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Natural metrics and composition operators in generalized hyperbolic function spaces

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Abstract

In this paper, we define some generalized hyperbolic function classes. We also introduce natural metrics in the generalized hyperbolic (p,α)-Bloch and in the generalized hyperbolic $Q^*(p,s)$ classes. These classes are shown to be complete metric spaces with respect to the corresponding metrics. Moreover, boundedness and compactness the composition operators C_ϕ acting from the generalized hyperbolic (p,α)-Bloch class to the class $Q^*(p,s)$ are characterized by conditions depending on an analytic self-map $\phi:\mathbb{D}\to\mathbb{D}$.

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1 Introduction

Let $\mathbb{D}=\{z:|z|<1\}$ be the open unit disc of the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary. Let $\mathcal{H}(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$ and let $B(\mathbb{D})$ be the subset of $\mathcal{H}(\mathbb{D})$ consisting of those $f\in\mathcal{H}(\mathbb{D})$ for which $|f(z)|<1$ for all $z\in\mathbb{D}$. Also, $dA(z)$ be the normalized area measure on $\mathbb{D}$ so that $A(\mathbb{D})\equiv 1$. The usual α-Bloch spaces $\mathcal{B}_\alpha$ and $\mathcal{B}_{\alpha,0}$ are defined as the sets of those $f\in\mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}_\alpha}=\sup_{z\in\mathbb{D}}\left|f'(z)\right|\left(1-|z|^2\right)^\alpha<\infty,$$

and

$$\lim_{|z|\to 1}\left|f'(z)\right|\left(1-|z|^2\right)^\alpha=0,$$

respectively. Now, we will give the following definition:

Definition 1.1 The (p,α)-Bloch spaces $\mathcal{B}_{p,\alpha}$ and $\mathcal{B}_{p,\alpha,0}$ are defined as the sets of those $f\in\mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}_{p,\alpha}}=\frac{p}{2}\sup_{z\in\mathbb{D}}\left|f(z)\right|^{\frac{p}{2}-1}\left|f'(z)\right|\left(1-|z|^2\right)^\alpha<\infty,$$

and

$$\lim_{|z|\to 1}\left|f(z)\right|^{\frac{p}{2}-1}\left|f'(z)\right|\left(1-|z|^2\right)^\alpha=0,$$

where $0<p,\alpha<\infty$.

Remark 1.1 The definition of (p,α)-Bloch spaces is introduced in the present paper for the first time. One should note that, if we put $p = 2$ in Definition 1.1, we will obtain the spaces $\mathcal{B}_\alpha$ and $\mathcal{B}_{\alpha,0}$.

Remark 1.2 (p,α)-Bloch space is very useful in some calculations in this paper and it can be also used to study some other operators like integral operators (see [12]).

If (X,d) is a metric space, we denote the open and closed balls with center x and radius $r > 0$ by $B(x,r) := \{y \in X : d(y,x) < r\}$ and $\bar{B}(x,r) := \{y \in X : d(y,x) = r\}$, respectively. The well-known hyperbolic derivative is defined by $f^*(z) = \frac{|f'(z)|}{1-|f(z)|^2}$ of $f \in B(\mathbb{D})$ and the hyperbolic distance is given by $\rho(f(z),0) := \frac{1}{2}\log(\frac{1+|f(z)|}{1-|f(z)|})$ between $f(z)$ and zero.

A function $f \in B(\mathbb{D})$ is said to belong to the hyperbolic α-Bloch class $\mathcal{B}^*_\alpha$ if

$$\|f\|_{\mathcal{B}^*_\alpha} = \sup_{z\in\mathbb{D}} f^*(z)\big(1-|z|^2\big)^\alpha < \infty.$$

The little hyperbolic Bloch-type class $\mathcal{B}^*_{\alpha,0}$ consists of all $f \in \mathcal{B}^*_\alpha$ such that

$$\lim_{|z|\to 1} f^*(z)\big(1-|z|^2\big)^\alpha = 0.$$

The Schwarz-Pick lemma implies $\mathcal{B}^*_\alpha = B(\mathbb{D})$ for all $\alpha \geq 1$ with $\|f\|_{\mathcal{B}^*_\alpha} \leq 1$, and therefore, the hyperbolic α-Bloch classes are of interest only when $0 < \alpha < 1$.

It is obvious that $\mathcal{B}^*_\alpha$ is not a linear space since the sum of two functions in $B(\mathbb{D})$ does not necessarily belong to $B(\mathbb{D})$.

Now, let $0 < p < \infty$, we define the hyperbolic derivative by $f^*_p(z) = \frac{p}{2}\frac{|f(z)|^{\frac{p}{2}-1}|f'(z)|}{1-|f(z)|^p}$ of $f \in B(\mathbb{D})$. When $p = 2$, we obtain the usual hyperbolic derivative as defined above.

A function $f \in B(\mathbb{D})$ is said to belong to the generalized hyperbolic (p,α)-Bloch class $\mathcal{B}^*_{p,\alpha}$ if

$$\|f\|_{\mathcal{B}^*_{p,\alpha}} = \sup_{z\in\mathbb{D}} f^*_p(z)\big(1-|z|^2\big)^\alpha < \infty.$$

The little generalized (p,α)-hyperbolic Bloch-type class $\mathcal{B}^*_{p,\alpha,0}$ consists of all $f \in \mathcal{B}^*_{p,\alpha}$ such that

$$\lim_{|z|\to 1} f^*_p(z)\big(1-|z|^2\big)^\alpha = 0.$$

Let the Green's function of $\mathbb{D}$ be defined as $g(z,a) = \log\frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation related to the point $a \in \mathbb{D}$. For $0 < p,s < \infty$, the hyperbolic class $Q^*(p,s)$ consists of those functions $f \in B(\mathbb{D})$ for which

$$\|f\|^p_{Q^*(p,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \big(f^*_p(z)\big)^2 g^s(z,a)\, dA(z) < \infty.$$

Moreover, we say that $f \in Q^*(p,s)$ belongs to the class $Q^*(p,s,0)$ if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} \big(f^*_p(z)\big)^2 g^s(z,a)\, dA(z) = 0.$$

When $p = 2$, we obtain the usual hyperbolic Q class as studied in [10, 11, 14].

Remark 1.3 The Schwarz-Pick lemma implies that $\mathcal{B}^*_{p,\alpha} = B(\mathbb{D})$ for all $\alpha \geq 1$ with $\|f\|_{\mathcal{B}^*_{p,\alpha}} \leq 1$ and therefore, the generalized hyperbolic (p,α)-classes are of interest only when $0 < \alpha < 1$. Also $Q^*(p,s) = B(\mathbb{D})$ for all $s > 1$, and hence, the generalized hyperbolic $Q(p,s)$-classes will be considered when $0 \leq s \leq 1$.

For any holomorphic self-mapping ϕ of $\mathbb{D}$, the symbol ϕ induces a linear composition operator $C_\phi(f) = f \circ \phi$ from $\mathcal{H}(\mathbb{D})$ or $B(\mathbb{D})$ into itself. The study of a composition operator C_ϕ acting on the spaces of analytic functions has engaged many analysts for many years (see, *e.g.*, [1–8, 11, 13, 16] and others).

Yamashita was probably the first to consider systematically hyperbolic function classes. He introduced and studied hyperbolic Hardy, BMOA and Dirichlet classes in [18–20] and others. More recently, Smith studied inner functions in the hyperbolic little Bloch-class [15], and the hyperbolic counterparts of the Q_p spaces were studied by Li in [10] and Li *et al.* in [11]. Further, hyperbolic Q_p classes and composition operators were studied by Pérez-González *et al.* in [14].

In this paper, we will study the generalized hyperbolic (p,α)-Bloch classes $\mathcal{B}^*_{p,\alpha}$ and the hyperbolic $Q^*(p,s)$ type classes. We will also give some results to characterize Lipschitz continuous and compact composition operators mapping from the generalized hyperbolic (p,α)-Bloch class $\mathcal{B}^*_{p,\alpha}$ to $Q^*(p,s)$ classes by conditions depending on the symbol ϕ only. Thus, the results are generalizations of the recent results of Pérez-González, Rättyä and Taskinen [14].

Recall that a linear operator $T : X \to Y$ is said to be bounded if there exists a constant $C > 0$ such that $\|T(f)\|_Y \leq C\|f\|_X$ for all maps $f \in X$. By elementary functional analysis, a linear operator between normed spaces is bounded if and only if it is continuous, and the boundedness is trivially also equivalent to the Lipschitz-continuity. Moreover, $T : X \to Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. For Banach spaces X and Y contained in $B(\mathbb{D})$ or $\mathcal{H}(\mathbb{D})$, $T : X \to Y$ is compact if and only if for each bounded sequence $(x_n) \in X$, the sequence $(Tx_n) \in Y$ contains a subsequence converging to a function $f \in Y$.

Throughout this paper, C stands for absolute constants which may indicate different constants from one occurrence to the next.

The following lemma follows by standard arguments similar to those outlined in [17]. Hence we omit the proof.

Lemma 1.1 *Assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself. Let $0 < p,s < \infty$, and $0 < \alpha < \infty$. Then $C_\phi : \mathcal{B}^*_{p,\alpha} \to Q^*(p,s)$ is compact if and only if for any bounded sequence $(f_n)_{n\in\mathbb{N}} \in \mathcal{B}^*_{p,\alpha}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, we have* $\lim_{n\to\infty} \|C_\phi f_n\|_{Q^*(p,s)} = 0$.

Using the standard arguments similar to those outlined in Lemma 1 of [9], we have the following lemma:

Lemma 1.2 *Let $0 < \alpha < \infty$, then there exist two functions $f, g \in \mathcal{B}^*_{p,\alpha}$ such that for some constant C,*

$$\left(\left|f_p^*(z)\right| + \left|g_p^*(z)\right|\right)\left(1 - |z|^2\right)^\alpha \geq C > 0, \quad \textit{for each } z \in \mathbb{D}.$$

2 Natural metrics in $\mathcal{B}^*_{p,\alpha}$ and $Q^*(p,s)$ classes

In this section we introduce natural metrics on generalized hyperbolic α-Bloch classes $\mathcal{B}^*_{p,\alpha}$ and the classes $Q^*(p,s)$.

Let $0 < p,s < \infty$, and $0 < \alpha < 1$. First, we can find a natural metric in $\mathcal{B}^*_{p,\alpha}$ (see [14]) by defining

$$d(f,g;\mathcal{B}^*_{p,\alpha}) := d_{\mathcal{B}^*_{p,\alpha}}(f,g) + \|f-g\|_{\mathcal{B}_{p,\alpha}} + \big|f(0)-g(0)\big|^{\frac{p}{2}}, \tag{1}$$

where

$$d_{\mathcal{B}^*_{p,\alpha}}(f,g) := \sup_{z\in\mathbb{D}}\left|\frac{f'(z)|f(z)|^{\frac{p}{2}-1}}{1-|f(z)|^p} - \frac{g'(z)|g(z)|^{\frac{p}{2}-1}}{1-|g(z)|^p}\right|\big(1-|z|^2\big)^\alpha.$$

For $f,g \in Q^*(p,s)$, define their distance by

$$d\big(f,g;Q^*(p,s)\big) := d_{Q^*}(f,g) + \|f-g\|_{Q(p,s)} + \big|f(0)-g(0)\big|^{\frac{p}{2}},$$

where

$$d_{Q^*}(f,g) := \left(\frac{p}{2}\sup_{z\in\mathbb{D}}\int_{\mathbb{D}}\left|\frac{f'(z)|f(z)|^{\frac{p}{2}-1}}{1-|f(z)|^p} - \frac{g'(z)|g(z)|^{\frac{p}{2}-1}}{1-|g(z)|^p}\right|^2 g^s(z,a)\,dA(z)\right)^{\frac{1}{2}}.$$

Now, we give a characterization of the complete metric space $d(\cdot,\cdot;\mathcal{B}^*_{p,\alpha})$.

Proposition 2.1 *The class $\mathcal{B}^*_{p,\alpha}$ equipped with the metric $d(\cdot,\cdot;\mathcal{B}^*_{p,\alpha})$ is a complete metric space. Moreover, $\mathcal{B}^*_{p,\alpha,0}$ is a closed (and therefore complete) subspace of $\mathcal{B}^*_{p,\alpha}$.*

Proof Clearly $d(f,g;\mathcal{B}^*_{p,\alpha}) \ge 0$, $d(f,g,\mathcal{B}^*_{p,\alpha}) = d(g,f;\mathcal{B}^*_{p,\alpha})$. Also,

$$d\big(f,h;\mathcal{B}^*_{p,\alpha}\big) \le d\big(f,g;\mathcal{B}^*_{p,\alpha}\big) + d\big(g,h;\mathcal{B}^*_{p,\alpha}\big).$$

Moreover, $d(f,f;\mathcal{B}^*_{p,\alpha}) = 0$ for all $f,g,h \in \mathcal{B}^*_{p,\alpha}$.

It follows from the presence of the usual (p,α)-Bloch term that $d(f,g;\mathcal{B}^*_{p,\alpha}) = 0$ implies $f = g$. Hence, $(\mathcal{B}^*_{p,\alpha},d)$ is a metric space. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in the metric space $(\mathcal{B}^*_{p,\alpha},d)$, that is, for any $\varepsilon > 0$, there is an $N = N(\varepsilon) \in \mathbb{N}$ such that

$$d\big(f_n,f_m;\mathcal{B}^*_{p,\alpha}\big) < \varepsilon$$

for all $n,m > N$. Since $(f_n) \subset B(\mathbb{D})$, the family (f_n) is uniformly bounded and hence normal in $\mathbb{D}$. Therefore, there exist $f \in B(\mathbb{D})$ and a subsequence $(f_{n_j})_{j=1}^\infty$ such that f_{n_j} converges to f uniformly on compact subsets, and by the Cauchy formula, the same also holds for the derivatives. Let $m > N$. Then the uniform convergence yields

$$\begin{aligned}&\left|\frac{f'(z)|f(z)|^{\frac{p}{2}-1}}{1-|f(z)|^p} - \frac{f_m'(z)|f_m(z)|^{\frac{p}{2}-1}}{1-|f_m(z)|^p}\right|\big(1-|z|^2\big)^\alpha \\ &\quad= \lim_{n\to\infty}\left|\frac{f_n'(z)|f_n(z)|^{\frac{p}{2}-1}}{1-|f_n(z)|^p} - \frac{f_m'(z)|f_m(z)|^{\frac{p}{2}-1}}{1-|f_m(z)|^p}\right|\big(1-|z|^2\big)^\alpha \le \lim_{n\to\infty} d\big(f_n,f_m;\mathcal{B}^*_{p,\alpha}\big) \le \varepsilon\end{aligned} \tag{2}$$

for all $z \in \mathbb{D}$, and it follows that

$$\|f\|_{\mathcal{B}^*_{p,\alpha}} \leq \|f_m\|_{\mathcal{B}^*_{p,\alpha}} + \varepsilon.$$

Thus, $f \in \mathcal{B}^*_{p,\alpha}$ as desired. Moreover, (2) and the completeness of the usual (p,α)-Bloch imply that $(f_n)_{n=1}^{\infty}$ converges to f with respect to the metric d. The second part of the assertion follows by (2). □

Next, we give a characterization of the complete metric space $d(\cdot,\cdot;Q^*(p,s))$.

Proposition 2.2 *The class $Q^*(p,s)$ equipped with the metric $d(\cdot,\cdot;Q^*(p,s))$ is a complete metric space. Moreover, $Q^*(p,s,0)$ is a closed (and therefore complete) subspace of $Q^*(p,s)$.*

Proof For $f,g,h \in Q^*(p,s)$, then clearly

- $d(f,g;Q^*(p,s)) \geq 0$,
- $d(f,f;Q^*(p,s)) = 0$,
- $d(f,g;Q^*(p,s)) = 0$ implies $f = g$,
- $d(f,g;Q^*(p,s)) = d(g,f;Q^*(p,s))$,
- $d(f,h;Q^*(p,s)) \leq d(f,g;Q^*(p,s)) + d(g,h;Q^*(p,s))$.

Hence, d is metric on $Q^*(p,s)$.

For the completeness proof, let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in the metric space $(Q^*(p,s),d)$, that is, for any $\varepsilon > 0$ there is an $N = N(\varepsilon) \in \mathbb{N}$ such that $d(f_n,f_m;Q^*(p,s)) < \varepsilon$, for all $n,m > N$. Since $f_n \in B(\mathbb{D})$ such that f_n converges to f uniformly on compact subsets of $\mathbb{D}$. Let $m > N$ and $0 < r < 1$. Then Fatou's lemma yields

$$\begin{aligned}
&\int_{D(0,r)} \left| \frac{|f(z)|^{\frac{p}{2}-1} f'(z)}{1-|f(z)|^p} - \frac{|f_m(z)|^{\frac{p}{2}-1} f_m'(z)}{1-|f_m(z)|^p} \right|^2 g^s(z,a)\,dA(z) \\
&\quad = \int_{D(0,r)} \lim_{n\to\infty} \left| \frac{|f_n(z)|^{\frac{p}{2}-1} f_n'(z)}{1-|f_n(z)|^p} - \frac{|f_m(z)|^{\frac{p}{2}-1} f_m'(z)}{1-|f_m(z)|^p} \right|^2 g^s(z,a)\,dA(z) \\
&\quad \leq \lim_{n\to\infty} \int_{\mathbb{D}} \left| \frac{|f_n(z)|^{\frac{p}{2}-1} f_n'(z)}{1-|f_n(z)|^p} - \frac{|f_m(z)|^{\frac{p}{2}-1} f_m'(z)}{1-|f_m(z)|^p} \right|^2 g^s(z,a)\,dA(z) \leq \varepsilon^2,
\end{aligned}$$

and by letting $r \to 1^-$, it follows that

$$\int_{\mathbb{D}} \left(f_p^*(z)\right)^2 g^s(z,a)\,dA(z) \leq 2\varepsilon^2 + 2\int_{\mathbb{D}} \left| \frac{|f_m(z)|^{\frac{p}{2}-1} f_m'(z)}{1-|f_m(z)|^p} \right|^2 g^s(z,a)\,dA(z). \tag{3}$$

This yields

$$\|f\|^p_{Q^*(p,s)} \leq 2\varepsilon^2 + 2\|f_m\|^2_{Q^*(p,s)},$$

and thus $f \in Q^*(p,s)$. We also find that $f_n \to f$ with respect to the metric of $Q^*(p,s)$. The second part of the assertion follows by (3). □

3 Lipschitz continuous and compactness of C_ϕ

Theorem 3.1 *Let $0 < p < \infty$, $0 \leq s \leq 1$, and $0 < \alpha \leq 1$. Assume that ϕ is a holomorphic mapping from $\mathbb{D}$ into itself. Then the following statements are equivalent:*

(i) $C_\phi : \mathcal{B}^*_{p,\alpha} \to Q^*(p,s)$ *is bounded;*
(ii) $C_\phi : \mathcal{B}^*_{p,\alpha} \to Q^*(p,s)$ *is Lipschitz continuous;*
(iii) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\phi'(z)|^2}{(1-|\phi(z)|^p)^{2\alpha}} g^s(z,a)\,dA(z) < \infty$.

Proof First, assume that (i) holds, then there exists a constant C such that

$$\|C_\phi f\|_{Q^*(p,s)} \le C\|f\|_{\mathcal{B}^*_{p,\alpha}}, \quad \text{for all } f \in \mathcal{B}^*_{p,\alpha}.$$

For given $f \in \mathcal{B}^*_{p,\alpha}$, the function $f_t(z) = f(tz)$, where $0 < t < 1$, belongs to $\mathcal{B}^*_{p,\alpha}$ with the property $\|f_t\|_{\mathcal{B}^*_{p,\alpha}} \le \|f\|_{\mathcal{B}^*_{p,\alpha}}$. Let f, g be the functions from Lemma 1.2 such that

$$\frac{1}{(1-|z|^2)^\alpha} \le \left|f^*_p(z)\right| + \left|g^*_p(z)\right|,$$

for all $z \in \mathbb{D}$, so that

$$\frac{|\phi'(z)|}{(1-|\phi(z)|)^\alpha} \le (f\circ\phi)^*(z) + (g\circ\phi)^*(z).$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{|t\phi'(z)|^2}{(1-|t\phi(z)|^2)^{2\alpha}} g^s(z,a)\,dA(z)\\
&\quad\le C\int_{\mathbb{D}} \left(\left((f\circ t\phi)^*_p(z)\right)^2 + \left((g\circ t\phi)^*_p(z)\right)^2\right) g^s(z,a)\,dA(z)\\
&\quad\le C\|C_\phi\|^2\left(\|f\|^2_{\mathcal{B}^*_{p,\alpha}} + \|g\|^2_{\mathcal{B}^*_{p,\alpha}}\right).
\end{aligned}$$

This estimate together with the Fatou's lemma implies (iii).

Conversely, assuming that (iii) holds and that $f \in \mathcal{B}^*_{p,\alpha}$, we see that

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left((f\circ\phi)^*_p(z)\right)^2 g^s(z,a)\,dA(z)\\
&\quad= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(f^*_p(\phi(z))\right)^2 \left|\phi'(z)\right|^2 g^s(z,a)\,dA(z)\\
&\quad\le \|f\|^2_{\mathcal{B}^*_{p,\alpha}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\phi'(z)|^2}{(1-|\phi(z)|^p)^{2\alpha}} g^s(z,a)\,dA(z).
\end{aligned}$$

Hence, it follows that (i) holds.

(ii) $\Longleftrightarrow$ (iii). Assume first that $C_\phi : \mathcal{B}^*_{p,\alpha} \to Q^*(p,s)$ is Lipschitz continuous, that is, there exists a positive constant C such that

$$d\left(f\circ\phi, g\circ\phi; Q^*(p,s)\right) \le Cd\left(f,g;\mathcal{B}^*_{p,\alpha}\right), \quad \text{for all } f,g \in \mathcal{B}^*_{p,\alpha}.$$

Taking $g = 0$, this implies

$$\|f\circ\phi\|_{Q^*(p,s)} \le C\left(\|f\|_{\mathcal{B}^*_{p,\alpha}} + \|f\|_{\mathcal{B}_{p,\alpha}} + \left|f(0)\right|^{\frac{p}{2}}\right), \quad \text{for all } f \in \mathcal{B}^*_{p,\alpha}. \tag{4}$$

The assertion (iii) for $\alpha = 1$ follows by choosing $f(z) = z$ in (4). If $0 < \alpha < 1$, then

$$\begin{aligned}\frac{2}{p}\left|f(z)\right|^{\frac{p}{2}} &= \left|\int_0^z \left|f(t)\right|^{\frac{p}{2}} f'(t)\,dt + \big(f(0)\big)^{\frac{p}{2}}\right| \\ &\le \|f\|_{\mathcal{B}_{p,\alpha}} \int_0^{|z|} \frac{dx}{(1-x^2)^\alpha} + \left|f(0)\right|^{\frac{p}{2}} \\ &\le \frac{\|f\|_{\mathcal{B}_\alpha}}{(1-\alpha)} + \left|f(0)\right|^{\frac{p}{2}},\end{aligned}$$

this yields

$$\frac{2}{p}\left|f\big(\phi(0)\big) - g\big(\phi(0)\big)\right|^{\frac{p}{2}} \le \frac{\|f-g\|_{\mathcal{B}_{p,\alpha}}}{(1-\alpha)} + \left|f(0)-g(0)\right|^{\frac{p}{2}}.$$

Moreover, Lemma 1.2 implies the existence of $f, g \in \mathcal{B}^*_{p,\alpha}$ such that

$$\left|f_p^*(z) + g_p^*(z)\right|\big(1-|z|^2\big)^\alpha \ge C > 0, \quad \text{for all } z \in \mathbb{D}. \tag{5}$$

Combining (4) and (5), we obtain

$$\begin{aligned}&\|f\|_{\mathcal{B}^*_{p,\alpha}} + \|g\|_{\mathcal{B}^*_{p,\alpha}} + \|f\|_{\mathcal{B}_{p,\alpha}} + \|g\|_{\mathcal{B}_{p,\alpha}} + \left|f(0)\right|^{\frac{p}{2}} + \left|g(0)\right|^{\frac{p}{2}} \\ &\quad \ge C\int_{\mathbb{D}} \frac{|\phi'(z)|^2}{(1-|\phi(z)|^p)^{2\alpha}} g^s(z,a)\,dA(z)\end{aligned}$$

for which the assertion (iii) follows.

Assume now that (iii) is satisfied, we have

$$\begin{aligned}d\big(f\circ\phi, g\circ\phi; Q^*(p,s)\big) &= d_{Q^*_{(p,s)}}(f\circ\phi, g\circ\phi) + \|f\circ\phi - g\circ\phi\|_{Q(p,s)} \\ &\quad + \left|f\big(\phi(0)\big) - g\big(\phi(0)\big)\right|^{\frac{p}{2}} \\ &\le d_{\mathcal{B}^*_{p,\alpha}}(f,g)\left(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{|\phi'(z)|^2}{(1-|\phi(z)|^p)^{2\alpha}} g^s(z,a)\,dA(z)\right)^{\frac{1}{2}} \\ &\quad + \|f-g\|_{\mathcal{B}_{p,\alpha}}\left(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{|\phi'(z)|^2}{(1-|\phi(z)|^p)^{2\alpha}} g^s(z,a)\,dA(z)\right)^{\frac{1}{2}} \\ &\quad + \frac{\|f-g\|_{\mathcal{B}_{p,\alpha}}}{(1-\alpha)} + \left|f(0)-g(0)\right|^{\frac{p}{2}} \\ &\le Cd\big(f,g;\mathcal{B}^*_{p,\alpha}\big).\end{aligned}$$

Thus $C_\phi : \mathcal{B}^*_{p,\alpha} \to Q^*(p,s)$ is Lipschitz continuous and the proof is completed. □

Remark 3.1 We know that a composition operator $C_\phi : \mathcal{B}^*_{p,\alpha} \to Q^*(p,s)$ is said to be bounded if there is a positive constant C such that $\|C_\phi f\|_{Q^*(p,s)} \le C\|f\|_{\mathcal{B}^*_{p,\alpha}}$ for all $f \in \mathcal{B}^*_{p,\alpha}$. Theorem 3.1 shows that $C_\phi : \mathcal{B}^*_{p,\alpha} \to Q^*(p,s)$ is bounded if and only if it is Lipschitz-continuous, that is, if there exists a positive constant C such that

$$d\big(f\circ\phi, g\circ\phi; Q^*(p,s)\big) \le Cd\big(f,g;\mathcal{B}^*_{p,\alpha}\big), \quad \text{for all } f, g \in \mathcal{B}^*_{p,\alpha}.$$

By elementary functional analysis, a linear operator between normed spaces is bounded if and only if it is continuous, and the boundedness is trivially also equivalent to the Lipschitz-continuity. So, our result for composition operators in hyperbolic spaces is the correct and natural generalization of the linear operator theory.

Recall that a composition operator $C_\phi : \mathcal{B}^*_{p,\alpha} \to Q^*(p,s)$ is compact if it maps any ball in $\mathcal{B}^*_{p,\alpha}$ onto a precompact set in $Q^*(p,s)$.

The following observation is sometimes useful.

Proposition 3.1 *Let* $0 < p < \infty$, $0 \le s \le 1$ *and* $0 < \alpha \le 1$. *Assume that* ϕ *is a holomorphic mapping from* $\mathbb{D}$ *into itself. If* $C_\phi : \mathcal{B}^*_{p,\alpha} \to Q^*(p,s)$ *is compact, it maps closed balls onto compact sets.*

Proof If $B \subset \mathcal{B}^*_{p,\alpha}$ is a closed ball and $g \in Q^*(p,s)$ belongs to the closure of $C_\phi(B)$, we can find a sequence $(f_n)_{n=1}^\infty \subset B$ such that $f_n \circ \phi$ converges to $g \in Q^*(p,s)$ as $n \to \infty$. But $(f_n)_{n=1}^\infty$ is a normal family, hence it has a subsequence $(f_{n_j})_{j=1}^\infty$ converging uniformly on the compact subsets of $\mathbb{D}$ to an analytic function f. As in earlier arguments of Proposition 2.1 in [14], we get a positive estimate which shows that f must belong to the closed ball B. On the other hand, also the sequence $(f_{n_j} \circ \phi)_{j=1}^\infty$ converges uniformly on compact subsets to an analytic function, which is $g \in Q^*(p,s)$. We get $g = f \circ \phi$, *i.e.*, g belongs to $C_\phi(B)$. Thus, this set is closed and also compact. □

Compactness of composition operators can be characterized in full analogy with the linear case.

Theorem 3.2 *Let* $0 < p < \infty$, $0 \le s \le 1$, *and* $0 < \alpha \le 1$. *Assume that* ϕ *is a holomorphic mapping from* $\mathbb{D}$ *into itself. Then the following statements are equivalent:*

(i) $C_\phi : \mathcal{B}^*_{p,\alpha} \to Q^*(p,s)$ *is compact.*

(ii)

$$\lim_{r\to 1^-} \sup_{a\in\mathbb{D}} \int_{|\phi|\ge r_j} \frac{|\phi'(z)|^2}{(1-|\phi(z)|^p)^{2\alpha}} g^s(z,a)\,dA(z) = 0.$$

Proof We first assume that (ii) holds. Let $B := \bar{B}(g,\delta) \subset \mathcal{B}^*_{p,\alpha}$, where $g \in \mathcal{B}^*_{p,\alpha}$ and $\delta > 0$, be a closed ball, and let $(f_n)_{n=1}^\infty \subset B$ be any sequence. We show that its image has a convergent subsequence in $Q^*(p,s)$, which proves the compactness of C_ϕ by definition.

Again, $(f_n)_{n=1}^\infty \subset B(\mathbb{D})$ is a normal family, hence there is a subsequence $(f_{n_j})_{j=1}^\infty$ which converges uniformly on the compact subsets of $\mathbb{D}$ to an analytic function f. By the Cauchy formula for the derivative of an analytic function, also the sequence $(f'_{n_j})_{j=1}^\infty$ converges uniformly to f'. It follows that also the sequences $(f_{n_j} \circ \phi)_{j=1}^\infty$ and $(f'_{n_j} \circ \phi)_{j=1}^\infty$ converge uniformly on the compact subsets of $\mathbb{D}$ to $f \circ \phi$ and $f' \circ \phi$, respectively. Moreover, $f \in B \subset \mathcal{B}^*_{p,\alpha}$ since for any fixed R, $0 < R < 1$, the uniform convergence yields

$$\sup_{|z|\le R} \left| \frac{f'(z)|f(z)|^{\frac{p}{2}-1}}{1-|f(z)|^p} - \frac{g'(z)|g(z)|^{\frac{p}{2}-1}}{1-|g(z)|^p} \right| \left(1-|z|^2\right)^\alpha$$
$$+ \sup_{|z|\le R} \left|f(z)-g(z)\right|^{\frac{p}{2}-1} \left|f'(z)-g'(z)\right| \left(1-|z|^2\right)^\alpha + \left|f(0)-g(0)\right|^{\frac{p}{2}-1}$$

$$= \lim_{j\to\infty} \sup_{|z|\le R} \left| \frac{f'_{n_j}(z)|f_{n_j}(z)|^{\frac{p}{2}-1}}{1-|f_{n_j}(z)|^p} - \frac{g'(z)|g(z)|^{\frac{p}{2}-1}}{1-|g(z)|^p} \right| \left(1-|z|^2\right)^\alpha$$
$$+ \sup_{|z|\le R} \left|f_{n_j}(z)-g(z)\right|^{\frac{p}{2}-1} \left|f'_{n_j}(z)-g'(z)\right| \left(1-|z|^2\right)^\alpha + \left|f(0)-g(0)\right|^{\frac{p}{2}-1} \le \delta.$$

Hence, $d(f,g;\mathcal{B}^*_{p,\alpha}) \le \delta$.

Let $\varepsilon > 0$. Since (ii) is satisfied, we may fix r, $0 < r < 1$, such that

$$\sup_{a\in\mathbb{D}} \int_{|\phi(z)|\ge r} \frac{|\phi(z)|^{p-2}|\phi'(z)|^2}{(1-|\phi(z)|^p)^{2\alpha}} g^s(z,a)\, dA(z) \le \varepsilon.$$

By the uniform convergence, we may fix $N_1 \in \mathbb{N}$ such that

$$\left|f_{n_j}\circ\phi(0) - f\circ\phi(0)\right| \le \varepsilon, \quad \text{for all } j \ge N_1. \tag{6}$$

The condition (ii) is known to imply the compactness of $C_\phi : \mathcal{B}_{p,\alpha} \to Q(p,s)$, hence possibly to passing once more to a subsequence and adjusting the notations, we may assume that

$$\|f_{n_j}\circ\phi - f\circ\phi\|_{Q(p,s)} \le \varepsilon, \quad \text{for all } j \ge N_2, \text{ for some } N_2 \in \mathbb{N}. \tag{7}$$

Now let

$$I_1(a,r) = \sup_{a\in\mathbb{D}} \int_{|\phi(z)|\ge r} \left[(f_{n_j}\circ\phi)^*_p(z) - (f\circ\phi)^*_p(z)\right]^2 g^s(z,a)\, dA(z),$$

and

$$I_2(a,r) = \sup_{a\in\mathbb{D}} \int_{|\phi(z)|\le r} \left[(f_{n_j}\circ\phi)^*_p(z) - (f\circ\phi)^*_p(z)\right]^2 g^s(z,a)\, dA(z).$$

Since $(f_{n_j})_{j=1}^\infty \subset B$ and $f \in B$, it follows that

$$\begin{aligned} I_1(a,r) &= \sup_{a\in\mathbb{D}} \int_{|\phi(z)|\ge r} \left[(f_{n_j}\circ\phi)^*_p(z) - (f\circ\phi)^*_p(z)\right]^2 g^s(z,a)\, dA(z) \\ &\le \frac{p}{2} \sup_{a\in\mathbb{D}} \int_{|\phi(z)|\ge r} \mathcal{L}(f_{n_j},f,\phi) g^s(z,a)\, dA(z) \\ &\le d_{\mathcal{B}^*_\alpha}(f_{n_j},f) \sup_{a\in\mathbb{D}} \int_{|\phi(z)|\ge r} \frac{|\phi'(z)|^2}{(1-|\phi(z)|^p)^{2\alpha}} g^s(z,a)\, dA(z), \end{aligned}$$

where

$$\mathcal{L}(f_{n_j},f,\phi) = \left| \frac{|(f_{n_j}\circ\phi)(z)|^{\frac{p}{2}-1}(f_{n_j}\circ\phi)'(z)}{1-|(f_{n_j}\circ\phi)(z)|^p} - \frac{|(f\circ\phi)(z)|^{\frac{p}{2}-1}(f\circ\phi)'(z)}{1-|(f\circ\phi)(z)|^p} \right|^2.$$

Hence,

$$I_1(a,r) \le C\varepsilon. \tag{8}$$

On the other hand, by the uniform convergence on compact subsets of $\mathbb{D}$, we can find an $N_3 \in \mathbb{N}$ such that for all $j \geq N_3$,

$$\mathcal{L}_1(f_{n_j}, f, \phi) = \left| \frac{|(f_{n_j} \circ \phi)(z)|^{\frac{p}{2}-1} f'_{n_j}(\phi(z))}{1 - |f_{n_j}(\phi(z))|^p} - \frac{|(f \circ \phi)(z)|^{\frac{p}{2}-1} f'(\phi(z))}{1 - |f(\phi(z))|^p} \right| \leq \varepsilon$$

for all $z \in \mathbb{D}$ with $|\phi(z)| \leq r$. Hence, for such j, we obtain

$$\begin{aligned} I_2(a,r) &= \sup_{a \in \mathbb{D}} \int_{|\phi(z)| \leq r} \big((f_{n_j} \circ \phi)^*_p(z) - (f \circ \phi)^*_p(z) \big)^2 g^s(z,a)\, dA(z) \\ &\leq \sup_{a \in \mathbb{D}} \int_{|\phi(z)| \leq r} \mathcal{L}_1(f_{n_j}, f, \phi) \big| \phi'(z) \big|^2 g^s(z,a)\, dA(z) \\ &\leq \varepsilon \left(\sup_{a \in \mathbb{D}} \int_{|\phi(z)| \leq r} \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^p)^{2\alpha}} g^s(z,a)\, dA(z) \right)^{\frac{1}{2}} \leq C\varepsilon, \end{aligned}$$

hence,

$$I_2(a,r) \leq C\varepsilon, \tag{9}$$

where C is the bound obtained from (iii) of Theorem 3.1. Combining (6), (7), (8) and (9), we deduce that $f_{n_j} \to f$ in $Q^*(p,s)$.

As for the converse direction, let $f_n(z) := \frac{1}{2} n^{\alpha-1} z^n$ for all $n \in \mathbb{N}$, $n \geq 2$.

$$\begin{aligned} \|f\|_{\mathcal{B}^*_{p,\alpha}} &= \frac{p}{2} \sup_{a \in \mathbb{D}} \frac{n^{\frac{\alpha p}{2}} |z|^{\frac{\alpha p}{2}-1} (1 - |z|^2)^\alpha}{1 - 2^{-p} n^{p(\alpha-1)} |z|^{np}} \\ &\leq \big(2^{p-1} + 1\big) \sup_{a \in \mathbb{D}} n^{\frac{\alpha p}{2}} |z|^{\frac{\alpha p}{2}-1} \big(1 - |z|^2\big)^\alpha. \end{aligned} \tag{10}$$

The function $r^{\frac{np}{2}-1}(1-r)^\alpha$ attains its maximum at the point $r = 1 - \frac{\alpha}{\alpha + \frac{\alpha p}{2} - 1}$. For simplicity, we see that (10) has the upper bound

$$\big(2^{p-1} + 1\big) n^\alpha \left(1 - \frac{\alpha}{\alpha + n - 1} \right)^{n-1} \left(\frac{\alpha}{\alpha + n - 1} \right)^\alpha \leq \big(2^{p-1} + 1\big).$$

Then the sequence $(f_n)_{n=1}^\infty$ belongs to the ball $\bar{B}(0, (2^{p-1} + 1)) \subset \mathcal{B}^*_{p,\alpha}$.

Suppose that C_ϕ maps the closed ball $\bar{B}(0, (2^{p-1} + 1)) \subset \mathcal{B}^*_{p,\alpha}$ into a compact subset of $Q^*(p,s)$; hence, there exists an unbounded increasing subsequence $(n_j)_{j=1}^\infty$ such that the image subsequence $(C_\phi f_{n_j})_{j=1}^\infty$ converges with respect to the norm. Since both $(f_n)_{n=1}^\infty$ and $(C_\phi f_{n_j})_{j=1}^\infty$ converge to the zero function uniformly on compact subsets of $\mathbb{D}$, the limit of the latter sequence must be zero. Hence,

$$\big\| n_j^{\alpha-1} \phi^{n_j} \big\|_{Q^*(p,s)} \to 0, \quad \text{as } j \to \infty. \tag{11}$$

Now let $r_j = 1 - \frac{1}{n_j}$. For all numbers a, $r_j \leq a < 1$, we have the estimate

$$\frac{n_j^\alpha a^{n_j-1}}{1 - a^{n_j}} \geq \frac{1}{e(1-a)^\alpha} \quad \big(\text{see [14]}\big). \tag{12}$$

Using (12), we deduce

$$\begin{aligned}\left\|n_j^{\alpha-1}\phi^{n_j}\right\|^2_{Q^*(p,s)} &\geq \frac{p}{2}\sup_{a\in\mathbb{D}}\int_{|\phi|\geq r_j}\left|\frac{n_j^{\alpha}(\phi(z))^{n_j-1}|\phi^{n_j}(z)|^{\frac{p}{2}-1}\phi'(z)}{1-|\phi^{n_j}(z)|^p}\right|^2 g^s(z,a)\,dA(z)\\ &\geq \frac{Cp}{8e^2}\sup_{a\in\mathbb{D}}\int_{|\phi|\geq r_j}\frac{|\phi'(z)|^2}{(1-|\phi(z)|^p)^{2\alpha}}g^s(z,a)\,dA(z). \qquad (13)\end{aligned}$$

From (11) and (13), the condition (ii) follows. This completes the proof. □

For $0 < p < \infty$ and $0 \leq s < \infty$, we define the weighted Dirichlet-class $\mathcal{D}(p,s)$ consists of those functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\int_{\mathbb{D}}\left|f(z)\right|^{p-2}\left|f'(z)\right|^2\left(1-|z|^2\right)^s dA(z) < \infty.$$

For $0 < p < \infty$ and $0 \leq s < \infty$, the generalized hyperbolic weighted Dirichlet-class $\mathcal{D}^*(p,s)$ consists of those functions $f \in B(\mathbb{D})$ for which

$$\int_{\mathbb{D}}\left(f_p^*(z)\right)^2\left(1-|z|^2\right)^s dA(z) < \infty.$$

The proof of Proposition 2.2 implies the following corollary:

Corollary 3.1 *For $f, g \in \mathcal{D}^*(p,s)$. Then, $\mathcal{D}^*(p,s)$ is a complete metric space with respect to the metric defined by*

$$d\left(f,g;\mathcal{D}^*(p,s)\right) := d_{\mathcal{D}^*(p,s)}(f,g) + \|f-g\|_{\mathcal{D}(p,s)} + \left|f(0)-g(0)\right|^{\frac{p}{2}},$$

where

$$d_{\mathcal{D}^*(p,s)}(f,g) := \left(\frac{p}{2}\sup_{z\in\mathbb{D}}\int_{\mathbb{D}}\left|\frac{f'(z)|f(z)|^{\frac{p}{2}-1}}{1-|f(z)|^p} - \frac{g'(z)|g(z)|^{\frac{p}{2}-1}}{1-|g(z)|^p}\right|^2\left(1-|z|^2\right)^s dA(z)\right)^{\frac{1}{2}}.$$

Moreover, the proofs of Theorems 3.1 and 3.2 yield the following result:

Theorem 3.3 *Let $0 < p < \infty$, $-1 < s \leq 1$, and $0 < \alpha \leq 1$. Assume that ϕ is a holomorphic mapping from $\mathbb{D}$ into itself. Then the following statements are equivalent:*

(i) $C_\phi : \mathcal{B}^*_{p,\alpha} \to \mathcal{D}^*(p,s)$ *is Lipschitz continuous;*
(ii) $C_\phi : \mathcal{B}^*_{p,\alpha} \to \mathcal{D}^*(p,s)$ *is compact;*
(iii)

$$\int_{\mathbb{D}}\frac{|\phi'(z)|^2}{(1-|\phi(z)|^p)^{2\alpha}}\left(1-|z|^2\right)^s dA(z) < \infty.$$

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