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A new note on generalized absolute matrix summability

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Abstract

This paper gives necessary and sufficient conditions in order that a series $\sum a_n \lambda_n$ should be summable $|B|_k$, $k \geq 1$, whenever $\sum a_n$ is summable $|A|$. Some new results have also been obtained.

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1 Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n) . Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^n p_v \rightarrow \infty \quad \text{as } (n \rightarrow \infty), \quad (P_{-i} = p_{-i} = 0, i \geq 1). \quad (1)$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v \quad (2)$$

defines the sequence (t_n) of the Riesz means of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [3]). The series $\sum a_n$ is said to be summable $|R, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty. \quad (3)$$

Let $A = (a_{nv})$ be a normal matrix, *i.e.*, a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^n a_{nv} s_v, \quad n = 0, 1, \dots \quad (4)$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see [7])

$$\sum_{n=1}^{\infty} n^{k-1} |\bar{\Delta}A_n(s)|^k < \infty, \quad (5)$$

where

$$\bar{\Delta}A_n(s) = A_n(s) - A_{n-1}(s).$$

If we take $a_{nv} = \frac{p_v}{p_n}$, then $|A|_k$ summability is the same as $|R, p_n|_k$ summability.

Before stating the main theorem we must first introduce some further notations.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^n a_{ni}, \quad n, v = 0, 1, \dots \quad (6)$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \dots \quad (7)$$

It may be noted that \bar{A} and \hat{A} are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^n a_{nv} s_v = \sum_{v=0}^n \bar{a}_{nv} a_v \quad (8)$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^n \hat{a}_{nv} a_v. \quad (9)$$

If A is a normal matrix, then $A' = (a'_{nv})$ will denote the inverse of A . Clearly if A is normal, then $\hat{A} = (\hat{a}_{nv})$ is normal and has two-sided inverse $\hat{A}' = (\hat{a}'_{nv})$, which is also normal (see [2]).

Sarıgöl [6] has proved the following theorem for $|R, p_n|_k$ summability method.

Theorem A *Suppose that (p_n) and (q_n) are positive sequences with $P_n \rightarrow \infty$ and $Q_n \rightarrow \infty$ as $n \rightarrow \infty$. Then $\sum a_n \lambda_n$ is summable $|R, q_n|_k$, $k \geq 1$ whenever $\sum a_n$ is summable $|R, p_n|_k$, if and only if*

$$\lambda_n = O\left\{n^{\frac{1}{k}-1} \frac{q_n P_n}{p_n Q_n}\right\}, \quad (10)$$

$$W_n \Delta(Q_{n-1} \lambda_n) = O\left(\frac{p_n}{P_n}\right), \quad (11)$$

$$Q_n \lambda_{n+1} W_n = O(1) \quad (12)$$

provided that

$$W_n = \left\{ \sum_{v=n+1}^{\infty} v^{k-1} \left(\frac{q_v}{Q_v Q_{v-1}} \right)^k \right\}^{\frac{1}{k}} < \infty.$$

Theorem B *The $|R, p_n|$ summability implies the $|R, q_n|_k$, $k \geq 1$, summability if and only if the following conditions hold:*

$$\frac{q_v}{Q_v} \cdot \frac{P_v}{p_v} = O\{v^{\frac{1}{k}-1}\}, \tag{13}$$

$$|\Delta Q_{v-1}| \cdot W_v = O\left(\frac{P_v}{p_v}\right), \tag{14}$$

$$Q_v W_v = O(1), \tag{15}$$

where

$$W_v = \left\{ \sum_{i=v+1}^{\infty} i^{k-1} \left(\frac{q_i}{Q_i Q_{i-1}} \right)^k \right\}^{\frac{1}{k}} < \infty$$

and we regarded that the above series converges for each v and Δ is the forward difference operator.

It may be remarked that the above theorem has been proved by Orhan and Sarigöl [5].

Lemma ([4]) *$A = (a_{nv}) \in (l_1, l_k)$ if and only if*

$$\sup_v \sum_{n=1}^{\infty} |a_{nv}|^k < \infty \tag{16}$$

for the cases $1 \leq k < \infty$, where (l_1, l_k) denotes the set of all matrices A which map l_1 into $l_k = \{x = (x_n) : \sum |x_n|^k < \infty\}$.

2 Main theorem

The aim of this paper is to generalize Theorem A for the $|A|$ and $|B|_k$ summabilities. Therefore we shall prove the following theorem.

Theorem *Let $k \geq 1$, $A = (a_{nv})$ and $B = (b_{nv})$ be two positive normal matrices such that*

$$a_{n-1,v} \geq a_{nv}, \quad \text{for } n \geq v + 1, \tag{17}$$

$$\bar{a}_{n0} = 1, \quad n = 0, 1, 2, \dots \tag{18}$$

Then, in order that $\sum a_n \lambda_n$ is summable $|B|_k$ whenever $\sum a_n$ is summable $|A|$, it is necessary that

$$|\lambda_n| = O\left(n^{\frac{1}{k}-1} \frac{a_{nn}}{b_{nn}}\right), \tag{19}$$

$$\sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_{\nu}(\hat{b}_{n\nu}\lambda_{\nu})|^k = O(a_{\nu\nu}^k), \tag{20}$$

$$\sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k = O(1). \tag{21}$$

Also (19)-(21) and

$$\bar{b}_{n0} = 1, \quad n = 0, 1, 2, \dots, \tag{22}$$

$$a_{nn} - a_{n+1,n} = O(a_{nn}a_{n+1,n+1}), \tag{23}$$

$$\sum_{\nu=r+2}^{\infty} |\hat{b}_{n\nu}\hat{a}'_{\nu r}\lambda_{\nu}| = O(|\hat{b}_{n,r+1}\lambda_{r+1}|) \tag{24}$$

are sufficient for the consequent to hold.

It should be noted that if we take $a_{n\nu} = \frac{p_{\nu}}{P_n}$ and $b_{n\nu} = \frac{q_{\nu}}{Q_n}$, then we get Theorem A. Also if we take $\lambda_n = 1$, then we get Theorem B.

Proof of the Theorem

Necessity. Let (x_n) and (y_n) denote A -transform and B -transform of the series $\sum a_n$ and $\sum a_n\lambda_n$, respectively. Then, by (8) and (9), we have

$$\bar{\Delta}x_n = \sum_{\nu=0}^n \hat{a}_{n\nu}a_{\nu} \quad \text{and} \quad \bar{\Delta}y_n = \sum_{\nu=0}^n \hat{b}_{n\nu}a_{\nu}\lambda_{\nu}. \tag{25}$$

For $k \geq 1$, we define

$$A = \left\{ (a_i) : \sum a_i \text{ is summable } |A| \right\}, \quad B = \left\{ (a_i\lambda_i) : \sum a_i\lambda_i \text{ is summable } |B|_k \right\}.$$

Then it is routine to verify that these are BK-spaces, if normed by

$$\|X\| = \left\{ \sum_{n=0}^{\infty} |\bar{\Delta}x_n| \right\} \tag{26}$$

and

$$\|Y\| = \left\{ \sum_{n=0}^{\infty} n^{k-1} |\bar{\Delta}y_n|^k \right\}^{\frac{1}{k}} \tag{27}$$

respectively. Since $\sum a_n$ is summable $|A|$ implies $\sum a_n\lambda_n$ is summable $|B|_k$, by the hypothesis of the theorem,

$$\|X\| < \infty \quad \Rightarrow \quad \|Y\| < \infty.$$

Now consider the inclusion map $c : A \rightarrow B$ defined by $c(x) = x$. This is continuous, which is immediate as A and B are BK-spaces. Thus there exists a constant M such that

$$\|Y\| \leq M\|X\|. \tag{28}$$

By applying (25) to $a_\nu = e_\nu - e_{\nu+1}$ (e_ν is the ν th coordinate vector), we have

$$\bar{\Delta}x_n = \begin{cases} 0, & \text{if } n < \nu, \\ \hat{a}_{n\nu}, & \text{if } n = \nu, \\ \Delta_\nu \hat{a}_{n\nu}, & \text{if } n > \nu \end{cases}$$

and

$$\bar{\Delta}y_n = \begin{cases} 0, & \text{if } n < \nu, \\ \hat{b}_{n\nu}\lambda_\nu, & \text{if } n = \nu, \\ \Delta_\nu(\hat{b}_{n\nu}\lambda_\nu), & \text{if } n > \nu. \end{cases}$$

So (26) and (27) give us

$$\|X\| = \left\{ a_{\nu\nu} + \sum_{n=\nu+1}^{\infty} |\Delta_\nu \hat{a}_{n\nu}| \right\}$$

and

$$\|Y\| = \left\{ \nu^{k-1} b_{\nu\nu} |\lambda_\nu|^k + \sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)|^k \right\}^{\frac{1}{k}}.$$

Hence it follows from (28) that

$$\nu^{k-1} b_{\nu\nu} |\lambda_\nu|^k + \sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu \hat{b}_{n\nu}\lambda_\nu|^k \leq M^k a_{\nu\nu}^k + M^k \sum_{n=\nu+1}^{\infty} |\Delta_\nu \hat{a}_{n\nu}|^k.$$

Using (17), we can find

$$\nu^{k-1} b_{\nu\nu} |\lambda_\nu|^k + \sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)|^k = O(a_{\nu\nu}^k).$$

The above inequality will be true if and only if each term on the left-hand side is $O(a_{\nu\nu}^k)$.

Taking the first term,

$$\nu^{k-1} b_{\nu\nu} |\lambda_\nu|^k = O(a_{\nu\nu}^k)$$

then

$$|\lambda_\nu| = O\left(\nu^{\frac{1}{k}-1} \frac{a_{\nu\nu}}{b_{\nu\nu}}\right)$$

which verifies that (19) is necessary. Using the second term, we have

$$\sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)|^k = O(a_{\nu\nu}^k)$$

which is condition (20). Now, if we apply (22) to $a_\nu = e_{\nu+1}$, we have

$$\bar{\Delta}x_n = \begin{cases} 0, & \text{if } n \leq \nu, \\ \hat{a}_{n,\nu+1}, & \text{if } n > \nu \end{cases}$$

and

$$\bar{\Delta}y_n = \begin{cases} 0, & \text{if } n \leq \nu, \\ \hat{b}_{n,\nu+1}\lambda_{\nu+1}, & \text{if } n > \nu, \end{cases}$$

respectively. Hence

$$\|X\| = \left\{ \sum_{n=\nu+1}^{\infty} |\hat{a}_{n,\nu+1}| \right\},$$

$$\|Y\| = \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k \right\}^{\frac{1}{k}}.$$

Hence it follows from (28) that

$$\sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k \leq M^k \left\{ \sum_{n=\nu+1}^{\infty} |\hat{a}_{n,\nu+1}| \right\}^k.$$

Using (18) we can find

$$\sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k = O(1)$$

which is condition (21).

Sufficiency. We use the notations of necessity. Then

$$\bar{\Delta}x_n = \sum_{\nu=0}^n \hat{a}_{n\nu} a_\nu \tag{29}$$

which implies

$$a_\nu = \sum_{r=0}^{\nu} \hat{a}'_{\nu r} \bar{\Delta}x_r. \tag{30}$$

In this case

$$\bar{\Delta}y_n = \sum_{\nu=0}^n \hat{b}_{n\nu} a_\nu \lambda_\nu = \sum_{\nu=0}^n \hat{b}_{n\nu} \lambda_\nu \sum_{r=0}^{\nu} \hat{a}'_{\nu r} \bar{\Delta}x_r.$$

On the other hand, since

$$\hat{b}_{n0} = \bar{b}_{n0} - \bar{b}_{n-1,0}$$

by (22), we have

$$\begin{aligned}
 \bar{\Delta}y_n &= \sum_{v=1}^n \hat{b}_{nv}\lambda_v \left\{ \sum_{r=0}^v \hat{a}'_{vr} \bar{\Delta}x_r \right\} \\
 &= \sum_{v=1}^n \hat{b}_{nv}\lambda_v \left\{ \hat{a}'_{vv} \bar{\Delta}x_v + \hat{a}'_{v,v-1} \bar{\Delta}x_{v-1} + \sum_{r=0}^{v-2} \hat{a}'_{vr} \bar{\Delta}x_r \right\} \\
 &= \sum_{v=1}^n \hat{b}_{nv}\lambda_v \hat{a}'_{vv} \bar{\Delta}x_v + \sum_{v=1}^n \hat{b}_{nv}\lambda_v \hat{a}'_{v,v-1} \bar{\Delta}x_{v-1} + \sum_{v=1}^n \hat{b}_{nv}\lambda_v \sum_{r=0}^{v-2} \hat{a}'_{vr} \bar{\Delta}x_r \\
 &= \hat{b}_{nn}\lambda_n \hat{a}'_{nn} \bar{\Delta}x_n + \sum_{v=1}^{n-1} (\hat{b}_{nv}\lambda_v \hat{a}'_{vv} + \hat{b}_{n,v+1}\lambda_{v+1} \hat{a}'_{v+1,v}) \bar{\Delta}x_v \\
 &\quad + \sum_{r=0}^{n-2} \bar{\Delta}x_r \sum_{v=r+2}^n \hat{b}_{nv}\lambda_v \hat{a}'_{vr}.
 \end{aligned} \tag{31}$$

By considering the equality

$$\sum_{k=v}^n \hat{a}'_{nk} \hat{a}_{kv} = \delta_{nv},$$

where δ_{nv} is the Kronecker delta, we have that

$$\begin{aligned}
 \hat{b}_{nv}\lambda_v \hat{a}'_{vv} + \hat{b}_{n,v+1}\lambda_{v+1} \hat{a}'_{v+1,v} &= \frac{\hat{b}_{nv}\lambda_v}{\hat{a}_{vv}} + \hat{b}_{n,v+1}\lambda_{v+1} \left(-\frac{\hat{a}_{v+1,v}}{\hat{a}_{vv}\hat{a}_{v+1,v+1}} \right) \\
 &= \frac{\hat{b}_{nv}\lambda_v}{a_{vv}} - \frac{\hat{b}_{n,v+1}\lambda_{v+1}(\bar{a}_{v+1,v} - \bar{a}_{v,v})}{a_{vv}a_{v+1,v+1}} \\
 &= \frac{\hat{b}_{nv}\lambda_v}{a_{vv}} - \frac{\hat{b}_{n,v+1}\lambda_{v+1}(a_{v+1,v+1} + a_{v+1,v} - a_{vv})}{a_{vv}a_{v+1,v+1}} \\
 &= \frac{\Delta_v(\hat{b}_{nv}\lambda_v)}{a_{vv}} + \hat{b}_{n,v+1}\lambda_{v+1} \frac{a_{vv} - a_{v+1,v}}{a_{vv}a_{v+1,v+1}}
 \end{aligned}$$

and so

$$\begin{aligned}
 \bar{\Delta}y_n &= \frac{b_{nn}\lambda_n}{a_{nn}} \bar{\Delta}x_n + \sum_{v=1}^{n-1} \frac{\Delta_v(\hat{b}_{nv}\lambda_v)}{a_{vv}} \bar{\Delta}x_v + \sum_{v=1}^{n-1} \hat{b}_{n,v+1}\lambda_{v+1} \frac{a_{vv} - a_{v+1,v}}{a_{vv}a_{v+1,v+1}} \bar{\Delta}x_v \\
 &\quad + \sum_{r=0}^{n-2} \bar{\Delta}x_r \sum_{v=r+2}^n \hat{b}_{nv}\lambda_v \hat{a}'_{vr}.
 \end{aligned}$$

Let

$$\begin{aligned}
 T_n(1) &= \frac{b_{nn}\lambda_n}{a_{nn}} \bar{\Delta}x_n + \sum_{v=1}^{n-1} \frac{\Delta_v(\hat{b}_{nv}\lambda_v)}{a_{vv}} \bar{\Delta}x_v + \sum_{v=1}^{n-1} \hat{b}_{n,v+1}\lambda_{v+1} \frac{a_{vv} - a_{v+1,v}}{a_{vv}a_{v+1,v+1}} \bar{\Delta}x_v, \\
 T_n(2) &= \sum_{r=0}^{n-2} \bar{\Delta}x_r \sum_{v=r+2}^n \hat{b}_{nv}\lambda_v \hat{a}'_{vr}.
 \end{aligned}$$

Since

$$|T_n(1) + T_n(2)|^k \leq 2^k (|T_n(1)|^k + |T_n(2)|^k)$$

to complete the proof of Theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} |T_n(i)|^k < \infty \quad \text{for } i = 1, 2.$$

Then

$$\begin{aligned} \overline{T_n(1)} &= n^{1-\frac{1}{k}} T_n(1) \\ &= n^{1-\frac{1}{k}} \frac{b_{nn}\lambda_n}{a_{nn}} \bar{\Delta}x_n + n^{1-\frac{1}{k}} \sum_{v=1}^{n-1} \frac{\Delta_v(\hat{b}_{nv}\lambda_v)}{a_{vv}} \bar{\Delta}x_v + n^{1-\frac{1}{k}} \sum_{v=1}^{n-1} \hat{b}_{n,v+1}\lambda_{v+1} \frac{a_{vv} - a_{v+1,v}}{a_{vv}a_{v+1,v+1}} \bar{\Delta}x_v \\ &= \sum_{v=1}^{\infty} c_{nv} \bar{\Delta}x_v, \end{aligned}$$

where

$$c_{nv} = \begin{cases} n^{1-\frac{1}{k}} \left(\frac{\Delta_v(\hat{b}_{nv}\lambda_v)}{a_{vv}} + \hat{b}_{n,v+1}\lambda_{v+1} \frac{a_{vv} - a_{v+1,v}}{a_{vv}a_{v+1,v+1}} \right), & \text{if } 1 \leq v \leq n-1, \\ n^{1-\frac{1}{k}} \frac{b_{nn}\lambda_n}{a_{nn}}, & \text{if } v = n, \\ 0, & \text{if } v > n. \end{cases}$$

Now

$$\sum |\overline{T_n(1)}|^k < \infty \quad \text{whenever} \quad \sum |\bar{\Delta}x_n| < \infty$$

is equivalently

$$\sup_v \sum_{n=1}^{\infty} |c_{nv}|^k < \infty \tag{32}$$

by Lemma. But it follows from conditions (20), (21) and (23) that

$$\begin{aligned} \sum_{n=v}^{\infty} |c_{nv}|^k &= O(1) \left\{ n^{k-1} \left| \frac{b_{nn}\lambda_n}{a_{nn}} \right|^k + \sum_{n=v+1}^{\infty} n^{k-1} \left| \frac{\Delta_v(\hat{b}_{nv}\lambda_v)}{a_{vv}} + \hat{b}_{n,v+1}\lambda_{v+1} \frac{a_{vv} - a_{v+1,v}}{a_{vv}a_{v+1,v+1}} \right|^k \right\} \\ &= O(1) \quad \text{as } v \rightarrow \infty. \end{aligned}$$

Finally,

$$\overline{T_n(2)} = n^{1-\frac{1}{k}} T_n(2) = n^{1-\frac{1}{k}} \sum_{r=0}^{n-2} \bar{\Delta}x_r \sum_{v=r+2}^n \hat{b}_{nv}\hat{a}'_{vr}\lambda_v = \sum_{r=0}^{\infty} d_{nr} \bar{\Delta}x_r,$$

where

$$d_{nr} = \begin{cases} n^{1-\frac{1}{k}} \sum_{v=r+2}^n \hat{b}_{nv}\hat{a}'_{vr}\lambda_v, & \text{if } 0 \leq r \leq n-2, \\ 0, & \text{if } r > n-2. \end{cases}$$

Now

$$\sum |\overline{T_n(2)}|^k < \infty \quad \text{whenever} \quad \sum |\bar{\Delta}x_n| < \infty$$

is equivalently

$$\sup_r \sum_{n=1}^{\infty} |d_{nr}|^k < \infty \tag{33}$$

by Lemma. But it follows from conditions (21) and (24) that

$$\begin{aligned} \sum_{n=r+2}^{\infty} |d_{nr}|^k &= O(1) \sum_{n=r+2}^{\infty} n^{k-1} \left\{ \sum_{v=r+2}^{\infty} |\hat{b}_{nv} \hat{a}'_{vr} \lambda_v| \right\}^k \\ &= O(1) \sum_{n=r+2}^{\infty} n^{k-1} |\hat{b}_{n,r+1} \lambda_{r+1}|^k \\ &= O(1) \quad \text{as } r \rightarrow \infty. \end{aligned}$$

Therefore, we have

$$\sum_{n=1}^{\infty} n^{k-1} |T_n(i)|^k < \infty \quad \text{for } i = 1, 2.$$

This completes the proof of the Theorem. □

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References

1. Bor, H: On the relative strength of two absolute summability methods. *Proc. Am. Math. Soc.* **113**, 1009-1012 (1991)
2. Cooke, RG: *Infinite Matrices and Sequence Spaces*. Macmillan & Co., London (1950)
3. Hardy, GH: *Divergent Series*. Oxford University Press, Oxford (1949)
4. Maddox, IJ: *Elements of Functional Analysis*. Cambridge University Press, Cambridge (1970)
5. Orhan, C, Sarigöl, MA: On absolute weighted mean summability. *Rocky Mt. J. Math.* **23**(3), 1091-1098 (1993)
6. Sarigöl, MA: On the absolute Riesz summability factors of infinite series. *Indian J. Pure Appl. Math.* **23**(12), 881-886 (1992)
7. Tanovič-Miller, N: On strong summability. *Glas. Mat.* **34**, 87-97 (1979)

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