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A supplement to the convergence rate in a theorem of Heyde

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Abstract

Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with zero mean, set $S_n = \sum_{k=1}^{n} X_k$, $EX^2 = \sigma^2 > 0$, and $\lambda(\epsilon) = \sum_{n=1}^{\infty} P(|S_n| \geq n\epsilon)$. In this paper, the authors discuss the rate of approximation of σ^2 by $\epsilon^2\lambda(\epsilon)$ under suitable conditions, improve the results of Klesov (Theory Probab. Math. Stat. 49:83-87, 1994), and extend the work He and Xie (Acta Math. Appl. Sin. 2012, doi:10.1007/s10255-012-0138-6).

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1 Introduction and main results

Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables, set $S_n = \sum_{k=1}^{n} X_k$, and $\lambda(\epsilon) = \sum_{n=1}^{\infty} P(|S_n| \geq n\epsilon)$. Heyde [1] proved that

$$\lim_{\epsilon \to 0} \epsilon^2 \lambda(\epsilon) = \sigma^2,$$

whenever $EX^2 = \sigma^2 < \infty$ and $EX = 0$.

There are various extensions of this result: Chen [2], Gut and Spătara [3], Lanzinger and Stadtmüller [4]. Liu and Lin [5] introduced a new kind of complete moment convergence; Klesov [6] studied the rate of approximation of σ^2 by $\epsilon^2\lambda(\epsilon)$ and proved the following Theorem A.

Theorem A *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with zero mean, if $EX^2 = \sigma^2 > 0$, and $E|X|^3 < \infty$, then*

$$\epsilon^2 \lambda(\epsilon) - \sigma^2 = o\left(\epsilon^{1/2}\right), \quad \text{as } \epsilon \to 0.$$

Recently, He and Xie [7] obtained Theorem B which improved Theorem A. Gut and Steinebach [8] extended the results of Klesov [6].

Theorem B *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables, and $0 < \delta \leq 1$, if*

$$EX = 0, \qquad EX^2 = \sigma^2 > 0 \quad \text{and} \quad E|X|^{2+\delta} < \infty,$$

then

$$\epsilon^2\lambda(\epsilon) - \sigma^2 = \begin{cases} O(\epsilon), & \delta = 1, \\ o(\epsilon^\delta), & 0 < \delta < 1. \end{cases}$$

Let G be the set of functions $g(x)$ that are defined for all real x and satisfy the following conditions: (a) $g(x)$ is nonnegative, even, nondecreasing in the interval $x > 0$, and $g(x) \neq 0$ for $x \neq 0$; (b) $\frac{x}{g(x)}$ is nondecreasing in the interval $x > 0$.

Let G_0 be the set of functions $g(x) \in G$ satisfying the supplementary condition (c) $\lim_{x\to\infty} \frac{g(x^2)}{xg(x)} = 0$. Obviously, the function $g(x) = |x|^\delta$ with $0 < \delta < 1$ belongs to G_0 and does not belong to G_0 if $\delta = 1$. The purpose of this paper is to generalize Theorem B to the case where the condition $E|X|^{2+\delta} < \infty$ is replaced by a more general condition $E|X|^2 g(X) < \infty$ in which the function g belongs to some subset of G. Denote $T_g(\nu) = EX^2 g(X)I(|X| > \nu)$, $T_g(\nu)$ is a nonnegative nonincreasing function in the interval $\nu > 0$, and $\lim_{\nu\to\infty} T_g(\nu) = 0$ with $EX^2 g(X) < \infty$. Now we state our results as follows.

Theorem 1.1 *Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. random variables with zero mean and $EX^2 = \sigma^2 > 0$, if $EX^2 g(X) < \infty$ for some function $g(x) \in G$, and*

$$\sum_{n=1}^{\infty} \frac{1}{ng(\sqrt{n})} < \infty, \tag{1.1}$$

then

$$\epsilon^2\lambda(\epsilon) - \sigma^2 = O(\epsilon^{1/2}) + o(1)\big(h_1(\epsilon) + f_1(\epsilon)\big), \quad \text{as } \epsilon \to 0, \tag{1.2}$$

where $f_1(\epsilon) = \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{ng(\sqrt{n})}$, $h_1(\epsilon) = \epsilon^2 \sum_{n=1}^{[\frac{1}{\epsilon^2}]} \frac{1}{g(\sqrt{n})}$.

Theorem 1.2 *Under the conditions of Theorem 1.1, and $g(x) \in G_0$, then*

$$\epsilon^2\lambda(\epsilon) - \sigma^2 = o(1)\big(h_1(\epsilon) + f_1(\epsilon)\big), \quad \text{as } \epsilon \to 0. \tag{1.3}$$

Throughout this paper, we suppose that C denotes a constant which only depends on some given numbers and may be different at each appearance, and that $[x]$ denotes the integer part of x.

2 Proofs of the main results

Before we prove the main results we state some lemmas. Lemma 2.1 is from [7]. $\Phi(x)$ is the standard normal distribution function, $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\,dt$.

Lemma 2.1 *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. standard normal distribution random variables. Then*

$$\epsilon^2\lambda(\epsilon) = \epsilon^2 \sum_{n=1}^{\infty} \frac{2}{\sqrt{2\pi}} \int_{\epsilon\sqrt{n}}^{\infty} e^{-t^2/2}\,dt = 1 - \frac{\epsilon^2}{2} + O(\epsilon^3), \quad \text{as } \epsilon \to 0. \tag{2.1}$$

If $\{X_n, n \geq 1\}$ is a sequence of independent random variables with zero mean and finite variance, and put $EX_j^2 = \sigma_j^2$, $B_n = \sum_{j=1}^n \sigma_j^2$, Bikelis [9] obtained the following inequality:

$$\begin{aligned}&\left|P\left(\frac{1}{\sqrt{B_n}}\sum_{j=1}^n X_j < x\right) - \Phi(x)\right|\\ &\quad\leq C\left\{B_n^{-1}\big(1+|x|\big)^{-2}\sum_{j=1}^n \int_{|u|>(1+|x|)B_n^{1/2}} u^2\,dV_j(u)\right.\\ &\qquad\left. + B_n^{-3/2}\big(1+|x|\big)^{-3}\sum_{j=1}^n \int_{|u|\leq(1+|x|)B_n^{1/2}} |u|^3\,dV_j(u)\right\},\end{aligned}$$

for every x, where $V_j(x) = P(X_j < x)$ is the distribution function of the random variable X_j. By applying the above inequality to the sequence of i.i.d. random variables with zero mean and variance 1, and letting $|x| = \epsilon\sqrt{n}$, we have the following lemma.

Lemma 2.2 *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with zero mean and $EX^2 = 1$. Then for any given $\epsilon > 0$, we have*

$$\begin{aligned}&\left|P\big(|S_n| > n\epsilon\big) - \frac{2}{\sqrt{2\pi}}\int_{\epsilon\sqrt{n}}^{\infty} e^{-t^2/2}\,dt\right|\\ &\quad\leq C(1+\epsilon\sqrt{n})^{-2}\int_{|u|>(1+\epsilon\sqrt{n})\sqrt{n}} u^2\,dV(u)\\ &\qquad + Cn^{-1/2}(1+\epsilon\sqrt{n})^{-3}\int_{|u|\leq(1+\epsilon\sqrt{n})\sqrt{n}} |u|^3\,dV(u),\end{aligned}$$

where $V(x) = P(X < x)$ is the distribution function of a random variable X.

Proof of Theorem 1.1 Without loss of generality, we suppose that $\sigma^2 = 1$, $0 < \epsilon < 1$, and write

$$\epsilon^2\lambda(\epsilon) = I + \epsilon^2\sum_{n=1}^{\infty}\frac{2}{\sqrt{2\pi}}\int_{\epsilon\sqrt{n}}^{\infty} e^{-t^2/2}\,dt,$$

where

$$I = \epsilon^2\sum_{n=1}^{\infty}\left(P\big(|S_n| > n\epsilon\big) - \frac{2}{\sqrt{2\pi}}\int_{\epsilon\sqrt{n}}^{\infty} e^{-t^2/2}\,dt\right).$$

Applying Lemma 2.1, we obtain

$$\epsilon^2\lambda(\epsilon) = I + 1 - \frac{\epsilon^2}{2} + O\big(\epsilon^3\big),$$

then

$$\epsilon^2\lambda(\epsilon) - 1 = -\frac{\epsilon^2}{2} + \epsilon^2\sum_{n=1}^{\infty} R_n + O\big(\epsilon^3\big),$$

here $R_n = P(|S_n| > n\epsilon) - \frac{2}{\sqrt{2\pi}} \int_{\epsilon\sqrt{n}}^{\infty} e^{-t^2/2}\,dt$. By Lemma 2.2,

$$|R_n| \le R_{1n} + R_{2n},$$

where

$$R_{1n} = C(1+\epsilon\sqrt{n})^{-2} \int_{|u|>(1+\epsilon\sqrt{n})\sqrt{n}} u^2\,dV(u),$$
$$R_{2n} = Cn^{-1/2}(1+\epsilon\sqrt{n})^{-3} \int_{|u|\le(1+\epsilon\sqrt{n})\sqrt{n}} |u|^3\,dV(u).$$

We obtain

$$\epsilon^2\lambda(\epsilon) - 1 = \epsilon^2 \sum_{n=1}^{\infty} R_{1n} + \epsilon^2 \sum_{n=1}^{\infty} R_{2n} + O(\epsilon^2). \tag{2.2}$$

Firstly, we estimate $\epsilon^2 \sum_{n=1}^{\infty} R_{1n}$. Note that

$$\epsilon^2 \sum_{n=1}^{\infty} R_{1n} = \epsilon^2 \sum_{n=1}^{[\frac{1}{\epsilon^2}]} R_{1n} + \epsilon^2 \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} R_{1n} =: T_1 + T_2.$$

Applying the condition $EX^2g(X) < \infty$, we have

$$\lim_{n\to\infty} \int_{|u|>\sqrt[4]{n}} u^2 g(u)\,dV(u) = 0.$$

Therefore, for any $\eta > 0$, there is an integer N_0 such that $\int_{|u|>\sqrt[4]{n}} u^2 g(u)\,dV(u) \le \eta$, whenever $n > N_0$. Hence

$$\begin{aligned}
T_1 &\le C\epsilon^2 \sum_{n=1}^{N_0} \int_{|u|>\sqrt{n}} u^2\,dV(u) + C\epsilon^2 \sum_{n=N_0+1}^{[\frac{1}{\epsilon^2}]} (1+\epsilon\sqrt{n})^{-2} \int_{|u|>(1+\epsilon\sqrt{n})\sqrt{n}} u^2\,dV(u) \\
&\le C\epsilon^2 N_0 + C\epsilon^2\eta \sum_{n=N_0+1}^{[\frac{1}{\epsilon^2}]} \frac{1}{(1+\epsilon\sqrt{n})^2 g(\sqrt{n}(1+\epsilon\sqrt{n}))} \\
&\le C\epsilon^2 \left(N_0 + \eta \sum_{n=1}^{[\frac{1}{\epsilon^2}]} \frac{1}{g(\sqrt{n})} \right) \\
&= Ch_1(\epsilon) \left(\frac{N_0}{\sum_{n=1}^{[\frac{1}{\epsilon^2}]} \frac{1}{g(\sqrt{n})}} + \eta \right) \\
&\le Ch_1(\epsilon)(N_0\epsilon + \eta) \\
&= o(h_1(\epsilon)),
\end{aligned} \tag{2.3}$$

where $h_1(\epsilon) = \epsilon^2 \sum_{n=1}^{[\frac{1}{\epsilon^2}]} \frac{1}{g(\sqrt{n})}$. For T_2, noting that $g(x) \in G$, we have the following inequality:

$$\begin{aligned}
T_2 &\le C\epsilon^2 \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{n\epsilon^2} \int_{|u|>\sqrt{n}(1+\epsilon\sqrt{n})} u^2\, dV(u) \\
&\le C \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{ng(\sqrt{n}(1+\epsilon\sqrt{n}))} \int_{|u|>\sqrt{n}(1+\epsilon\sqrt{n})} u^2 g(u)\, dV(u) \\
&\le C \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{ng(\sqrt{n})} \int_{|u|>\frac{1}{\epsilon}} u^2 g(u)\, dV(u) \\
&\le CT_g\left(\frac{1}{\epsilon}\right) f_1(\epsilon).
\end{aligned} \tag{2.4}$$

Next, we estimate the second term of (2.2). Note that

$$\begin{aligned}
\epsilon^2 \sum_{n=1}^{\infty} R_{2n} &= C\epsilon^2 \sum_{n=1}^{\infty} n^{-1/2}(1+\epsilon\sqrt{n})^{-3} \int_{|u|\le(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}} |u|^3\, dV(u) \\
&\quad + C\epsilon^2 \sum_{n=1}^{\infty} n^{-1/2}(1+\epsilon\sqrt{n})^{-3} \int_{(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}<|u|<\sqrt{n}(1+\epsilon\sqrt{n})} |u|^3\, dV(u) \\
&=: J_1 + J_2.
\end{aligned}$$

For J_1, we can write

$$\begin{aligned}
J_1 &= C\epsilon^2 \left(\sum_{n=1}^{[\frac{1}{\epsilon^2}]} + \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \right) n^{-1/2}(1+\epsilon\sqrt{n})^{-3} \int_{|u|\le(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}} |u|^3\, dV(u) \\
&=: J_{11} + J_{12}.
\end{aligned}$$

Noting that $\frac{x}{g(x)}$ is nondecreasing in the interval $x > 0$, we have

$$\begin{aligned}
J_{11} &= C\epsilon^2 \sum_{n=1}^{[\frac{1}{\epsilon^2}]} \frac{1}{\sqrt{n}(1+\epsilon\sqrt{n})^3} \int_{|u|\le(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}} |u|^3\, dV(u) \\
&\le C\epsilon^2 \sum_{n=1}^{[\frac{1}{\epsilon^2}]} \frac{1}{n^{1/4}(1+\epsilon\sqrt{n})^{5/2} g((\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2})} \int_{|u|\le(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}} u^2 g(u)\, dV(u) \\
&\le C\epsilon^2 \sum_{n=1}^{[\frac{1}{\epsilon^2}]} \frac{1}{n^{1/4} g(n^{1/4})} \\
&= Ch_2(\epsilon),
\end{aligned} \tag{2.5}$$

where $h_2(\epsilon) = \epsilon^2 \sum_{n=1}^{[\frac{1}{\epsilon^2}]} \frac{1}{n^{1/4} g(n^{1/4})}$.

Similarly, we can obtain

$$\begin{aligned}
J_{12} &= C\epsilon^2 \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{\sqrt{n}(1+\epsilon\sqrt{n})^3} \int_{|u|\le(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}} |u|^3\, dV(u) \\
&\le C\epsilon^2 \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{n^{1/4}(1+\epsilon\sqrt{n})^{5/2} g((\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2})} \int_{|u|\le(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}} u^2 g(u)\, dV(u) \\
&\le C\epsilon^2 \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{\epsilon^{5/2} n^{3/2} g(n^{1/4})} \\
&= C\frac{1}{\sqrt{\epsilon}} f_2(\epsilon), \qquad (2.6)
\end{aligned}$$

where $f_2(\epsilon) = \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{n^{3/2} g(n^{1/4})}$.

For J_2, we write

$$\begin{aligned}
J_2 &= C\epsilon^2 \left(\sum_{n=1}^{[\frac{1}{\epsilon^2}]} + \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \right) n^{-1/2}(1+\epsilon\sqrt{n})^{-3} \int_{(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}<|u|<\sqrt{n}(1+\epsilon\sqrt{n})} |u|^3\, dV(u) \\
&=: J_{21} + J_{22}.
\end{aligned}$$

Using the properties of $g(x)$ by simple calculation, it follows that

$$\begin{aligned}
J_{21} &= C\epsilon^2 \sum_{n=1}^{[\frac{1}{\epsilon^2}]} n^{-1/2}(1+\epsilon\sqrt{n})^{-3} \int_{(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}<|u|<\sqrt{n}(1+\epsilon\sqrt{n})} |u|^3\, dV(u) \\
&\le C\epsilon^2 \left(\sum_{n=1}^{N_0} + \sum_{n=N_0+1}^{[\frac{1}{\epsilon^2}]} \right) \frac{1}{(1+\epsilon\sqrt{n})^2 g(\sqrt{n}(1+\epsilon\sqrt{n}))} \\
&\quad \times \int_{(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}<|u|<\sqrt{n}(1+\epsilon\sqrt{n})} u^2 g(u)\, dV(u) \\
&\le C\epsilon^2 \left(\sum_{n=1}^{N_0} + \sum_{n=N_0+1}^{[\frac{1}{\epsilon^2}]} \right) \frac{1}{g(\sqrt{n})} \int_{|u|>n^{1/4}} u^2 g(u)\, dV(u) \\
&\le C\epsilon^2 \left(N_0 + \eta \sum_{n=1}^{[\frac{1}{\epsilon^2}]} \frac{1}{g(\sqrt{n})} \right) \\
&= o\big(h_1(\epsilon)\big), \qquad (2.7)
\end{aligned}$$

and

$$\begin{aligned}
J_{22} &\le C\epsilon^2 \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} n^{-\frac{1}{2}}(1+\epsilon\sqrt{n})^{-3} \int_{(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}<|u|<\sqrt{n}(1+\epsilon\sqrt{n})} |u|^3\, dV(u) \\
&\le C \sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty} \frac{1}{ng(\sqrt{n})} \int_{(\sqrt{n}(1+\epsilon\sqrt{n}))^{1/2}<|u|<\sqrt{n}(1+\epsilon\sqrt{n})} u^2 g(u)\, dV(u)
\end{aligned}$$

$$\leq CT_g\left(\frac{1}{\sqrt{\epsilon}}\right)\sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty}\frac{1}{ng(\sqrt{n})}$$

$$\leq CT_g\left(\frac{1}{\sqrt{\epsilon}}\right)f_1(\epsilon). \tag{2.8}$$

From (2.2) to (2.8), we conclude that

$$\epsilon^2\lambda(\epsilon)-1\leq C\frac{1}{\sqrt{\epsilon}}f_2(\epsilon)+CT_g\left(\frac{1}{\sqrt{\epsilon}}\right)f_1(\epsilon)+o(1)h_1(\epsilon)+Ch_2(\epsilon). \tag{2.9}$$

Since

$$\frac{1}{\sqrt{\epsilon}}f_2(\epsilon)\leq\frac{C}{\sqrt{\epsilon}}\sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty}\frac{1}{n^{3/2}}\leq C\sqrt{\epsilon},$$

and

$$h_2(\epsilon)=\epsilon^2\sum_{n=1}^{[\frac{1}{\epsilon^2}]}\frac{1}{\sqrt[4]{n}g(\sqrt[4]{n})}\leq C\epsilon^2\sum_{n=1}^{[\frac{1}{\epsilon^2}]}\frac{1}{\sqrt[4]{n}}\leq C\sqrt{\epsilon},$$

by (2.9), we have

$$\epsilon^2\lambda(\epsilon)-1=O\bigl(\epsilon^{1/2}\bigr)+o(1)\bigl(f_1(\epsilon)+h_1(\epsilon)\bigr).$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 By the conditions $g(x)\in G_0$, and $\lim_{x\to\infty}\frac{g(x^2)}{xg(x)}=0$, for any $\eta>0$, there is an integer N_1 such that $\frac{g(\sqrt{n})}{\sqrt[4]{n}g(\sqrt[4]{n})}\leq\eta$, whenever $n>N_1$. We have

$$h_2(\epsilon)\leq\epsilon^2\sum_{n=1}^{N_1}\frac{1}{\sqrt[4]{n}g(\sqrt[4]{n})}+\epsilon^2\sum_{n=N_1}^{[\frac{1}{\epsilon^2}]}\frac{\eta}{g(\sqrt{n})}$$

$$\leq C\epsilon^2N_1+\epsilon^2\sum_{n=N_1+1}^{[\frac{1}{\epsilon^2}]}\frac{\eta}{g(\sqrt{n})}$$

$$\leq C\epsilon^2N_1+\epsilon^2\sum_{n=1}^{[\frac{1}{\epsilon^2}]}\frac{\eta}{g(\sqrt{n})}$$

$$=o(1)h_1(\epsilon), \tag{2.10}$$

and

$$\frac{1}{\sqrt{\epsilon}}f_2(\epsilon)\leq\frac{1}{\sqrt{\epsilon}}\sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty}\frac{\eta}{n^{5/4}g(\sqrt{n})}$$

$$\leq\sum_{n=[\frac{1}{\epsilon^2}]+1}^{\infty}\frac{\eta}{ng(\sqrt{n})}=o(1)f_1(\epsilon). \tag{2.11}$$

By (2.9)-(2.11), note that $T_g(\frac{1}{\sqrt{\epsilon}}) = o(1)$, as $\epsilon \to 0$, we have

$$\epsilon^2\lambda(\epsilon) - \sigma^2 = o(1)\big(h_1(\epsilon) + f_1(\epsilon)\big), \quad \text{as } \epsilon \to 0.$$

This completes the proof of Theorem 1.2. □

Remark 2.1 If $g(x) = |x|^\delta$, $0 < \delta < 1$, then $f_1(\epsilon) = O(\epsilon^\delta)$, $h_1(\epsilon) = O(\epsilon^\delta)$. By Theorem 1.2, we get

$$\epsilon^2\lambda(\epsilon) - \sigma^2 = o\big(\epsilon^\delta\big), \quad \text{as } \epsilon \to 0.$$

Remark 2.2 If $g(x) = |x|$, $\delta = 1$, then $\frac{1}{\sqrt{\epsilon}}f_2(\epsilon) = O(\epsilon)$, $f_1(\epsilon) = O(\epsilon)$, $h_1(\epsilon) = O(\epsilon)$, $h_2(\epsilon) = O(\epsilon)$. By (2.9), we get

$$\epsilon^2\lambda(\epsilon) - \sigma^2 = O(\epsilon), \quad \text{as } \epsilon \to 0.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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