4.16}$$

At the same time, from (4.15), we can see that for $2 < m < n$ ($n \geq 3$), we have $2 < m < m^*$, where

$$m^* = \begin{cases} \frac{mn}{n-m}, & \text{if } n > m, \\ m^* > m, & \text{if } m \geq n. \end{cases} \quad \text{with } \frac{1}{m^*} < \frac{1}{m} < \frac{1}{2}.$$

Therefore, we can find $t \in [0,1]$ such that $\frac{1}{m} = \frac{1-t}{2} + \frac{t}{m^*}$.

Using Sobolev's, Caccioppoli's and Young's inequalities with (4.15), we have

$$(\theta\rho)^{-n}(\theta\rho)^{-m+2} \int_{B^+_{\theta\rho}(y)} \left| u - u'_{y,\theta\rho} \right|^m \, dx$$

$$\leq \frac{1}{(\theta\rho)^{m-2}} \left((\theta\rho)^{-n} \int_{B^+_{\theta\rho}(y)} \left| u - u'_{y,\theta\rho} \right|^2 \, dx \right)^{\frac{(1-t)m}{2}} \left((\theta\rho)^{-n} \int_{B^+_{\theta\rho}(y)} \left| u - u'_{y,\theta\rho} \right|^{m^*} \, dx \right)^{\frac{mt}{m^*}}$$

$$\leq \frac{1}{(\theta\rho)^{m-2}}\left(C_7\left(\theta^{-n}\varepsilon + \theta^2 + \theta^{2(1-\frac{n}{s})}\right)I^{\frac{2}{m}}\right)^{\frac{(1-t)m}{2}}\left((\theta\rho)^{m-n}\int_{B_{\theta\rho}^+(y)}|Du|^m\,dx\right)^t$$

$$\leq \left(\frac{\alpha_n}{2}\right)^t C_7^{\frac{(1-t)m}{2}}\left[\left(\theta^{-n}\varepsilon + \theta^2 + \theta^{2(1-\frac{n}{s})}\right)\right]^{\frac{(1-t)m}{2}}\left[(\theta\rho)^{(m-2)(t-1)}\right]I$$

$$\leq \left(\frac{\alpha_n}{2}\right)^t C_7^{\frac{(1-t)m}{2}}\left(\frac{(1-t)m}{2}\left(\theta^{-n}\varepsilon + \theta^2 + \theta^{2(1-\frac{n}{s})}\right) + \frac{mt}{m^*}(\theta\rho)^{\frac{(m-2)(t-1)m^*}{mt}}\right)I$$

$$\leq C_8\left(\theta^{-n}\varepsilon + \theta^2 + \theta^{2(1-\frac{n}{s})}\right)I, \tag{4.17}$$

provided $(\theta\rho)^{\frac{(m-2)(1-t)m^*}{mt}} \leq \theta^2$ and where $C_8 = \left(\frac{\alpha_n}{2}\right)^t C_7^{\frac{(1-t)m}{2}} + 1$.

We then fix $\varepsilon = \theta^{n+2}$, note that this also fixed δ. Since $\rho \leq 1$, from the definition of I, we can get $[\|g\|_{H^{1,s}}^2 + \|g\|_{H^{1,s}}^m](\theta\rho)^{2(1-\frac{n}{s})} \leq \theta^{2(1-\frac{n}{s})}I$, and $(\theta\rho)^{2\sigma} \leq \theta^{2\sigma}I$.

Combining these estimates with (4.16) and (4.17), we can get

$$I(y,\theta\rho) \leq \left[3(C_7 + C_8)\theta^{2(1-\frac{n}{s})} + \theta^{2\sigma}\right]I. \tag{4.18}$$

Choosing $\theta \in (0, \frac{1}{4}]$ sufficiently small that there holds $3(C_7 + C_8)\theta^{2(1-\frac{n}{s})} \leq \theta^{2\sigma}$, we can see from (4.18)

$$I(y,\theta\rho) \leq \theta^{2\sigma}I. \tag{4.19}$$

We now choose $s_0 > 0$ such that $0 < \omega(s_0) < (\frac{\delta}{2})$ and define C_9 by $C_9 = \max\{2^{n+2m-1}, 2C_6^2 + 1, 2^{m-1}C_s^m + 1\}$. Suppose that we have

$$I(x_0,R) \leq C_9^{-1}\min\left\{\left(\frac{\delta}{2}\right)^{\frac{m}{m-1}}, s_0\right\} \tag{4.20}$$

for some $R \in (0, R_0]$, where $R_0 = \min\{\sqrt{2s_0}, 1/\sqrt{2b}, 1 - |x_0|\}$.

For any $y \in D_{\frac{R}{2}}(x_0)$, we use the Sobolev inequality to calculate

$$\frac{\alpha_n R^n}{2^{n+1}}\left|u'_{x_0,R} - u'_{y,\frac{R}{2}}\right|^2 \leq 2\int_{B_{\frac{R}{2}}^+}\left|g - g'_{x_0,R}\right|^2 dx + 2\int_{B_{\frac{R}{2}}^+}\left|g - g'_{y,\frac{R}{2}}\right|^2 dx$$

$$\leq 2\alpha_n C_s^2\|g\|_{H^{1,s}}^2 R^{n+2(1-\frac{n}{s})},$$

and

$$\frac{\alpha_n R^n}{2^{n+1}}\left|u'_{x_0,R} - u'_{y,\frac{R}{2}}\right|^m \leq 2^{m-1}\int_{B_{\frac{R}{2}}^+}\left|g - g'_{x_0,R}\right|^m dx + 2^{m-1}\int_{B_{\frac{R}{2}}^+}\left|g - g'_{y,\frac{R}{2}}\right|^m dx$$

$$\leq 2^m\alpha_n C_s^m\|g\|_{H^{1,s}}^m R^{n+m(1-\frac{n}{s})}.$$

And then we can calculate

$$I\left(y,\frac{1}{2}R\right) \leq 2^{n+2m-1}\left[\fint_{B_{\frac{R}{2}}^+(y)}\left|u - u'_{x_0,R}\right|^2 dx + \frac{1}{R^{m-2}}\fint_{B_{\frac{R}{2}}^+(y)}\left|u - u'_{x_0,R}\right|^m dx\right]$$

$$+ \left(2C_s^2 + 1\right)\|g\|_{H^{1,s}}^2 R^{2(1-\frac{n}{s})} + \left(2^{m-1}C_s^m + 1\right)\|g\|_{H^{1,s}}^m R^{2(1-\frac{n}{s})} + \frac{1}{4}R^{2\sigma}$$

$$\leq C_9 I(x_0,R). \tag{4.21}$$

Then we have

$$I^{\frac{m-1}{m}}\left(y,\frac{1}{2}R\right) + \omega\left(I\left(y,\frac{1}{2}R\right)\right) \le \left[C_9 I(x_0,R)\right]^{\frac{m-1}{m}} + \omega\left(C_9 I(x_0,R)\right) \le \frac{1}{2}\delta + \omega(s_0) \le \delta,$$

which means that the condition (4.20) is sufficient to guarantee the smallness condition (4.10) for $\rho = \frac{R}{2}$, for all $y \in D_{\frac{R}{2}}(x_0)$. We can thus conclude that (4.18) holds in this situation. From (4.18), we thus have

$$I^{\frac{m-1}{m}}(y,\theta\rho/2) + \omega\big(I(y,\theta\rho/2)\big) \le I^{\frac{m-1}{m}}\left(y,\frac{1}{2}R\right) + \omega\left(I\left(y,\frac{1}{2}R\right)\right) \le \delta,$$

meaning that we can apply (4.18) on $B^+_{\theta\rho/2}(y)$ as well, yielding

$$I\big(y,\theta^2 R/2\big) \le \theta^{4\sigma} I(y,R/2),$$

and inductively,

$$I\big(y,\theta^k R/2\big) \le \theta^{2k\sigma} I(y,R/2). \tag{4.22}$$

The next step is to go from a discrete to a continuous version of the decay estimate. Given $\rho \in (0,R/2]$, we can find $k \in N_0$ such that $\theta^{k+1}R/2 < \rho \le \theta^k R/2$. We calculate in a similar manner to above. Firstly, we use the Sobolev inequality, to see

$$\int_{B^+_\rho(y)} \left|u'_{y,\rho} - u'_{y,\theta^k R/2}\right|^2 dx \le 2\alpha_n \big(1/2\theta^k R\big)^n C_s^2 \|g\|^2_{H^{1,s}} \big(1/2\theta^k R\big)^{2(1-\frac{n}{s})},$$

which allows us to deduce

$$\int_{B^+_\rho(y)} \left|u - u'_{y,\rho}\right|^2 dx \le 2\int_{B^+_\rho(y)} \left|u - u_{y,\theta^k R/2'}\right|^2 dx + 4\alpha_n \big(1/2\theta^k R\big)^n C_s^2 \|g\|^2_{H^{1,s}} \big(1/2\theta^k R\big)^{2(1-\frac{n}{s})},$$

and similarly, we can get

$$\int_{B^+_\rho(y)} \left|u'_{y,\rho} - u'_{y,\theta^k R/2}\right|^m dx \le 2^{m-1}\alpha_n \big(1/2\theta^k R\big)^n C_s^m \|g\|^m_{H^{1,s}} \big(1/2\theta^k R\big)^{m(1-\frac{n}{s})},$$

and

$$\int_{B^+_\rho(y)} \left|u - u'_{y,\rho}\right|^m dx$$
$$\le 2^{m-1}\int_{B^+_\rho(y)} \left|u - u'_{y,\theta^k R/2}\right|^m dx + 2^{2m-2}\alpha_n \big(1/2\theta^k R\big)^n C_s^m \|g\|^m_{H^{1,s}} \big(1/2\theta^k R\big)^{m(1-\frac{n}{s})},$$

and hence, finally,

$$I(y,\rho) \le C_{10} I\big(y,\theta^k R/2\big)$$

for $C_{10} = 8\theta^{-n}C_s^2 + 2^{2m-1}\theta^{-n}C_s^m + 1$. Combining this with (4.22) and (4.21), we have

$$I(y,\rho) \le C_{10}\theta^{2k\sigma} I(y,R/2) \le C_{10}C_9\theta^{-2\sigma}\left(\frac{2\rho}{R}\right)^{2\sigma} I(x_0,R) \le C_{10}C_9\left(\frac{2}{\theta}\right) I(x_0,R)\left(\frac{\rho}{R}\right)^{2\sigma},$$

and more particularly

$$\inf_{\mu \in R^N} \int_{B_\rho^+(y)} |u - \mu|^2 \, dx \le C_{11} I(x_0, R) \left(\frac{\rho}{R} \right)^{2\sigma} \tag{4.23}$$

for C_{11} given by $C_{11} = C_{10} C_9 (\frac{2}{\theta})^{2\sigma}$. Recalling that this estimate is valid for all $y \in D$ and ρ with $D_\rho(y) \subset D_{R/2}(x_0)$, assuming only the condition (4.20) on $I(x_0, R)$. This yields - after replacing R by $6R$ - the boundary estimate (2.7) requiring to apply Lemma 2.2.

The analogous interior estimate to (4.22) is standard, and we obtain the desired interior estimate (2.8) by a standard argument. Hence, we can apply Lemma 2.2 and conclude the desired Hölder continuity. $\qquad\square$

Authors' contributions
SC participated in design of the study and drafted the manuscript. ZT participated in conceived of the study and the amendment of the paper. All authors read and approved the final manuscript.

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