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Refinements of the Heinz inequalities

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Abstract

This article aims to discuss Heinz inequalities involving unitarily invariant norms. We obtain refinements of the Heinz inequalities. In particular, our results refine some results given in Kittaneh.

Keywords: refinements, Heinz inequality, convex function, Hermite-Hadamard inequality, unitarily invariant norm

1. Introduction

If A,B,X are operators on a complex separable Hilbert space such that A and B are positive, then for every unitarily invariant norm $|||\cdot|||$, the function $f(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$ is convex on the interval $[0, 1]$, attains its minimum at $\nu = \frac{1}{2}$, and attains its maximum at $\nu = 0$ and $\nu = 1$. Moreover, $f(\nu) = f(1 - \nu)$ for $0 \le \nu \le 1$. Thus, for every unitarily invariant norm, we have the Heinz inequalities (see [1])

$$2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| \le \left|\left|\left|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\right|\right|\right| \le |||AX + XB|||. \tag{1.1}$$

In this article, we use the convexity of the function

$$f(\nu) = \left|\left|\left|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\right|\right|\right|$$

on $[0, 1]$ to obtain new refinements of the inequalities (1.1). Our analysis enables us to discuss the equality conditions in (1.1) for certain unitarily invariant norms. When we consider $|||T|||$, we are implicitly assuming that the operator T belongs to the norm ideal associated with $|||\cdot|||$. Our results are better than those in [2].

2. Main results

The following Hermite-Hadamard integral inequality for convex functions is well known (see p. 122 in [3], also see Lemma 1 in [2]).

Lemma 1 (Hermite-Hadamard Integral Inequality). *Let f be a real-valued function which is convex on the interval $[a, b]$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}.$$

In [2], Kittaneh obtained several refinements of the Heinz inequalities by using the previous lemma. In the following, we will use the following lemma to obtain several better refinements of the Heinz inequalities.

The following lemma can be proved by using the previous lemma.

Lemma 2. *Let f be a real-valued function which is convex on the interval* [*a, b*]. *Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{1}{4}\left(f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right) \le \frac{f(a)+f(b)}{2}.$$

Proof. Using the previous lemma, we can easily verify the inequality

$$\frac{1}{4}\left(f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right) \le \frac{f(a)+f(b)}{2}.$$

Next, we will prove the following inequality.

$$\frac{1}{b-a}\int_a^b f(t)dt \le \frac{1}{4}\left(f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right).$$

From the previous lemma, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)dt &= \frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}} f(t)dt + \int_{\frac{a+b}{2}}^b f(t)dt\right)\\ &\le \frac{1}{b-a}\left(\frac{f(a)+f(\frac{a+b}{2})}{2}\cdot\frac{b-a}{2}+\frac{f(\frac{a+b}{2})+f(a)}{2}\cdot\frac{b-a}{2}\right)\\ &= \frac{1}{4}\left(f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right).\end{aligned}$$

Applying the previous lemma to the function $f(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$ on the interval $[\mu, 1-\mu]$ when $0 \le \mu \le \frac{1}{2}$, and on the interval $[1-\mu, \mu]$ when $\frac{1}{2} \le \mu \le 1$, we obtain refinement of the first inequality in (1.1).

Theorem 1. *Let A, B, X be operators such that A, B are positive. Then for* $0 \le \mu \le 1$ *and for every unitarily invariant norm, we have*

$$\begin{aligned}2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| &\le \frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu}\left|\left|\left|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}\right|\right|\right|d\nu\right|\\ &\le \frac{1}{2}\left(\left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right)\\ &\le \left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|.\end{aligned} \tag{2.1}$$

Proof. First assume that $0 \le \mu \le \frac{1}{2}$. Then it follows by the previous lemma that

$$\begin{aligned}f\left(\frac{1-\mu+\mu}{2}\right) &\le \frac{1}{1-2\mu}\int_{\mu}^{1-\mu} f(t)dt\\ &\le \frac{1}{4}\left(f(\mu)+2f\left(\frac{1-\mu+\mu}{2}\right)+f(1-\mu)\right)\\ &\le \frac{f(\mu)+f(1-\mu)}{2},\end{aligned}$$

and so

$$\begin{aligned} f\left(\frac{1}{2}\right) &\le \frac{1}{1-2\mu}\int_{\mu}^{1-\mu} f(t)dt \\ &\le \frac{1}{2}\left(f(\mu)+f\left(\frac{1}{2}\right)\right) \\ &\le f(\mu). \end{aligned}$$

Thus,

$$\begin{aligned} 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| &\le \frac{1}{1-2\mu}\int_{\mu}^{1-\mu}\left|\left|\left|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right|\right|\right|dv \\ &\le \frac{1}{2}\left(\left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\ &\le \left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|. \end{aligned} \tag{2.2}$$

Now, assume that $\frac{1}{2} \le \mu \le 1$. Then by applying (2.2) to 1 - μ, it follows that

$$\begin{aligned} 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| &\le \frac{1}{2\mu-1}\int_{1-\mu}^{\mu}\left|\left|\left|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right|\right|\right|dv \\ &\le \frac{1}{2}\left(\left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\ &\le \left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|. \end{aligned} \tag{2.3}$$

Since

$$\begin{aligned} &\lim_{\mu\to\frac{1}{2}}\frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu}\left|\left|\left|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right|\right|\right|dv\right| \\ &= \lim_{\mu\to\frac{1}{2}}\frac{1}{2}\left(\left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\ &= 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|, \end{aligned}$$

the inequalities in (2.1) follow by combining (2.2) and (2.3).

Applying the previous lemma to the function $f(v) = |||A^{v}XB^{1-v} + A^{1-v}XB^{v}|||$ on the interval $[\mu, \frac{1}{2}]$ when $0 \le \mu \le \frac{1}{2}$, and on the interval $[\frac{1}{2}, \mu]$ when $\frac{1}{2} \le \mu \le 1$, we obtain the following.

Theorem 2. *Let A, B, X be operators such that A, B are positive. Then for* $0 \le \mu \le 1$ *and for every unitarily invariant norm, we have*

$$\begin{aligned} &\left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right| \\ &\le \frac{2}{|1-2\mu|}\left|\int_{\mu}^{\frac{1}{2}}\left|\left|\left|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right|\right|\right|dv\right| \\ &\le \frac{1}{4}\left(\left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|+2\left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right|+\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\ &\le \frac{1}{2}\left(\left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right). \end{aligned} \tag{2.4}$$

The inequality (2.4) and the first inequality in (1.1) yield the following refinement of the first inequality in (1.1).

Corollary 1. *Let A,B,X be operators such that A, B are positive. Then for* $0 \le \mu \le 1$ *and for every unitarily invariant norm, we have*

$$\begin{aligned}
&2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| \\
&\le \left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}} + A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right| \\
&\le \frac{2}{|1-2\mu|}\left|\int_{\mu}^{\frac{1}{2}} \left|\left|\left|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\right|\right|\right| d\nu\right| \\
&\le \frac{1}{4}\left(\left|\left|\left|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\right|\right|\right| + 2\left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}} + A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right| + \left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\
&\le \frac{1}{2}\left(\left|\left|\left|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\right|\right|\right| + 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right). \\
&\le \left|\left|\left|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\right|\right|\right|.
\end{aligned} \tag{2.5}$$

Applying the previous lemma to the function $f(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$ on the interval $[0, \mu]$ when $0 \le \mu \le \frac{1}{2}$, and on the interval $[\mu, 1]$ when $\frac{1}{2} \le \mu \le 1$, we obtain the following theorem.

Theorem 3. *Let A, B, X be operators such that A, B are positive. Then*

(1) for $0 \le \mu \le \frac{1}{2}$ *and for every unitarily norm,*

$$\begin{aligned}
&\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right| \\
&\le \frac{1}{\mu}\int_{0}^{\mu} \left|\left|\left|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\right|\right|\right| d\nu \\
&\le \frac{1}{4}\left(|||AX + XB||| + 2\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right| + \left|\left|\left|A^{\mu} + XB^{1-\mu} + A^{1-\mu}XB^{\mu}\right|\right|\right|\right) \\
&\le \frac{1}{2}\left(|||AX + XB||| + \left|\left|\left|A^{\mu} + XB^{1-\mu} + A^{1-\mu}XB^{\mu}\right|\right|\right|\right).
\end{aligned} \tag{2.6}$$

(2) for $\frac{1}{2} \le \mu \le 1$ *and for every unitarily norm,*

$$\begin{aligned}
&\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}} + A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right| \\
&\le \frac{1}{1-\mu}\int_{\mu}^{1} \left|\left|\left|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\right|\right|\right| d\nu \\
&\le \frac{1}{4}\left(|||AX + XB||| + 2\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}} + A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right| + \left|\left|\left|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\right|\right|\right|\right) \\
&\le \frac{1}{2}\left(|||AX + XB||| + \left|\left|\left|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\right|\right|\right|\right).
\end{aligned} \tag{2.7}$$

Since the function $f(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$ is decreasing on the interval $[0, \frac{1}{2}]$ and increasing on the interval $[\frac{1}{2}, 1]$, and using the inequalities (2.6) and (2.7), we obtain the refinement of the second inequality in (1.1).

Corollary 2. *Let A, B, X be operators such that A, B are positive. Then for* $0 \le \mu \le 1$ *and for every unitarily invariant norm, we have*

(1) for $0 \le \mu \le \frac{1}{2}$ *and for every unitarily norm,*

$$\begin{aligned}
&|||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\\
&\le \left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right|\\
&\le \frac{1}{\mu}\int_0^{\mu} |||A^{v}XB^{1-v} + A^{1-v}XB^{v}||| \, dv\\
&\le \frac{1}{4}\left(|||AX + XB||| + 2\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right| + |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\right)\\
&\le \frac{1}{2}\left(|||AX + XB||| + |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\right).\\
&\le |||AX + XB|||.
\end{aligned} \tag{2.8}$$

(2) for $\frac{1}{2} \le \mu \le 1$ *and for every unitarily norm,*

$$\begin{aligned}
&|||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\\
&\le \left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}} + A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right|\\
&\le \frac{1}{1-\mu}\int_{\mu}^{1} |||A^{v}XB^{1-v} + A^{1-v}XB^{v}||| \, dv\\
&\le \frac{1}{4}\left(|||AX + XB||| + 2\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}} + A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right| + |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\right)\\
&\le \frac{1}{2}\left(|||AX + XB||| + |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\right)\\
&\le |||AX + XB|||.
\end{aligned} \tag{2.9}$$

It should be noticed that in the inequalities (2.6) to (2.9),

$$\begin{aligned}
&\lim_{\mu\to 0}\frac{1}{\mu}\int_0^{\mu} |||A^{v}XB^{1-v} + A^{1-v}XB^{v}||| \, dv\\
&\lim_{\mu\to 1}\frac{1}{1-\mu}\int_{\mu}^{1} |||A^{v}XB^{1-v} + A^{1-v}XB^{v}||| \, dv\\
&= |||AX + XB|||.
\end{aligned}$$

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Competing interests
The authors declare that they have no competing interests.

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