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Some new Gronwall-Bellman type nonlinear dynamic inequalities containing integration on infinite intervals on time scales

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Abstract

In this paper, some new Gronwall-Bellman-type nonlinear dynamic inequalities containing integration on infinite intervals on time scales are established, which provides new bounds on unknown functions and can be used as a handy tool in the qualitative analysis of solutions of certain dynamic equations on time scales.

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1 Introduction

In the analysis of solutions of certain differential, integral and difference equations, if the solutions are unknown, then it is necessary to make estimate for their bounds. The Gronwall-Bellman inequality [1, 2] and its various generalizations which provide explicit bounds for solutions of differential, integral and difference equations have proved to be of particular importance in this aspect, and much effort has been made to establish such inequalities over the years (for example, see [3–10]). On the other hand, since Hilger [11] initiated the theory of time scales as a theory capable to contain both difference and differential calculus in a consistent way, many authors have expounded on various aspects of the theory of dynamic equations on time scales (for example, see [12–20] and the references therein). In these investigations, many authors have paid considerable attention to inequalities on time scales, and a lot of inequalities including Gronwall-Bellman type inequalities on time scales have been established (for example, see [15–27] and the references therein). But Gronwall-Bellman type nonlinear delay dynamic inequalities on time scales have been paid little attention in literature so far. Recent results in this direction include the works of Li [18], Ma *et al.* [23], Saker [24], and Feng *et al.* [25, 26], while nobody has undertaken research into Gronwall-Bellman type nonlinear dynamic inequalities containing integration on infinite intervals on time scales to our best knowledge. Besides, in order to fulfill the analysis of boundedness of the solutions of some dynamic equations, for example, the equations $u^p(t) = C + \int_t^\infty F(s, u(\tau(s)))\Delta s$ and $(u^p(t))^\Delta = F(t, u(\tau_1(t)), \int_t^\infty M(\xi, u(\tau_2(\xi)))\Delta\xi)$, which contain integration on infinite intervals on time scales, it is necessary to seek some new Gronwall-Bellman type nonlinear dynamic inequalities containing integration on infinite intervals on time scales so as to obtain desired results.

In this paper, we will establish some new Gronwall-Bellman type nonlinear dynamic inequalities containing integration on infinite intervals on time scales, which provide new bounds on unknown functions in some certain dynamic equations on time scales.

First, we will give some preliminaries on time scales and some universal symbols for further use. Throughout the paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0,\infty)$, while $\mathbb{Z}$ denotes the set of integers. For two given sets G, H, we denote the set of maps from G to H by (G,H). A time scale is an arbitrary nonempty closed subset of the real numbers. In this paper, $\mathbb{T}$ denotes an arbitrary time scale and $\mathbb{T}_0 = [t_0,\infty) \cap \mathbb{T}$, where $t_0 \in \mathbb{T}$. On $\mathbb{T}$ we define the forward and backward jump operators $\sigma \in (\mathbb{T},\mathbb{T})$ and $\rho \in (\mathbb{T},\mathbb{T})$ such that $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$, $\rho(t) = \sup\{s \in \mathbb{T}, s < t\}$. The graininess $\mu \in (\mathbb{T},\mathbb{R}_+)$ is defined by $\mu(t) = \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t \neq \inf\mathbb{T}$ and right-dense if $\sigma(t) = t$ and $t \neq \sup\mathbb{T}$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum, otherwise it is $\mathbb{T}$ without the left-scattered maximum. A function $f \in (\mathbb{T},\mathbb{R})$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while f is called regressive if $1 + \mu(t)f(t) \neq 0$. C_{rd} denotes the set of rd-continuous functions, while $\mathfrak{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathfrak{R}^+ = \{f | f \in \mathfrak{R}, 1 + \mu(t)f(t) > 0, \forall t \in \mathbb{T}\}$.

Definition 1.1 For some $t \in \mathbb{T}^\kappa$, and a function $f \in (\mathbb{T},\mathbb{R})$, the *delta derivative* of f is denoted by $f^\Delta(t)$ and satisfies

$$\left|f\big(\sigma(t)\big) - f(s) - f^\Delta(t)\big(\sigma(t) - s\big)\right| \le \varepsilon\left|\sigma(t) - s\right| \quad \forall\varepsilon > 0,$$

where $s \in \mathfrak{U}$, and $\mathfrak{U}$ is a neighborhood of t. The function f is called *delta differential* at t.

The *nabla derivative* of f is denoted by $f^\nabla(t)$ and satisfies

$$\left|f\big(\rho(t)\big) - f(s) - f^\nabla(t)\big(\rho(t) - s\big)\right| \le \varepsilon\left|\rho(t) - s\right| \quad \forall\varepsilon > 0,$$

where $s \in \mathfrak{U}$, and $\mathfrak{U}$ is a neighborhood of t.

Remark 1.1 If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t)$ and $f^\nabla(t)$ become the usual derivative $f'(t)$, while $f^\Delta(t) = f(t+1) - f(t)$, $f^\nabla(t) = f(t) - f(t-1)$ if $\mathbb{T} = \mathbb{Z}$, which represent the forward and backward difference respectively.

Definition 1.2 If $F^\Delta(t) = f(t)$, $t \in \mathbb{T}^\kappa$, then F is called an *antiderivative* of f, and the Cauchy integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad \text{where } a, b \in \mathbb{T}.$$

Similarly, if $F^\nabla(t) = f(t)$, then

$$\int_a^b f(t)\nabla t = F(b) - F(a), \quad \text{where } a, b \in \mathbb{T}.$$

The following two theorems include some important properties for *delta derivative*, *nabla derivative*, and the Cauchy integral on time scales.

Theorem 1.1 ([27]) *If $f, g \in (\mathbb{T}, \mathbb{R})$, and $t \in \mathbb{T}^\kappa$, then*

(i)

$$f^{\Delta}(t) = \begin{cases} \frac{f(\sigma(t))-f(t)}{\sigma(t)-t} & \text{if } \rho(t) \neq t, \\ \lim_{s\to t} \frac{f(t)-f(s)}{t-s} & \text{if } \mu(t) = 0, \end{cases} \qquad f^{\nabla}(t) = \begin{cases} \frac{f(\rho(t))-f(t)}{\rho(t)-t} & \text{if } \rho(t) = t, \\ \lim_{s\to t} \frac{f(t)-f(s)}{t-s} & \text{if } \mu(t) = 0. \end{cases}$$

(ii) *If f, g are delta differentials at t, then fg is also delta differential at t, and*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f\bigl(\sigma(t)\bigr)g^{\Delta}(t), \qquad (fg)^{\nabla}(t) = f^{\nabla}(t)g(t) + f\bigl(\rho(t)\bigr)g^{\nabla}(t).$$

Theorem 1.2 ([27]) *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{\mathrm{rd}}$, then*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$,
(ii) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t$,
(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$,
(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,
(v) $\int_a^a f(t)\Delta t = 0$,
(vi) *if $f(t) \geq 0$ for all $a \leq t \leq b$, then $\int_a^b f(t)\Delta t \geq 0$.*

Remark 1.2 Theorem 1.2 also holds for *nabla derivative*. If $b = \infty$, then all the conclusions of Theorem 1.2 still hold.

Definition 1.3 The cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ is defined by

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h} & \text{if } h \neq 0 \text{ (for } z \neq -\frac{1}{h}), \\ z & \text{if } h = 0, \end{cases}$$

where Log is the principal logarithm function.

Definition 1.4 For $p \in \mathfrak{R}$, the *exponential function* is defined by

$$e_p(t,s) = \exp\biggl(\int_s^t \xi_{\mu(\tau)}\bigl(p(\tau)\bigr)\Delta\tau\biggr) \quad \text{for } s, t \in \mathbb{T}.$$

Definition 1.5 If $\sup_{t\in\mathbb{T}} t = \infty$, $p \in \mathfrak{R}$, we define

$$e_p(\infty,t) = \exp\biggl(\int_t^\infty \xi_{\mu(\tau)}\bigl(p(\tau)\bigr)\Delta\tau\biggr) \quad \text{for } t \in \mathbb{T}.$$

Remark 1.3 If $\mathbb{T} = \mathbb{R}$, then for $t \in \mathbb{R}$,

$$\begin{cases} e_p(t,s) = \exp(\int_s^t p(\tau)\,d\tau), & \text{for } s \in \mathbb{T}, \\ e_p(\infty,s) = \exp(\int_s^\infty p(\tau)\,d\tau). \end{cases}$$

If $\mathbb{T} = \mathbb{Z}$, then for $t \in \mathbb{Z}$,

$$\begin{cases} e_p(t,s) = \prod_{\tau=s}^{t-1}[1 + p(\tau)], & \text{for } s \in \mathbb{T} \text{ and } s < t, \\ e_p(\infty,s) = \prod_{\tau=s}^{\infty}[1 + p(\tau)]. \end{cases}$$

The following two theorems include some known properties on the *exponential function*.

Theorem 1.3 ([28]) *If $p \in \mathfrak{R}$, then the following conclusions hold*:

(i) $e_p(t,t) \equiv 1$, *and* $e_0(t,s) \equiv 1$,

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$,

(iii) *if $p \in \mathfrak{R}^+$, then $e_p(t,s) > 0$ $\forall s,t \in \mathbb{T}$*,

(iv) *if $p \in \mathfrak{R}^+$, then $\ominus p \in \mathfrak{R}^+$*,

(v) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$,

where $\ominus p = -\frac{p}{1+\mu p}$.

Remark 1.4 If $s = \infty$, then Theorem 1.3(v) still holds.

Theorem 1.4 ([28]) *If $p \in \mathfrak{R}$, and fix $t_0 \in \mathbb{T}$, then the exponential function $e_p(t,t_0)$ is the unique solution of the following initial value problem*:

$$\begin{cases} y^{\Delta}(t) = p(t)y(t), \\ y(t_0) = 1. \end{cases}$$

For more details about the calculus of time scales, we refer the reader to [29].

2 Main results

For the sake of convenience, we always assume $\mathbb{T}_0 \subseteq \mathbb{T}^{\kappa}$ and $\sup_{t\in\mathbb{T}_0} t = \infty$ in the rest of this paper.

Lemma 2.1 *Suppose $u,b \in C_{\mathrm{rd}}(\mathbb{T}_0,\mathbb{R}^+)$. $-m \in \mathfrak{R}^+$, that is, $1-\mu(t)m(t) > 0$, $\forall t \in \mathbb{T}_0$, and furthermore, assume $\prod_{\tau=t_0}^{\infty} \frac{1-\mu m(\tau)+m(\tau)}{1-\mu m(\tau)} < \infty$. u is* delta differential *at $t \in \mathbb{T}_0$, and $m(t) \geq 0$, $t \in \mathbb{T}_0$. Then*

$$u(t) \leq b(t) + \int_t^{\infty} m(s)u(s)\Delta s, \quad t \in \mathbb{T}_0 \tag{2.1}$$

implies

$$u(t) \leq b(t) + \int_t^{\infty} m(s)b(s)e_{-m}\bigl(t,\sigma(s)\bigr)\Delta s, \quad t \in \mathbb{T}_0. \tag{2.2}$$

Proof Let

$$v(t) = \int_t^{\infty} m(s)u(s)\Delta s,$$

then we have

$$u(t) \leq b(t) + v(t), \quad t \in \mathbb{T}_0, \tag{2.3}$$

and

$$v^{\Delta}(t) = -m(t)u(t) \geq -m(t)\bigl[b(t)+v(t)\bigr],$$

that is,

$$v^{\Delta}(t) + m(t)v(t) \geq -m(t)b(t). \tag{2.4}$$

Since $-m \in \mathfrak{R}^{+}$, then from Theorem 1.3(iv), we have $\ominus(-m) \in \mathfrak{R}^{+}$, and furthermore, from Theorem 1.3(iii), we obtain $e_{\ominus(-m)}(t,\alpha) > 0$ $\forall t, \alpha \in \mathbb{T}_0$.

According to Theorem 1.1(ii),

$$\left[v(t)e_{\ominus(-m)}(t,\alpha)\right]^{\Delta} = \left[e_{\ominus(-m)}(t,\alpha)\right]^{\Delta} v(t) + e_{\ominus(-m)}\left(\sigma(t),\alpha\right)v^{\Delta}(t). \tag{2.5}$$

On the other hand, from Theorem 1.4, we have

$$\left[e_{\ominus(-m)}(t,\alpha)\right]^{\Delta} = \left(\ominus(-m)\right)(t)e_{\ominus(-m)}(t,\alpha). \tag{2.6}$$

So combining (2.5), (2.6) and Theorem 1.3 yields

$$\begin{aligned}\left[v(t)e_{\ominus(-m)}(t,\alpha)\right]^{\Delta} &= \left(\ominus(-m)\right)(t)e_{\ominus(-m)}(t,\alpha)v(t) + e_{\ominus(-m)}\left(\sigma(t),\alpha\right)v^{\Delta}(t) \\ &= e_{\ominus(-m)}\left(\sigma(t),\alpha\right)\left[\frac{(\ominus(-m))(t)}{1+\mu(t)(\ominus(-m))(t)}v(t) + v^{\Delta}(t)\right] \\ &= e_{\ominus(-m)}\left(\sigma(t),\alpha\right)\left[v^{\Delta}(t) + m(t)v(t)\right].\end{aligned} \tag{2.7}$$

Substituting t with s and an integration for (2.7) with respect to s from α to ∞ yield

$$\begin{aligned}&v(\infty)e_{\ominus(-m)}(\infty,\alpha) - v(\alpha)e_{\ominus(-m)}(\alpha,\alpha) \\ &\quad= \int_{\alpha}^{\infty} e_{\ominus(-m)}\left(\sigma(s),\alpha\right)\left[v^{\Delta}(s) + m(s)v(s)\right]\Delta s.\end{aligned} \tag{2.8}$$

Since $\prod_{\tau=t_0}^{\infty} \frac{1-\mu m(\tau)+m(\tau)}{1-\mu m(\tau)} < \infty$, then $e_{\ominus(-m)}(\infty,\alpha) < \infty$. Considering $e_{\ominus p}(\alpha,\alpha) = 1$, and $v(\infty) = 0$, by (2.4) and (2.8), we have

$$-v(\alpha) \geq -\int_{\alpha}^{\infty} e_{\ominus(-m)}\left(\sigma(s),\alpha\right)m(s)b(s)\Delta s = -\int_{\alpha}^{\infty} e_{-m}\left(\alpha,\sigma(s)\right)m(s)b(s)\Delta s,$$

which is followed by

$$v(\alpha) \leq \int_{\alpha}^{\infty} e_{-m}\left(\alpha,\sigma(s)\right)m(s)b(s)\Delta s. \tag{2.9}$$

Since $\alpha \in \mathbb{T}_0$ is arbitrary, then substituting α with t and combining (2.3), we can obtain the desired inequality. □

Lemma 2.2 *Under the conditions of Lemma* 2.1, *furthermore, if* $b(t)$ *is decreasing, then we have*

$$u(t) \leq b(t)e_{-m}(t,\infty), \quad t \in \mathbb{T}_0. \tag{2.10}$$

Proof Since $b(t)$ is decreasing on $\mathbb{T}_0$, then from (2.2), we have

$$\begin{aligned} u(t) &\le b(t) + \int_t^\infty m(s)b(s)e_{-m}\bigl(t,\sigma(s)\bigr)\Delta s \\ &\le b(t)\left[1 + \int_t^\infty m(s)e_{-m}\bigl(t,\sigma(s)\bigr)\Delta s\right]. \end{aligned} \tag{2.11}$$

On the other hand, from [23, Theorems 2.39 and 2.36(i)], for some $\bar{t} \in \mathbb{T}_0$, we have

$$\int_t^{\bar{t}} m(s)e_{-m}\bigl(t,\sigma(s)\bigr)\Delta s = \int_{\bar{t}}^t -m(s)e_{-m}\bigl(t,\sigma(s)\bigr)\Delta s = e_{-m}(t,\bar{t}) - 1. \tag{2.12}$$

Then letting $\bar{t} \to \infty$ in (2.12), we obtain

$$\int_t^\infty m(s)e_{-m}\bigl(t,\sigma(s)\bigr)\Delta s = e_{-m}(t,\infty) - 1. \tag{2.13}$$

Combining (2.11) and (2.13), we obtain the desired inequality. □

Lemma 2.3 ([30]) *Assume that $a \ge 0$, $p \ge q \ge 0$ and $p \ne 0$, then for any $K > 0$,*

$$a^{\frac{q}{p}} \le \frac{q}{p}K^{\frac{q-p}{p}}a + \frac{p-q}{p}K^{\frac{q}{p}}.$$

Now, we consider the delay dynamic inequality of the following form:

$$\begin{aligned} u^p(t) &\le a(t) + \int_t^\infty f(s)u^p\bigl(\tau_1(s)\bigr)\Delta s \\ &\quad + \int_t^\infty \left[m(s) + g(s)\omega\bigl(u\bigl(\tau_2(s)\bigr)\bigr) + \int_s^\infty h(\xi)\omega\bigl(u\bigl(\tau_2(\xi)\bigr)\bigr)\nabla\xi\right]\nabla s, \end{aligned} \tag{2.14}$$

where $u, m, a, f, g, h \in C_{\mathrm{rd}}(\mathbb{T}_0, \mathbb{R}_+)$, and a is decreasing. $\tau_i \in (\mathbb{T}_0, \mathbb{T})$ with $\tau_i(t) \ge t$, $i = 1, 2$, $p > 0$ is a constant, $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing, and ω is submultiplicative, that is, $\omega(\alpha\beta) \le \omega(\alpha)\omega(\beta)$ holds $\forall\alpha \ge 0$, $\beta \ge 0$.

Theorem 2.1 *If $u(t)$ satisfies inequality* (2.14), *then we have*

$$\begin{aligned} u(t) \le \Bigg\{ G^{-1}\Bigg\{ G\left[a(t) + \int_t^\infty m(s)\nabla s\right] \\ + \int_t^\infty \left[g(s) + \int_s^\infty h(\xi)\nabla\xi\right]\omega\bigl(e_{-f}^{\frac{1}{p}}(s,\infty)\bigr)\nabla s\Bigg\} e_{-f}(t,\infty)\Bigg\}^{\frac{1}{p}}, \quad t \in \mathbb{T}_0, \end{aligned} \tag{2.15}$$

provided that $-f \in \mathfrak{R}_+$, and $\prod_{\tau=t_0}^{\infty} \frac{1-\mu f(\tau)+f(\tau)}{1-\mu f(\tau)} < \infty$, where G is an increasing bijective function, and

$$G(v) = \int_1^v \frac{1}{\omega(r^{\frac{1}{p}})}\,dr, \quad v > 0 \quad \textit{with} \quad G(\infty) = \infty.$$

Proof Let the right side of (2.14) be $v(t)$. Then

$$u(t) \le v^{\frac{1}{p}}(t), \quad t \in \mathbb{T}_0, \tag{2.16}$$

and

$$u\bigl(\tau_i(t)\bigr) \le v^{\frac{1}{p}}\bigl(\tau_i(t)\bigr) \le v^{\frac{1}{p}}(t). \tag{2.17}$$

Furthermore,

$$v(t) \le a(t) + \int_t^\infty \biggl[m(s) + g(s)\omega\bigl(v^{\frac{1}{p}}(s)\bigr) + \int_s^\infty h(\xi)\omega\bigl(v^{\frac{1}{p}}(\xi)\bigr)\nabla\xi\biggr]\nabla s + \int_t^\infty f(s)v(s)\Delta s. \tag{2.18}$$

According to $-f \in \mathfrak{R}_+$, a suitable application of Lemma 2.2 to (2.18) yields

$$v(t) \le \biggl\{a(t) + \int_t^\infty \biggl[m(s) + g(s)\omega\bigl(v^{\frac{1}{p}}(s)\bigr) + \int_s^\infty h(\xi)\omega\bigl(v^{\frac{1}{p}}(\xi)\bigr)\nabla\xi\biggr]\nabla s\biggr\} e_{-f}(t,\infty). \tag{2.19}$$

Fix $T \in \mathbb{T}_0$, and let $t \in [T,\infty) \cap \mathbb{T}$. Define

$$c(t) = a(T) + \int_T^\infty m(s)\nabla s + \int_t^\infty \biggl[g(s)\omega\bigl(v^{\frac{1}{p}}(s)\bigr) + \int_s^\infty h(\xi)\omega\bigl(v^{\frac{1}{p}}(\xi)\bigr)\nabla\xi\biggr]\nabla s. \tag{2.20}$$

Then

$$v(t) \le c(t)e_{-f}(t,\infty), \quad t \in [T,\infty) \cap \mathbb{T}, \tag{2.21}$$

and

$$\begin{aligned} c^\nabla(t) &= -\biggl[g(t)\omega\bigl(v^{\frac{1}{p}}(t)\bigr) + \int_t^\infty h(\xi)\omega\bigl(v^{\frac{1}{p}}(\xi)\bigr)\nabla\xi\biggr] \\ &\ge -\biggl[g(t) + \int_t^\infty h(\xi)\nabla\xi\biggr]\omega\bigl(v^{\frac{1}{p}}(t)\bigr) \\ &\ge -\biggl[g(t) + \int_t^\infty h(\xi)\nabla\xi\biggr]\omega\bigl(c^{\frac{1}{p}}(t)e_{-f}^{\frac{1}{p}}(t,\infty)\bigr) \\ &\ge -\biggl[g(t) + \int_t^\infty h(\xi)\nabla\xi\biggr]\omega\bigl(c^{\frac{1}{p}}(t)\bigr)\omega\bigl(e_{-f}^{\frac{1}{p}}(t,\infty)\bigr), \end{aligned}$$

that is,

$$\frac{c^\nabla(t)}{\omega(c^{\frac{1}{p}}(t))} \ge -\biggl[g(t) + \int_t^\infty h(\xi)\nabla\xi\biggr]\omega\bigl(e_{-f}^{\frac{1}{p}}(t,\infty)\bigr), \quad t \in [T,\infty) \cap \mathbb{T}.$$

On the other hand, for $t \in [T,\infty] \cap \mathbb{T}$, if $\rho(t) < t$, then

$$\begin{aligned} \bigl[G\bigl(c(t)\bigr)\bigr]^\nabla &= \frac{G(c(\rho(t))) - G(c(t))}{\rho(t) - t} = \frac{1}{\rho(t) - t}\int_{c(t)}^{c(\rho(t))} \frac{1}{\omega(r^{\frac{1}{p}})}\,dr \\ &\ge \frac{c(\rho(t)) - c(t)}{\rho(t) - t}\frac{1}{\omega(c^{\frac{1}{p}}(t))} = \frac{c^\nabla(t)}{\omega(c^{\frac{1}{p}}(t))}. \end{aligned}$$

If $\rho(t) = t$, then

$$\begin{aligned}\left[G\big(c(t)\big)\right]^{\nabla} &= \lim_{s\to t}\frac{G(c(t)) - G(c(s))}{t-s} = \lim_{s\to t}\frac{1}{t-s}\int_{c(s)}^{c(t)}\frac{1}{\omega(r^{\frac{1}{p}})}\,dr \\ &= \lim_{s\to t}\frac{c(t)-c(s)}{t-s}\frac{1}{\omega(\xi^{\frac{1}{p}})} = \frac{c^{\nabla}(t)}{\omega(c^{\frac{1}{p}}(t))},\end{aligned}$$

where ξ lies between $c(s)$ and $c(t)$. So we always have $[G(c(t))]^{\nabla} \ge \frac{c^{\nabla}(t)}{\omega(c^{\frac{1}{p}}(t))}$. Using the statements above, we deduce that

$$\left[G\big(c(t)\big)\right]^{\nabla} \ge -\left[g(t) + \int_t^{\infty} h(\xi)\nabla\xi\right]\omega\big(e_{-f}^{\frac{1}{p}}(t,\infty)\big), \quad t \in [T,\infty)\cap\mathbb{T}. \tag{2.22}$$

Substituting t with s in (2.22) and an integration with respect to s from t to ∞ yield

$$\begin{aligned}&G\big(c(\infty)\big) - G\big(c(t)\big) \\ &\quad \ge -\int_t^{\infty}\left[g(s) + \int_s^{\infty} h(\xi)\nabla\xi\right]\omega\big(e_{-f}^{\frac{1}{p}}(s,\infty)\big)\nabla s, \quad t \in [T,\infty)\cap\mathbb{T}.\end{aligned}$$

Consider G is strictly increasing and $c(\infty) = a(T) + \int_T^{\infty} m(s)\nabla s$, then it follows

$$\begin{aligned}c(t) \le G^{-1}\Bigg\{&G\left[a(T) + \int_T^{\infty} m(s)\nabla s\right] \\ &+ \int_t^{\infty}\left[g(s) + \int_s^{\infty} h(\xi)\nabla\xi\right]\omega\big(e_{-f}^{\frac{1}{p}}(s,\infty)\big)\nabla s\Bigg\}, \quad t \in [T,\infty)\cap\mathbb{T}.\end{aligned} \tag{2.23}$$

Combining (2.16), (2.21) and (2.23), we have for $t \in [T,\infty)\cap\mathbb{T}$

$$\begin{aligned}u(t) \le \Bigg\{G^{-1}\Bigg\{&G\left[a(T) + \int_T^{\infty} m(s)\nabla s\right] \\ &+ \int_t^{\infty}\left[g(s) + \int_s^{\infty} h(\xi)\nabla\xi\right]\omega\big(e_{-f}^{\frac{1}{p}}(s,\infty)\big)\nabla s\Bigg\}e_{-f}(t,\infty)\Bigg\}^{\frac{1}{p}}.\end{aligned} \tag{2.24}$$

Setting $t = T$ in (2.24) yields

$$\begin{aligned}u(T) \le \Bigg\{G^{-1}\Bigg\{&G\left[a(T) + \int_T^{\infty} m(s)\nabla s\right] \\ &+ \int_T^{\infty}\left[g(s) + \int_s^{\infty} h(\xi)\nabla\xi\right]\omega\big(e_{-f}^{\frac{1}{p}}(s,\infty)\big)\nabla s\Bigg\}e_{-f}(T,\infty)\Bigg\}^{\frac{1}{p}}.\end{aligned} \tag{2.25}$$

Since $T \in \mathbb{T}_0$ is selected arbitrarily, after substituting T with t in (2.25) the proof is complete. □

If we let $p = 1$ in Theorem 2.1, then we have the following corollary.

Corollary 2.1 *Under the conditions of Theorem* 2.1 *with* $p = 1$, *if for* $t \in \mathbb{T}_0$, $u(t)$ *satisfies the following inequality*:

$$u(t) \le a(t) + \int_t^\infty f(s)u(\tau_1(s))\Delta s + \int_t^\infty \left[m(s) + g(s)\omega(u(\tau_2(s))) + \int_s^\infty h(\xi)\omega(u(\tau_2(\xi)))\nabla\xi \right]\nabla s, \tag{2.26}$$

then

$$u(t) \le G^{-1}\left\{ G\left[a(t) + \int_t^\infty m(s)\nabla s\right] + \int_t^\infty \left[g(s) + \int_s^\infty h(\xi)\nabla\xi\right]\omega(e_{-f}(s,\infty))\nabla s \right\} e_{-f}(t,\infty), \quad t \in \mathbb{T}_0, \tag{2.27}$$

provided that $-f \in \mathfrak{R}_+$, *and* $\prod_{\tau=t_0}^{\infty} \frac{1-\mu f(\tau)+f(\tau)}{1-\mu f(\tau)} < \infty$.

Since $\mathbb{T}$ is an arbitrary time scale, then if we take $\mathbb{T}$ for some peculiar cases in Theorem 2.1, then we can obtain some corollaries immediately. Especially, if we take $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, then we obtain continuous and discrete analyses respectively, which are shown in the following two corollaries.

Corollary 2.2 *Suppose* $\mathbb{T} = \mathbb{R}$, $t_0 \in \mathbb{R}$, *and* $\mathbb{R}_0 = [t_0, \infty) \cap \mathbb{R}$. $u, m, a, b, f \in (\mathbb{R}_0, \mathbb{R}^+)$, *and* a, b *are decreasing on* $\mathbb{R}_0$. $\tau_i \in (\mathbb{R}_0, \mathbb{R})$, $\tau_i(t) \ge t$, $i = 1, 2$, ω *is defined the same as in Theorem* 2.1. *If for* $t \in \mathbb{R}_0$, $u(t)$ *satisfies*

$$u^p(t) \le a(t) + \int_t^\infty \left[m(s) + f(s)u^p(\tau_1(s)) + g(s)\omega(u(\tau_2(s))) + \int_s^\infty h(\xi)\omega(u(\tau_2(\xi)))\,d\xi \right] ds,$$

then

$$u(t) \le \left\{ G^{-1}\left\{ G\left[a(t) + \int_t^\infty m(s)\,ds\right] + \int_t^\infty \left[g(s) + \int_s^\infty h(\xi)\,d\xi\right]\omega\left(e_{-f}^{\frac{1}{p}}(s,\infty)\right) ds \right\} e_{-f}(t,\infty) \right\}^{\frac{1}{p}}, \quad t \in \mathbb{R}_0,$$

provided that $\prod_{\tau=t_0}^{\infty} f(\tau) < \infty$.

Corollary 2.3 *Suppose* $\mathbb{T} = \mathbb{Z}$, $n_0 \in \mathbb{Z}$, *and* $\mathbb{Z}_0 = [n_0, \infty) \cap \mathbb{Z}$. $u, m, a, b, f \in (\mathbb{Z}_0, \mathbb{R}^+)$, *and* a, b *are decreasing on* $\mathbb{Z}_0$. $\tau_i \in (\mathbb{Z}_0, \mathbb{Z})$, $\tau_i(n) \ge n$, $i = 1, 2$, ω *is defined the same as in Theorem* 2.1. *If for* $n \in \mathbb{Z}_0$, $u(n)$ *satisfies*

$$u^p(n) \le a(n) + \sum_{s=n}^{\infty} f(s)u^p(\tau_1(s)) + \sum_{s=n+1}^{\infty} \left[m(s) + g(s)\omega(u(\tau_2(s))) + \sum_{\xi=s}^{\infty} h(\xi)\omega(u(\tau_2(\xi))) \right], \quad \textit{then for } n \in \mathbb{Z}_0,$$

$$u(n) \le \left\{ G^{-1} \left\{ G\left[a(n) + \sum_{s=n+1}^{\infty} m(s) \right] \right.\right.$$
$$\left.\left. + \sum_{s=n+1}^{\infty} \left\{ \left[g(s) + \sum_{\xi=s+1}^{\infty} h(\xi) \right] \omega\left(\left(\prod_{\tau=s}^{\infty} \frac{1}{1-f(\tau)} \right)^{\frac{1}{p}} \right) \right\} \right\} \prod_{\tau=n}^{\infty} \frac{1}{1-f(\tau)} \right\}^{\frac{1}{p}},$$

provided that $f(n) < 1$, *and* $\prod_{\tau=n_0}^{\infty} \frac{1}{1-f(\tau)} < \infty$.

Second, we study the following delay dynamic inequality on time scales:

$$u^p(t) \le a(t) + b(t) \int_t^{\infty} f(s)\omega\big(u(\tau(s))\big)\nabla s, \quad t \in \mathbb{T}_0, \tag{2.28}$$

where $b \in C_{\mathrm{rd}}(\mathbb{T}_0, \mathbb{R}_+)$, and b is decreasing, u, a, f are defined the same as in Theorem 2.1, $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing, $\tau \in (\mathbb{T}_0, \mathbb{T})$, $\tau(t) \ge t$.

Theorem 2.2 *If* $u(t)$ *satisfies inequality* (2.28), *then*

$$u(t) \le \left\{ G^{-1}\left[G\big(a(t)\big) + b(t) \int_t^{\infty} f(s)\nabla s \right] \right\}^{\frac{1}{p}}, \quad t \in \mathbb{T}_0, \tag{2.29}$$

where G is defined the same as in Theorem 2.1.

Proof Let the right side of (2.28) be $v(t)$. Then we have

$$u(t) \le v^{\frac{1}{p}}(t), \quad t \in \mathbb{T}_0, \tag{2.30}$$

and

$$u\big(\tau(t)\big) \le v^{\frac{1}{p}}\big(\tau(t)\big) \le v^{\frac{1}{p}}(t), \quad t \in \mathbb{T}_0. \tag{2.31}$$

Furthermore,

$$v(t) \le a(t) + b(t) \int_t^{\infty} f(s)\omega\big(v^{\frac{1}{p}}(s)\big)\nabla s. \tag{2.32}$$

Let $T \in \mathbb{T}_0$ be fixed, and denote

$$z(t) = a(T) + b(T) \int_t^{\infty} f(s)\omega\big(v^{\frac{1}{p}}(s)\big)\nabla s. \tag{2.33}$$

Consider a, b are decreasing on $\mathbb{T}$, then for $t \in [T, \infty) \cap \mathbb{T}$, we have

$$v(t) \le z(t). \tag{2.34}$$

Moreover,

$$\begin{aligned} z^{\nabla}(t) &= -b(T)f(t)\omega\big(v^{\frac{1}{p}}(t)\big) \\ &\ge -b(T)f(t)\omega\big(z^{\frac{1}{p}}(t)\big). \end{aligned}$$

Similar to Theorem 2.1, we have

$$\left[G\big(z(t)\big)\right]^{\nabla} \geq \frac{z^{\nabla}(t)}{\omega(z^{\frac{1}{p}}(t))} \geq -b(T)f(t). \tag{2.35}$$

Substituting t with s in (2.35) and an integration with respect to s from t to ∞ yield

$$G\big(z(\infty)\big) - G\big(z(t)\big) \geq -\int_t^{\infty} b(T)f(s)\nabla s = -b(T)\int_t^{\infty} f(s)\nabla s. \tag{2.36}$$

Consider G is strictly increasing, and $z(\infty) = a(T)$, then (2.36) is followed by

$$z(t) \leq G^{-1}\left[G\big(a(T)\big) + b(T)\int_t^{\infty} f(s)\nabla s\right], \quad t \in [T,\infty) \cap \mathbb{T}. \tag{2.37}$$

Combining (2.30), (2.34) and (2.37) yields

$$u(t) \leq \left\{G^{-1}\left[G\big(a(T)\big) + b(T)\int_t^{\infty} f(s)\nabla s\right]\right\}^{\frac{1}{p}}, \quad t \in [T,\infty) \cap \mathbb{T}. \tag{2.38}$$

Setting $t = T$ in (2.38), we obtain

$$u(T) \leq \left\{G^{-1}\left[G\big(a(T)\big) + b(T)\int_T^{\infty} f(s)\nabla s\right]\right\}^{\frac{1}{p}}. \tag{2.39}$$

Since $T \in \mathbb{T}_0$ is selected arbitrarily, then after substituting T with t in (2.39), we obtain the desired inequality (2.29). □

Considering $\mathbb{T}$ is an arbitrary time scale, if we take $\mathbb{T}$ for some peculiar cases in Theorem 2.2, we immediately get the following two corollaries.

Corollary 2.4 *Suppose $\mathbb{T} = \mathbb{R}$, $t_0 \in \mathbb{R}$, and $\mathbb{R}_0 = [t_0,\infty) \cap \mathbb{R}$. $u, a, b, f \in (\mathbb{R}_0, \mathbb{R}^+)$, and a, b are decreasing on $\mathbb{R}_0$. $\tau \in (\mathbb{R}_0, \mathbb{R})$, $\tau(t) \geq t$, ω is defined as in Theorem 2.2. If $u(t)$ satisfies*

$$u^p(t) \leq a(t) + b(t)\int_t^{\infty} f(s)\omega\big(u\big(\tau(s)\big)\big)\,ds, \quad t \in \mathbb{R}_0, \tag{2.40}$$

then

$$u(t) \leq \left\{G^{-1}\left[G\big(a(t)\big) + b(t)\int_t^{\infty} f(s)\,ds\right]\right\}^{\frac{1}{p}}, \quad t \in \mathbb{R}_0. \tag{2.41}$$

Corollary 2.5 *Suppose $\mathbb{T} = \mathbb{Z}$, $n_0 \in \mathbb{Z}$, and $\mathbb{Z}_0 = [n_0,\infty) \cap \mathbb{Z}$. $u, a, b, f \in (\mathbb{Z}_0, \mathbb{R}^+)$, and a, b are decreasing on $\mathbb{Z}_0$. $\tau \in (\mathbb{Z}_0, \mathbb{Z})$, $\tau(n) \geq n$, ω is the same as in Theorem 2.2. If $u(n)$ satisfies*

$$u^p(n) \leq a(n) + b(n)\sum_{s=n+1}^{\infty} f(s)\omega\big(u\big(\tau(s)\big)\big), \quad n \in \mathbb{Z}_0, \tag{2.42}$$

then

$$u(n) \leq \left\{G^{-1}\left[G\big(a(n)\big) + b(n)\sum_{s=n+1}^{\infty} f(s)\right]\right\}^{\frac{1}{p}}, \quad n \in \mathbb{Z}_0. \tag{2.43}$$

In Theorem 2.2, if we change the conditions for ω, then we can obtain another bound for the function $u(t)$, which is shown in the following theorem.

Theorem 2.3 *Suppose $u, a, b, f \in C_{\mathrm{rd}}(\mathbb{T}_0, \mathbb{R}_+)$ with a, b decreasing, $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing, subadditive and submultiplicative, that is, $\forall \alpha \geq 0$, $\beta \geq 0$, we always have $\omega(\alpha + \beta) \leq \omega(\alpha) + \omega(\beta)$ and $\omega(\alpha\beta) \leq \omega(\alpha)\omega(\beta)$. τ, α, ϕ are the same as in Theorem 2.2. If $\widetilde{G}$ is a strictly increasing bijective function, and $u(t)$ satisfies inequality (2.28), then*

$$u(t) \leq \left\{a(t) + b(t)\widetilde{G}^{-1}\left[\widetilde{G}\big(A(t)\big) + \int_t^\infty f(s)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}} b(s)\right)\nabla s\right]\right\}^{\frac{1}{p}}, \quad t \in \mathbb{T}_0, \tag{2.44}$$

where $K > 0$ is an arbitrary constant, $\widetilde{G}$ is an increasing bijective function, and

$$\begin{cases} \widetilde{G}(v) = \int_1^v \frac{1}{\omega(r)}\,dr, \quad v > 0 \quad \text{with} \quad \widetilde{G}(\infty) = \infty, \\ A(t) = \int_t^\infty f(s)\omega(\frac{1}{p}K^{\frac{1-p}{p}} a(s) + \frac{p-1}{p}K^{\frac{1}{p}})\nabla s. \end{cases} \tag{2.45}$$

Proof Let

$$v(t) = \int_t^\infty f(s)\omega\big(u\big(\tau(s)\big)\big)\nabla s, \quad t \in \mathbb{T}_0. \tag{2.46}$$

Then

$$u(t) \leq \big(a(t) + b(t)v(t)\big)^{\frac{1}{p}}, \quad t \in \mathbb{T}_0, \tag{2.47}$$

and

$$u\big(\tau(t)\big) \leq \big(a\big(\tau(t)\big) + b\big(\tau(t)\big)v\big(\tau(t)\big)\big)^{\frac{1}{p}} \leq \big(a(t) + b(t)v(t)\big)^{\frac{1}{p}}, \quad t \in \mathbb{T}_0. \tag{2.48}$$

Considering ω is nondecreasing, subadditive and submultiplicative, combining (2.46), (2.48) and Lemma 2.3, we obtain

$$\begin{aligned} v(t) &\leq \int_t^\infty f(s)\omega\big(\big(a(s) + b(s)v(s)\big)^{\frac{1}{p}}\big)\nabla s \\ &\leq \int_t^\infty f(s)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}}\big(a(s) + b(s)v(s)\big) + \frac{p-1}{p}K^{\frac{1}{p}}\right)\nabla s \\ &\leq \int_t^\infty f(s)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}} a(s) + \frac{p-1}{p}K^{\frac{1}{p}}\right)\nabla s + \int_t^\infty f(s)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}} b(s)v(s)\right)\nabla s \\ &\leq \int_t^\infty f(s)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}} a(s) + \frac{p-1}{p}K^{\frac{1}{p}}\right)\nabla s + \int_t^\infty f(s)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}} b(s)\right)\omega\big(v(s)\big)\nabla s \\ &= A(t) + \int_t^\infty f(s)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}} b(s)\right)\omega\big(v(s)\big)\nabla s, \quad \forall K > 0, t \in \mathbb{T}_0, \end{aligned} \tag{2.49}$$

where $K > 0$ is a constant, and $A(t)$ is defined in (2.45).

Let T be fixed in $\mathbb{T}_0$, and $t \in [T, \infty) \cap \mathbb{T}$. Denote

$$z(t) = A(T) + \int_t^\infty f(s)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}} b(s)\right)\omega\big(v(s)\big)\nabla s. \tag{2.50}$$

Considering $A(t)$ is decreasing, we have

$$v(t) \le z(t), \quad t \in [T, \infty) \cap \mathbb{T}. \tag{2.51}$$

Furthermore,

$$\begin{aligned} z^{\nabla}(t) &= -f(t)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}}b(t)\right)\omega\big(v(t)\big) \\ &\ge -f(t)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}}b(t)\right)\omega\big(z(t)\big). \end{aligned}$$

Similar to Theorem 2.1, we have

$$\big[\widetilde{G}\big(z(t)\big)\big]^{\nabla} \ge \frac{z^{\nabla}(t)}{\omega(z(t))} \ge -f(t)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}}b(t)\right). \tag{2.52}$$

Substituting t with s in (2.52) and an integration with respect to s from t to ∞ yield

$$\widetilde{G}\big(z(\infty)\big) - \widetilde{G}\big(z(t)\big) \ge -\int_t^{\infty} f(s)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}}b(s)\right)\nabla s,$$

which is followed by

$$\begin{aligned} z(t) &\le \widetilde{G}^{-1}\left[\widetilde{G}\big(z(\infty)\big) + \int_t^{\infty} f(s)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}}b(s)\right)\nabla s\right] \\ &= \widetilde{G}^{-1}\left[\widetilde{G}\big(A(T)\big) + \int_t^{\infty} f(s)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}}b(s)\right)\nabla s\right]. \end{aligned} \tag{2.53}$$

Combining (2.47), (2.51) and (2.53), we obtain

$$u(t) \le \left\{a(t) + b(t)\widetilde{G}^{-1}\left[\widetilde{G}\big(A(T)\big) + \int_t^{\infty} f(s)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}}b(s)\right)\nabla s\right]\right\}^{\frac{1}{p}}, \quad t \in [T, \infty) \cap \mathbb{T}. \tag{2.54}$$

Setting $t = T$ in (2.54), yields

$$u(T) \le \left\{a(T) + b(T)\widetilde{G}^{-1}\left[\widetilde{G}\big(A(T)\big) + \int_T^{\infty} f(s)\omega\left(\frac{1}{p}K^{\frac{1-p}{p}}b(s)\right)\nabla s\right]\right\}^{\frac{1}{p}}. \tag{2.55}$$

Since T is selected from $\mathbb{T}_0$ arbitrarily, then substituting T with t, we can obtain the desired inequality (2.44). □

Next, we consider the following delay dynamic inequality on time scales:

$$u^p(t) \le C + \int_t^{\infty} \big[f(s)u^q\big(\tau_1(s)\big) + g(s)u^q\big(\tau_2(s)\big)\omega\big(u\big(\tau_2(s)\big)\big)\big]\nabla s, \quad t \in \mathbb{T}_0, \tag{2.56}$$

where u, f, g, τ_1, τ_2 are defined the same as in Theorem 2.1, $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing, $C > 0$ is a constant, $p \ge q > 0$ are constants.

Theorem 2.4 *If $u(t)$ satisfies* (2.56), *then*

$$u(t) \le \left\{\overline{G}^{-1}\left\{H^{-1}\left[H\left(\overline{G}(C) + \int_t^\infty f(s)\nabla s\right) + \int_t^\infty g(s)\nabla s\right]\right\}\right\}^{\frac{1}{p}}, \quad t \in \mathbb{T}_0, \tag{2.57}$$

where $\overline{G}$, H are two increasing bijective functions, and

$$\begin{aligned} &\overline{G}(v) = \int_1^v \frac{1}{r^{\frac{q}{p}}}\,dr, \quad v > 0, \\ &H(z) = \int_1^z \frac{1}{\omega((\overline{G}^{-1}(r))^{\frac{1}{p}})}\,dr, \quad z > 0 \quad \textit{with} \quad H(\infty) = \infty. \end{aligned} \tag{2.58}$$

Proof Let the right side of (2.56) be $v(t)$. Then

$$u(t) \le v^{\frac{1}{p}}(t), \quad t \in \mathbb{T}_0, \tag{2.59}$$

and

$$u(\tau_i(t)) \le v^{\frac{1}{p}}(\tau_i(t)) \le v^{\frac{1}{p}}(t), \quad i = 1, 2, t \in \mathbb{T}_0. \tag{2.60}$$

Furthermore,

$$\begin{aligned} v^\nabla(t) &= -\left[f(t)u^q(\tau_1(t)) + g(t)u^q(\tau_2(t))\omega(u(\tau_2(t)))\right] \\ &\ge -\left[f(t)v^{\frac{q}{p}}(t) + g(t)v^{\frac{q}{p}}(t)\omega(v^{\frac{1}{p}}(t))\right]. \end{aligned}$$

Then similar to Theorem 2.1, we have

$$\left[\overline{G}(v(t))\right]^\nabla \ge \frac{v^\nabla(t)}{v^{\frac{q}{p}}(t)} \ge -\left[f(t) + g(t)\omega(v^{\frac{1}{p}}(t))\right]. \tag{2.61}$$

An integration for (2.61) from t to ∞ yields

$$\overline{G}(v(\infty)) - \overline{G}(v(t)) \ge -\int_t^\infty \left[f(s) + g(s)\omega(v^{\frac{1}{p}}(s))\right]\nabla s. \tag{2.62}$$

Consider $\overline{G}$ is increasing, and $v(\infty) = C$, then (2.62) implies

$$v(t) \le \overline{G}^{-1}\left[\overline{G}(C) + \int_t^\infty \left[f(s) + g(s)\omega(v^{\frac{1}{p}}(s))\right]\nabla s\right]. \tag{2.63}$$

Given a fixed number T in $\mathbb{T}_0$, and $t \in [T, \infty) \cap \mathbb{T}$. Denote

$$z(t) = \overline{G}(C) + \int_T^\infty f(s)\nabla s + \int_t^\infty g(s)\omega(v^{\frac{1}{p}}(s))\nabla s. \tag{2.64}$$

Then

$$v(t) \le \overline{G}^{-1}(z(t)), \quad t \in [T, \infty) \cap \mathbb{T}, \tag{2.65}$$

and furthermore,

$$\begin{aligned} z^{\nabla}(t) &= -g(t)\omega\bigl(v^{\frac{1}{p}}(t)\bigr) \\ &\geq -g(t)\omega\bigl(\bigl(\overline{G}^{-1}\bigl(z(t)\bigr)\bigr)^{\frac{1}{p}}\bigr), \end{aligned}$$

which is followed by

$$\bigl[H\bigl(z(t)\bigr)\bigr]^{\nabla} \geq \frac{z^{\nabla}(t)}{\omega((\overline{G}^{-1}(z(t)))^{\frac{1}{p}})} \geq -g(t). \tag{2.66}$$

Integrating (2.66) from t to ∞ yields

$$H\bigl(z(\infty)\bigr) - H\bigl(z(t)\bigr) \geq -\int_t^{\infty} g(s)\nabla s. \tag{2.67}$$

Consider H is increasing and $z(\infty) = \overline{G}(C) + \int_T^{\infty} f(s)\nabla s$, then combining (2.59), (2.65) and (2.67), we have

$$u(t) \leq \left\{\overline{G}^{-1}\left\{H^{-1}\left[H\left(\overline{G}(C) + \int_T^{\infty} f(s)\nabla s\right) + \int_t^{\infty} g(s)\nabla s\right]\right\}\right\}^{\frac{1}{p}}, \quad t \in [T,\infty) \cap \mathbb{T}. \tag{2.68}$$

Setting $t = T$ in (2.68) and considering T is an arbitrary number in $\mathbb{T}_0$, we can obtain the desired inequality after substituting T with t. □

Finally, we study the following delay dynamic inequality on time scales:

$$\begin{aligned} \eta\bigl(u(t)\bigr) \leq a(t) + b(t)\int_t^{\infty}\Bigl[f(s)\omega\bigl(u\bigl(\tau_1(s)\bigr)\bigr) \\ + g(s)\int_s^{\infty} h(\xi)\omega\bigl(u\bigl(\tau_2(\xi)\bigr)\bigr)\nabla\xi\Bigr]\nabla s, \quad t \in \mathbb{T}_0, \end{aligned} \tag{2.69}$$

where $u, a, b, f, g, h, \alpha, \tau_i, i = 1, 2$ are defined the same as in Theorem 2.1, b is defined as in Theorem 2.2, $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing, $\eta \in C_{\mathrm{rd}}([\alpha, t_0] \cap \mathbb{T}, \mathbb{R}_+)$ is strictly increasing.

Theorem 2.5 *If for $t \in \mathbb{T}_0$, $u(t)$ satisfies inequality* (2.69), *then*

$$u(t) \leq \eta^{-1}\left\{\widehat{G}^{-1}\left\{\widehat{G}\left(a(t) + b(t)\int_t^{\infty}\left[f(s) + g(s)\int_s^{\infty} h(\xi)\nabla\xi\right]\nabla s\right)\right\}\right\}, \quad t \in \mathbb{T}_0, \tag{2.70}$$

where $\widehat{G}$ is an increasing bijective function, and

$$\widehat{G}(v) = \int_1^{v} \frac{1}{\omega(\eta^{-1}(r))}\,dr, \quad v > 0 \quad \textit{with} \quad \widehat{G}(\infty) = \infty.$$

Proof Let the right side of (2.69) be $v(t)$. Then

$$\eta\bigl(u(t)\bigr) \leq v(t), \quad t \in \mathbb{T}_0, \tag{2.71}$$

and

$$\eta\big(u\big(\tau_i(t)\big)\big) \le \nu\big(\tau_i(t)\big) \le \nu(t), \quad i = 1, 2\ \forall t \in \mathbb{T}_0. \tag{2.72}$$

Furthermore, considering η is increasing, we have

$$\nu(t) \le a(t) + b(t)\int_t^\infty \left[f(s)\omega\big(\eta^{-1}\big(\nu(s)\big)\big) + g(s)\int_s^\infty h(\xi)\omega\big(\eta^{-1}\big(\nu(\xi)\big)\big)\nabla\xi\right]\nabla s, \quad t \in \mathbb{T}_0. \tag{2.73}$$

Fix $T \in \mathbb{T}_0$, and let $t \in [T, \infty) \cap \mathbb{T}$. Define

$$c(t) = a(T) + b(T)\int_t^\infty \left[f(s)\omega\big(\eta^{-1}\big(\nu(s)\big)\big) + g(s)\int_s^\infty h(\xi)\omega\big(\eta^{-1}\big(\nu(\xi)\big)\big)\nabla\xi\right]\nabla s. \tag{2.74}$$

Consider a, b are decreasing on $\mathbb{T}_0$, then it follows

$$\nu(t) \le c(t), \quad t \in [T, \infty) \cap \mathbb{T}. \tag{2.75}$$

On the other hand,

$$\begin{aligned} c^\nabla(t) &= -b(T)\left[f(t)\omega\big(\eta^{-1}\big(\nu(t)\big)\big) + g(t)\int_t^\infty h(\xi)\omega\big(\eta^{-1}\big(\nu(\xi)\big)\big)\nabla\xi\right] \\ &\ge -b(T)\left[f(t)\omega\big(\eta^{-1}\big(c(t)\big)\big) + g(t)\int_t^\infty h(\xi)\omega\big(\eta^{-1}\big(c(\xi)\big)\big)\nabla\xi\right] \\ &\ge -b(T)\left[f(t) + g(t)\int_t^\infty h(\xi)\nabla\xi\right]\omega\big(\eta^{-1}\big(c(t)\big)\big). \end{aligned}$$

Similar to Theorem 2.1, we have

$$\big[\widehat{G}\big(c(t)\big)\big]^\nabla \ge \frac{c^\nabla(t)}{\omega(\eta^{-1}(c(t)))} \ge -b(T)\left[f(t) + g(t)\int_t^\infty h(\xi)\nabla\xi\right]. \tag{2.76}$$

Substituting t with s in (2.76) and an integration for (2.76) with respect to s from t to ∞ yield

$$\widehat{G}\big(c(\infty)\big) - \widehat{G}\big(c(t)\big) \ge -b(T)\int_t^\infty \left[f(s) + g(s)\int_s^\infty h(\xi)\nabla\xi\right]\nabla s. \tag{2.77}$$

Since $c(\infty) = a(T)$, and $\widehat{G}$ is strictly increasing, then it follows

$$c(t) \le \widehat{G}^{-1}\left\{\widehat{G}\big(a(T)\big) + b(T)\int_t^\infty \left[f(s) + g(s)\int_s^\infty h(\xi)\nabla\xi\right]\nabla s\right\}. \tag{2.78}$$

Combining (2.71), (2.75) and (2.78), we have

$$u(t) \le \eta^{-1}\left\{\widehat{G}^{-1}\left\{\widehat{G}\big(a(T)\big) + b(T)\int_t^\infty \left[f(s) + g(s)\int_s^\infty h(\xi)\nabla\xi\right]\nabla s\right\}\right\}, \\ t \in [T, \infty) \cap \mathbb{T}. \tag{2.79}$$

Setting $t = T$ in (2.79), we can obtain

$$u(T) \le \eta^{-1}\left\{\widehat{G}^{-1}\left\{\widehat{G}\big(a(T)\big) + b(T)\int_T^\infty \left[f(s) + g(s)\int_s^\infty h(\xi)\nabla\xi\right]\nabla s\right\}\right\}. \tag{2.80}$$

Since $T \in \mathbb{T}_0$ is selected arbitrarily, then substituting T with t in (2.80), we can obtain the desired inequality (2.70). □

Remark 2.1 If we take $\mathbb{T}$ for some peculiar cases in Theorems 2.3, 2.4 and 2.5 such as $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, we can obtain some corollaries respectively, which are omitted here due to the limited space.

3 Some simple applications

In this section, we will present some applications for the established results above. Some new bounds for the solutions of certain delay dynamic equations on time scales will be derived in the following examples.

Example 1 Consider the delay dynamic integral equation on time scales

$$u^p(t) = C + \int_t^\infty F\big(s, u\big(\tau(s)\big)\big)\nabla s, \quad t \in \mathbb{T}_0, \tag{3.1}$$

where $u \in C_{\mathrm{rd}}(\mathbb{T}_0, \mathbb{R})$, $C = u^p(t_0)$, p is a positive number with $p \ge 1$, τ is defined as in Theorem 2.1.

Theorem 3.1 *Suppose $u(t)$ is a solution of* (3.1) *and assume $|F(t,u)| \le f(t)|u|$, where $f \in C_{\mathrm{rd}}(\mathbb{T}_0, \mathbb{R}_+)$, then we have*

$$\big|u(t)\big| \le \left\{G^{-1}\left[G\big(|C|\big) + \int_t^\infty f(s)\nabla s\right]\right\}^{\frac{1}{p}}, \quad t \in \mathbb{T}_0, \tag{3.2}$$

where

$$G(v) = \int_1^v \frac{1}{r^{\frac{1}{p}}}\,dr, \quad v > 0. \tag{3.3}$$

Proof From (3.1), we obtain

$$\big|u(t)\big|^p \le |C| + \int_t^\infty \big|F\big(s, u\big(\tau(s)\big)\big)\big|\nabla s \le |C| + \int_t^\infty f(s)\big|u\big(\tau(s)\big)\big|\nabla s. \tag{3.4}$$

Let $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$, and $\omega(v) = v$. Then (3.4) can be rewritten as

$$\big|u(t)\big|^p \le |C| + \int_t^\infty f(s)\omega\big(\big|u\big(\tau(s)\big)\big|\big)\nabla s. \tag{3.5}$$

A suitable application of Theorem 2.2 to (3.5) yields the desired inequality. □

Remark 3.1 In the proof of Theorem 3.1, if we apply Theorem 2.3 instead of Theorem 2.2 to (3.5), then we obtain another bound for $u(t)$ as follows:

$$\bigl|u(t)\bigr| \le \left\{|C| + A(t)\exp\left(\int_t^\infty f(s)\frac{1}{p}K^{\frac{1-p}{p}}\nabla s\right)\right\}^{\frac{1}{p}}, \quad t \in \mathbb{T}_0, \tag{3.6}$$

where $K > 0$ is an arbitrary constant, and

$$\begin{cases} \widetilde{G}(v) = \int_1^v \frac{1}{r}\,dr = \ln v, \quad v > 0, \\ A(t) = \int_t^\infty f(s)(\frac{1}{p}K^{\frac{1-p}{p}}|C| + \frac{p-1}{p}K^{\frac{1}{p}})\nabla s. \end{cases} \tag{3.7}$$

Example 2 Consider the following delay dynamic differential equation on time scales:

$$\bigl(u^p(t)\bigr)^\nabla = F\left(t, u\bigl(\tau_1(t)\bigr), \int_t^\infty M\bigl(\xi, u\bigl(\tau_2(\xi)\bigr)\bigr)\nabla\xi\right), \quad t \in \mathbb{T}_0, \tag{3.8}$$

where $u \in C_{\mathrm{rd}}(\mathbb{T}_0, \mathbb{R})$, $C = u^p(\infty)$, p is a positive number with $p \ge 1$, τ_i, $i = 1, 2$ are defined as in Theorem 2.5.

Theorem 3.2 *Suppose* $u(t)$ *is a solution of* (3.8), *and assume* $|F(t,u,v)| \le f(t)|u| + |v|$, $|M(t,u)| \le h(t)|u|$, *where* $f, h \in C_{\mathrm{rd}}(\mathbb{T}_0, \mathbb{R}_+)$, *then we have*

$$\bigl|u(t)\bigr| \le \left\{G^{-1}\left\{G\bigl(|C|\bigr) + \int_t^\infty \left[f(s) + \int_s^\infty h(\xi)\nabla\xi\right]\nabla s\right\}\right\}^{\frac{1}{p}}, \quad t \in \mathbb{T}_0, \tag{3.9}$$

where G *is defined as in Theorem* 3.1.

Proof The equivalent integral form of (3.8) can be denoted by

$$u^p(t) = C + \int_t^\infty F\left(s, u\bigl(\tau_1(s)\bigr), \int_s^\infty M\bigl(\xi, u\bigl(\tau_2(\xi)\bigr)\bigr)\nabla\xi\right)\nabla s. \tag{3.10}$$

Then

$$\begin{aligned} \bigl|u(t)\bigr|^p &= \left|C + \int_t^\infty F\left(s, u\bigl(\tau_1(s)\bigr), \int_s^\infty M\bigl(\xi, u\bigl(\tau_2(\xi)\bigr)\bigr)\nabla\xi\right)\nabla s\right| \\ &\le |C| + \left|\int_t^\infty F\left(s, u\bigl(\tau_1(s)\bigr), \int_s^\infty M\bigl(\xi, u\bigl(\tau_2(\xi)\bigr)\bigr)\nabla\xi\right)\nabla s\right| \\ &\le |C| + \int_t^\infty \left|F\left(s, u\bigl(\tau_1(s)\bigr), \int_s^\infty M\bigl(\xi, u\bigl(\tau_2(\xi)\bigr)\bigr)\nabla\xi\right)\right|\nabla s \\ &\le |C| + \int_t^\infty \left[f(s)\bigl|u\bigl(\tau_1(s)\bigr)\bigr| + \left|\int_s^\infty M\bigl(\xi, u\bigl(\tau_2(\xi)\bigr)\bigr)\nabla\xi\right|\right]\nabla s \\ &\le |C| + \int_t^\infty \left[f(s)\bigl|u\bigl(\tau_1(s)\bigr)\bigr| + \int_s^\infty \bigl|M\bigl(\xi, u\bigl(\tau_2(\xi)\bigr)\bigr)\bigr|\nabla\xi\right]\nabla s \\ &\le |C| + \int_t^\infty \left[f(s)\bigl|u\bigl(\tau_1(s)\bigr)\bigr| + \int_s^\infty h(\xi)\bigl|u\bigl(\tau_2(\xi)\bigr)\bigr|\nabla\xi\right]\nabla s \\ &= |C| + \int_t^\infty \left[f(s)\omega\bigl(\bigl|u\bigl(\tau_1(s)\bigr)\bigr|\bigr) + \int_s^\infty h(\xi)\omega\bigl(\bigl|u\bigl(\tau_2(\xi)\bigr)\bigr|\bigr)\nabla\xi\right]\nabla s, \end{aligned} \tag{3.11}$$

where $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$, and $\omega(u) = u$.

A suitable application of Theorem 2.5 to (3.11) yields the desired inequality. □

4 Conclusions

In this paper, some new Gronwall-Bellman type nonlinear dynamic inequalities containing integration at infinite intervals on time scales have been established. As one can see through the present examples, the established results are useful in dealing with the boundedness of solutions of certain dynamic equations on time scales. Finally, we note that the process of Theorems 2.1-2.5 can be applied to establish delay dynamic inequalities with two independent variables on time scales.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
QF carried out the main part of this article. All authors read and approved the final manuscript.

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