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Statistical summability $(C, 1)$ and a Korovkin type approximation theorem

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Abstract

The concept of statistical summability $(C, 1)$ has recently been introduced by Móricz [Jour. Math. Anal. Appl. **275**, 277-287 (2002)]. In this paper, we use this notion of summability to prove the Korovkin type approximation theorem by using the test functions $1, e^{-x}, e^{-2x}$. We also give here the rate of statistical summability $(C, 1)$ and apply the classical Baskakov operator to construct an example in support of our main result.

MSC: 41A10; 41A25; 41A36; 40A30; 40G15

Keywords: statistical convergence; statistical summability $(C, 1)$; positive linear operator; Korovkin type approximation theorem

1 Introduction and preliminaries

In 1951, Fast [6] presented the following definition of statistical convergence for sequences of real numbers. Let $K \subseteq \mathbb{N}$, the set of all natural numbers and $K_n = \{k \le n : k \in K\}$. Then the *natural density* of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where the vertical bars indicate the number of elements in the enclosed set. The sequence $x = (x_k)$ is said to be *statistically convergent* to L if for every $\epsilon > 0$, the set $K_\epsilon := \{k \in \mathbb{N} : |x_k - L| \ge \epsilon\}$ has natural density zero, *i.e.*, for each $\epsilon > 0$,

$$\lim_n \frac{1}{n}\bigl|\{j \le n : |x_j - L| \ge \epsilon\}\bigr| = 0.$$

In this case, we write $L = st\text{-}\lim x$. Note that every convergent sequence is statistically convergent but not conversely. Define the sequence $w = (w_n)$ by

$$w_n = \begin{cases} 1 & \text{if } n = k^2, k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Then x is statistically convergent to 0 but not convergent.

Recently, Móricz [12] has defined the concept of statistical summability $(C, 1)$ as follows:

For a sequence $x = (x_k)$, let us write $t_n = \frac{1}{n+1}\sum_{k=0}^{n} x_k$. We say that a sequence $x = (x_k)$ is *statistically summable* $(C, 1)$ if $st\text{-}\lim_{n\to\infty} t_n = L$. In this case, we write $L = C_1(st)\text{-}\lim x$.

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In the following example, we exhibit that a sequence is statistically summable $(C,1)$ but not statistically convergent. Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 1 & \text{if } k = m^2 - m, m^2 - m + 1, \ldots, m^2 - 1, \\ -m & \text{if } k = m^2, m = 2,3,4,\ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Then

$$\begin{aligned} t_n &= \frac{1}{n+1} \sum_{k=0}^{n} x_k \\ &= \begin{cases} \frac{s+1}{n+1} & \text{if } n = m^2 - m + s; s = 0,1,2,\ldots,m-1; m = 2,3,\ldots, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{1.3}$$

It is easy to see that $\lim_{n\to\infty} t_n = 0$ and hence $st\text{-}\lim_{n\to\infty} t_n = 0$, *i.e.*, a sequence $x = (x_k)$ is statistically summable $(C,1)$ to 0. On the other hand $st\text{-}\liminf_{k\to\infty} x_k = 0$ and $st\text{-}\limsup_{k\to\infty} x_k = 1$, since the sequence $(m^2)_{m=2}^{\infty}$ is statistically convergent to 0. Hence $x = (x_k)$ is not statistically convergent.

Let $C[a,b]$ be the space of all functions f continuous on $[a,b]$. We know that $C[a,b]$ is a Banach space with norm

$$\|f\|_\infty := \sup_{x\in[a,b]} |f(x)|, \quad f \in C[a,b].$$

The classical Korovkin approximation theorem is stated as follows [9]:

Let (T_n) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then $\lim_n \|T_n(f,x) - f(x)\|_\infty = 0$, for all $f \in C[a,b]$ if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_\infty = 0$, for $i = 0,1,2$, where $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$.

Recently, Mohiuddine [10] has obtained an application of almost convergence for single sequences in Korovkin-type approximation theorem and proved some related results. For the function of two variables, such type of approximation theorems are proved in [1] by using almost convergence of double sequences. Quite recently, in [13] and [14] the Korovkin type theorem is proved for statistical λ-convergence and statistical lacunary summability, respectively. For some recent work on this topic, we refer to [5, 7, 8, 11, 15, 16]. Boyanov and Veselinov [3] have proved the Korovkin theorem on $C[0,\infty)$ by using the test functions 1, e^{-x}, e^{-2x}. In this paper, we generalize the result of Boyanov and Veselinov by using the notion of statistical summability $(C,1)$ and the same test functions 1, e^{-x}, e^{-2x}. We also give an example to justify that our result is stronger than that of Boyanov and Veselinov [3].

2 Main result

Let $C(I)$ be the Banach space with the uniform norm $\|\cdot\|_\infty$ of all real-valued two dimensional continuous functions on $I = [0,\infty)$; provided that $\lim_{x\to\infty} f(x)$ is finite. Suppose that $L_n : C(I) \to C(I)$. We write $L_n(f;x)$ for $L_n(f(s);x)$; and we say that L is a positive operator if $L(f;x) \geq 0$ for all $f(x) \geq 0$.

The following statistical version of Boyanov and Veselinov's result can be found in [4].

Theorem A *Let* (T_k) *be a sequence of positive linear operators from* $C(I)$ *into* $C(I)$. *Then for all* $f \in C(I)$

$$st\text{-}\lim_{k\to\infty} \| T_k(f;x) - f(x) \|_\infty = 0$$

if and only if

$$st\text{-}\lim_{k\to\infty} \| T_k(1;x) - 1 \|_\infty = 0,$$

$$st\text{-}\lim_{k\to\infty} \| T_k(e^{-s};x) - e^{-x} \|_\infty = 0,$$

$$st\text{-}\lim_{k\to\infty} \| T_k(e^{-2s};x) - e^{-2x} \|_\infty = 0.$$

Now we prove the following result by using the notion of statistical summability $(C,1)$.

Theorem 2.1 *Let* (T_k) *be a sequence of positive linear operators from* $C(I)$ *into* $C(I)$. *Then for all* $f \in C(I)$

$$C_1(st)\text{-}\lim_{k\to\infty} \| T_k(f;x) - f(x) \|_\infty = 0 \tag{2.1}$$

if and only if

$$C_1(st)\text{-}\lim_{k\to\infty} \| T_k(1;x) - 1 \|_\infty = 0, \tag{2.2}$$

$$C_1(st)\text{-}\lim_{k\to\infty} \| T_k(e^{-s};x) - e^{-x} \|_\infty = 0, \tag{2.3}$$

$$C_1(st)\text{-}\lim_{k\to\infty} \| T_k(e^{-2s};x) - e^{-2x} \|_\infty = 0. \tag{2.4}$$

Proof Since each of 1, e^{-x}, e^{-2x} belongs to $C(I)$, conditions (2.2)-(2.4) follow immediately from (2.1). Let $f \in C(I)$. Then there exists a constant $M > 0$ such that $|f(x)| \le M$ for $x \in I$. Therefore,

$$|f(s) - f(x)| \le 2M, \quad -\infty < s,x < \infty. \tag{2.5}$$

It is easy to prove that for a given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(s) - f(x)| < \varepsilon, \tag{2.6}$$

whenever $|e^{-s} - e^{-x}| < \delta$ for all $x \in I$.

Using (2.5), (2.6), and putting $\psi_1 = \psi_1(s,x) = (e^{-s} - e^{-x})^2$, we get

$$|f(s) - f(x)| < \varepsilon + \frac{2M}{\delta^2}(\psi_1), \quad \forall |s-x| < \delta.$$

This is,

$$-\varepsilon - \frac{2M}{\delta^2}(\psi_1) < f(s) - f(x) < \varepsilon + \frac{2M}{\delta^2}(\psi_1).$$

Now, applying the operator $T_k(1;x)$ to this inequality, since $T_k(f;x)$ is monotone and linear, we obtain

$$T_k(1;x)\left(-\varepsilon - \frac{2M}{\delta^2}(\psi_1)\right) < T_k(1;x)\bigl(f(s) - f(x)\bigr) < T_k(1;x)\left(\varepsilon + \frac{2M}{\delta^2}(\psi_1)\right).$$

Note that x is fixed and so $f(x)$ is a constant number. Therefore,

$$-\varepsilon T_k(1;x) - \frac{2M}{\delta^2} T_{j,k}(\psi_1;x) < T_k(f;x) - f(x)T_k(1;x) < \varepsilon T_k(1;x) + \frac{2M}{\delta^2} T_k(\psi_1;x). \tag{2.7}$$

Also,

$$\begin{aligned} T_k(f;x) - f(x) &= T_k(f;x) - f(x)T_k(1;x) + f(x)T_k(1;x) - f(x) \\ &= T_k(f;x) - f(x)T_k(1;x) + f(x)\bigl[T_k(1;x) - 1\bigr]. \end{aligned} \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$T_k(f;x) - f(x) < \varepsilon T_k(1;x) + \frac{2M}{\delta^2} T_k(\psi_1;x) + f(x)\bigl[T_k(1;x) - 1\bigr]. \tag{2.9}$$

Now

$$\begin{aligned} T_k(\psi_1;x) &= T_k\bigl(\bigl(e^{-s} - e^{-x}\bigr)^2;x\bigr) \\ &= T_k\bigl(e^{-2s} - 2e^{-s}e^{-x} + e^{-2x};x\bigr) \\ &= T_k\bigl(e^{-2s};x\bigr) - 2e^{-x}T_k\bigl(e^{-s};x\bigr) + e^{-2x}T_k(1;x) \\ &= \bigl[T_k\bigl(e^{-2s};x\bigr) - e^{-2x}\bigr] - 2e^{-x}\bigl[T_k\bigl(e^{-s};x\bigr) - e^{-x}\bigr] + e^{-2x}\bigl[T_k(1;x) - 1\bigr]. \end{aligned}$$

Using (2.9), we obtain

$$\begin{aligned} T_k(f;x) - f(x) &< \varepsilon T_k(1;x) + \frac{2M}{\delta^2}\bigl\{\bigl[T_k\bigl(e^{-2s};x\bigr) - e^{-2x}\bigr] - 2e^{-x}\bigl[T_k\bigl(e^{-s};x\bigr) - e^{-x}\bigr] \\ &\quad + e^{-2x}\bigl[T_k(1;x) - 1\bigr]\bigr\} + f(x)\bigl[T_k(1;x) - 1\bigr] \\ &= \varepsilon\bigl[T_k(1;x) - 1\bigr] + \varepsilon + \frac{2M}{\delta^2}\bigl\{\bigl[T_k\bigl(e^{-2s};x\bigr) - e^{-2x}\bigr] - 2e^{-x}\bigl[T_k\bigl(e^{-s};x\bigr) - e^{-x}\bigr] \\ &\quad + e^{-2x}\bigl[T_k(1;x) - 1\bigr]\bigr\} + f(x)\bigl[T_k(1;x) - 1\bigr]. \end{aligned}$$

Therefore,

$$\begin{aligned} \bigl|T_k(f;x) - f(x)\bigr| &\leq \varepsilon + (\varepsilon + M)\bigl|T_k(1;x) - 1\bigr| + \frac{2M}{\delta^2}\bigl|T_k\bigl(e^{-2s};x\bigr) - e^{-2x}\bigr| \\ &\quad + \frac{4M}{\delta^2}\bigl|e^{-x}\bigr|\bigl|T_k\bigl(e^{-s};x\bigr) - e^{-x}\bigr| + \frac{2M}{\delta^2}\bigl|e^{-2x}\bigr|\bigl|T_k(1;x) - 1\bigr| \\ &\leq \varepsilon + \left(\varepsilon + M + \frac{2M}{\delta^2}\right)\bigl|T_k(1;x) - 1\bigr| + \frac{2M}{\delta^2}\bigl|T_k\bigl(e^{-2s};x\bigr) - e^{-2x}\bigr| \\ &\quad + \frac{4M}{\delta^2}\bigl|T_k\bigl(e^{-s};x\bigr) - e^{-x}\bigr|, \end{aligned}$$

since $|e^{-x}| \leq 1$ for all $x \in I$. Now taking $\sup_{x\in I}$, we get

$$\left\| T_k(f;x) - f(x)\right\|_\infty \leq \varepsilon + K\Big(\left\| T_k(1;x) - 1\right\|_\infty + \left\| T_k\big(e^{-s};x\big) - e^{-x}\right\|_\infty + \left\| T_k\big(e^{-2s};x\big) - e^{-2x}\right\|_\infty\Big), \tag{2.10}$$

where $K = \max\{\varepsilon + M + \frac{2M}{\delta^2}, \frac{2M}{\delta^2}, \frac{4M}{\delta^2}\}$.

Now replacing $T_k(\cdot,x)$ by $\frac{1}{m+1}\sum_{k=0}^{m} T_k(\cdot,x)$ and then by $B_m(\cdot,x)$ in (2.10) on both sides. For a given $r > 0$ choose $\varepsilon' > 0$ such that $\varepsilon' < r$. Define the following sets

$$D = \left\{m \leq n : \left\|B_m(f,x) - f(x)\right\|_\infty \geq r\right\},$$

$$D_1 = \left\{m \leq n : \left\|B_m(1,x) - 1\right\|_\infty \geq \frac{r-\varepsilon'}{3K}\right\},$$

$$D_2 = \left\{m \leq n : \left\|B_m(t,x) - e^{-x}\right\|_\infty \geq \frac{r-\varepsilon'}{3K}\right\},$$

$$D_3 = \left\{m \leq n : \left\|B_m\big(t^2,x\big) - e^{-2x}\right\|_\infty \geq \frac{r-\varepsilon'}{3K}\right\}.$$

Then $D \subset D_1 \cup D_2 \cup D_3$, and so $\delta(D) \leq \delta(D_1) + \delta(D_2) + \delta(D_3)$. Therefore, using conditions (2.2)-(2.4), we get

$$C_1(st)\text{-}\lim_n \left\| T_n(f,x) - f(x)\right\|_\infty = 0.$$

This completes the proof of the theorem. □

3 Rate of statistical summability (*C*, 1)

In this section, we study the rate of weighted statistical convergence of a sequence of positive linear operators defined from $C(I)$ into $C(I)$.

Definition 3.1 Let (a_n) be a positive nonincreasing sequence. We say that the sequence $x = (x_k)$ is statistically summable $(C,1)$ to the number L with the rate $o(a_n)$ if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{a_n}\left|\left\{m \leq n : |t_m - L| \geq \varepsilon\right\}\right| = 0.$$

In this case, we write $x_k - L = C_1(st) - o(a_n)$.

Now, we recall the notion of modulus of continuity. The modulus of continuity of $f \in C(I)$, denoted by $\omega(f,\delta)$ is defined by

$$\omega(f,\delta) = \sup_{|x-y|<\delta} \left|f(x) - f(y)\right|.$$

It is well known that

$$\left|f(x) - f(y)\right| \leq \omega(f,\delta)\left(\frac{|e^{-y} - e^{-x}|}{\delta} + 1\right). \tag{3.1}$$

Then we have the following result.

Theorem 3.2 *Let* (T_k) *be a sequence of positive linear operators from* $C(I)$ *into* $C(I)$. *Suppose that*

(i) $\|T_k(1;x) - 1\|_\infty = C_1(st) - o(a_n)$,

(ii) $\omega(f,\lambda_k) = C_1(st) - o(b_n)$, *where* $\lambda_k = \sqrt{T_k(\varphi_x;x)}$ *and* $\varphi_x(y) = (e^{-y} - e^{-x})^2$.

Then for all $f \in C(I)$, *we have*

$$\|T_k(f;x) - f(x)\|_\infty = C_1(st) - o(c_n),$$

where $c_n = \max\{a_n, b_n\}$.

Proof Let $f \in C(I)$ and $x \in I$. From (2.8) and (3.1), we can write

$$\begin{aligned}
|T_k(f;x) - f(x)| &\le T_k(|f(y) - f(x)|;x) + |f(x)||T_k(1;x) - 1| \\
&\le T_k\left(\frac{|e^{-y} - e^{-x}|}{\delta} + 1;x\right)\omega(f,\delta) + |f(x)||T_k(1;x) - 1| \\
&\le T_k\left(1 + \frac{1}{\delta^2}(e^{-y} - e^{-x})^2;x\right)\omega(f,\delta) + |f(x)||T_k(1;x) - 1| \\
&\le \left(T_k(1;x) + \frac{1}{\delta^2}T_k(\varphi_x;x)\right)\omega(f,\delta) + |f(x)||T_k(1;x) - 1| \\
&\le \omega(f,\delta)|T_k(1;x) - 1| + |f||T_k(1;x) - 1| + \omega(f,\delta) \\
&\quad + \frac{1}{\delta^2}\omega(f,\delta)T_k(\varphi_x;x).
\end{aligned}$$

Put $\delta = \lambda_k = \sqrt{T_k(\varphi_x;x)}$. Hence, we get

$$\begin{aligned}
\|T_k(f;x) - f(x)\|_\infty &\le \|f\|\|T_k(1;x) - 1\|_\infty + 2\omega(f,\lambda_k) + \omega(f,\lambda_k)\|T_k(1;x) - 1\|_\infty \\
&\le K\{\|T_k(1;x) - 1\|_\infty + \omega(f,\lambda_k) + \omega(f,\lambda_k)\|T_k(1;x) - 1\|_\infty\},
\end{aligned}$$

where $K = \max\{\|f\|, 2\}$. Now replacing $T_k(\cdot;x)$ by $\frac{1}{n+1}\sum_{k=0}^{n} T_k(\cdot;x) = L_n(\cdot;x)$

$$\|L_n(f;x) - f(x)\|_\infty \le K\{\|L_n(1;x) - 1\|_\infty + \omega(f,\lambda_k) + \omega(f,\lambda_k)\|T_k(1;x) - 1\|_\infty\}.$$

Using the Definition 3.1, and conditions (i) and (ii), we get the desired result. □

This completes the proof of the theorem.

4 Example and the concluding remark

In the following, we construct an example of a sequence of positive linear operators satisfying the conditions of Theorem 2.1 but does not satisfy the conditions of the Korovkin approximation theorem due to of Boyanov and Veselinov [3] and the conditions of Theorem A.

Consider the sequence of classical Baskakov operators [2]

$$V_n(f;x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n-1+k}{k}x^k(1+x)^{-n-k},$$

where $0 \le x, y < \infty$.

Let $L_n : C(I) \to C(I)$ be defined by

$$L_n(f;x) = (1 + x_n)V_n(f;x),$$

where the sequence $x = (x_n)$ is defined by (1.2). Note that this sequence is statistically summable $(C,1)$ to 0 but neither convergent nor statistically convergent. Now

$$V_n(1;x) = 1,$$
$$V_n\left(e^{-s};x\right) = \left(1 + x - xe^{-\frac{1}{n}}\right)^{-n},$$
$$V_n\left(e^{-2s};x^2\right) = \left(1 + x^2 - x^2 e^{-\frac{1}{n}}\right)^{-n},$$

we have that the sequence (L_n) satisfies the conditions (2.2), (2.3), and (2.4). Hence, by Theorem 2.1, we have

$$C_1(st)\text{-}\lim_{n\to\infty} \left\| L_n(f) - f \right\|_\infty = 0.$$

On the other hand, we get $L_n(f;0) = (1 + x_n)f(0)$, since $V_n(f;0) = f(0)$, and hence

$$\left\| L_n(f;x) - f(x) \right\|_\infty \geq \left| L_n(f;0) - f(0) \right| = x_n \left| f(0) \right|.$$

We see that (L_n) does not satisfy the conditions of the theorem of Boyanov and Veselinov as well as of Theorem A, since (x_n) is neither convergent nor statistically convergent.

Hence, our Theorem 2.1 is stronger than that of Boyanov and Veselinov [3] as well as Theorem A.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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References
1. Anastassiou, GA, Mursaleen, M, Mohiuddine, SA: Some approximation theorems for functions of two variables through almost convergence of double sequences. J. Comput. Anal. Appl. **13**(1), 37-40 (2011)
2. Becker, M: Global approximation theorems for Szasz-Mirakjan and Baskakov operators in polynomial weight spaces. Indiana Univ. Math. J. **27**(1), 127-142 (1978)
3. Boyanov, BD, Veselinov, VM: A note on the approximation of functions in an infinite interval by linear positive operators. Bull. Math. Soc. Sci. Math. Roum. **14**(62), 9-13 (1970)
4. Duman, O, Demirci, K, Karakuş, S: Statistical approximation for infinite intervals. Preprint
5. Edely, OHH, Mohiuddine, SA, Noman, AK: Korovkin type approximation theorems obtained through generalized statistical convergence. Appl. Math. Lett. **23**, 1382-1387 (2010)
6. Fast, H: Sur la convergence statistique. Colloq. Math. **2**, 241-244 (1951)
7. Gadjiev, AD, Orhan, C: Some approximation theorems via statistical convergence. Rocky Mt. J. Math. **32**, 129-138 (2002)

8. Karakuş, S, Demirci, K, Duman, O: Equi-statistical convergence of positive linear operators. J. Math. Anal. Appl. **339**, 1065-1072 (2008)
9. Korovkin, PP: Linear Operators and Approximation Theory. Hindustan Publ. Co., Delhi (1960)
10. Mohiuddine, SA: An application of almost convergence in approximation theorems. Appl. Math. Lett. **24**, 1856-1860 (2011)
11. Mohiuddine, SA, Alotaibi, A: Statistical convergence and approximation theorems for functions of two variables. J. Computational Analy. Appl. (accepted)
12. Móricz, F: Tauberian conditions under which statistical convergence follows from statistical summability $(C,1)$. J. Math. Anal. Appl. **275**, 277-287 (2002)
13. Mursaleen, M, Alotaibi, A: Statistical summability and approximation by de la Vallée-Poussin mean. Appl. Math. Lett. **24**, 320-324 (2011)
14. Mursaleen, M, Alotaibi, A: Statistical lacunary summability and a Korovkin type approximation theorem. Ann. Univ. Ferrara **57**(2), 373-381 (2011)
15. Mursaleen, M, Karakaya, V, Ertürk, M, Gürsoy, F: Weighted statistical convergence and its application to Korovkin type approximation theorem. Appl. Math. Comput. **218**, 9132-9137 (2012)
16. Srivastava, HM, Mursaleen, M, Khan, A: Generalized equi-statistical convergence of positive linear operators and associated approximation theorems. Math. Comput. Model. **55**, 2040-2051 (2012)

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