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Hypersingular integral operators on modulation spaces for $0 < p < 1$

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Abstract

The purpose of this paper is to investigate the mapping properties of the hypersingular integral operators on general weighted modulation spaces $M_s^{p,q}(\mathbb{R}^n)$ for $0 < p < 1$. Our results show that modulation spaces are good substitutions for Lebesgue spaces.

MSC: 42B20; 42B25

Keywords: hypersingular integral operator; modulation space; Fourier multiplier

1 Introduction

The modulation spaces $M^{p,q}$, $p,q \in [1,\infty]$ were first introduced by Feichtinger in [10] and [11]. Their classical definition is based on the notion of short time Fourier transform (STFT). Recently, Kobayashi [15] extended the classical definition to general case for $0 < p,q \le \infty$. For the more general definitions, involving different kinds of weight functions, both in the time and the frequency variables, we refer the readers to [13]. During the last ten years, modulation spaces have not only become useful function spaces for time-frequency analysis, they have also been employed to study boundedness properties of pseudo-differential operators, Fourier multipliers, Fourier integral operators and well-posedness of solutions to PDE's. For more details of the applications of these spaces, the readers can see [1–3, 9, 12, 16, 18–21] and references therein.

In this paper, we are mainly concerned about the boundedness of the hypersingular integral operators along curves on weighted modulation spaces $M_s^{p,q}(\mathbb{R}^n)$ for $0 < p < 1$, $0 < q \le \infty$ and $s \in \mathbb{R}$. From our results, we will see that modulation spaces are good substitutions for Lebesgue spaces.

Suppose $\alpha, \beta > 0$ and

$$K_\alpha(x) = \frac{e^{i|x|^{-\beta}}}{|x|^{n+\alpha}}\eta(x), \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}, \tag{1.1}$$

where $\eta(x)$ is a smooth, compactly supported, radial function which is equal to 1 in the unit ball.

Define the convolution operator

$$Tf(x) = (K_\alpha * f)(x), \quad \text{for } f \in C_0^\infty(\mathbb{R}^n). \tag{1.2}$$

It is well known that the L^p-boundedness of the operator T defined by (1.2) is closely related to the values of the parameters α and β. In [23], Wainger proved that T is a bounded operator on L^2 if and only if $\alpha \le \frac{n\beta}{2}$. Interpolating this result with the result of weak type (1,1), Wainger obtained the L^p-boundedness of the operator T. Inspired by Wainger, in [7], the authors considered the mapping properties of the operator T on weighted modulation $M_s^{p,q}(\mathbb{R}^n)$ for $1 \le p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. In this paper, we extend such result to the case $0 < p < 1$. Our result is as follows.

Theorem 1.1 *Assume* $0 < \alpha \le \frac{\beta}{2}$. *Then the operator* T *defined by (1.2) is bounded on weighted modulation spaces* $M_s^{p,q}(\mathbb{R}^n)$ *for* $0 < p < 1$, $0 < q \le \infty$ *and* $s \in \mathbb{R}$.

The hypersingular integral operator along curves which is defined by

$$T_{\alpha,\beta}f(x,y) = p.v. \int_{-1}^{1} f\bigl(x-t, y-\gamma(t)\bigr)\frac{e^{-2\pi i|t|^{-\beta}}}{t|t|^{\alpha}}\,dt \tag{1.3}$$

has been studied by many mathematicians (see [4–6, 8, 17, 22, 24]), where $x, y \in \mathbb{R}$ and $\alpha, \beta > 0$. This operator is initially studied by Zielinski in his PhD thesis (see [24]). He showed that if $\gamma(t) = t^2$, then $T_{\alpha,\beta}$ is bounded on $L^2(\mathbb{R}^2)$ if and only if $\beta \ge 3\alpha$. This result was later improved by Chandrana in [4]. He considered the general homogeneous curves $\gamma(t) = |t|^k$ or $\gamma(t) = |t|^k \operatorname{sgn} t$ for $k \ge 2$ and proved that if $\beta > 3\alpha > 0$, then the operator $T_{\alpha,\beta}$ is bounded on $L^p(\mathbb{R}^2)$ for

$$1 + \frac{3\alpha(\beta+1)}{\beta(\beta+1)+(\beta-3\alpha)} < p < \frac{\beta(\beta+1)+(\beta-3\alpha)}{3\alpha(\beta+1)} + 1.$$

Recently, in [8], the authors have considered the mapping properties of the operator $T_{\alpha,\beta}$ defined by (1.3) on weighted modulation spaces $M_s^{p,q}(\mathbb{R}^n)$ for $1 \le p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. For the case $0 < p < 1$, we prove the following result.

Theorem 1.2 *Let* $\gamma(t) = |t|^k$ *or* $\gamma(t) = |t|^k \operatorname{sgn} t$ *for* $k \ge 2$ *and the operator* $T_{\alpha,\beta}$ *be defined by (1.3). If* $\beta \ge 3\alpha$, *then the operator* $T_{\alpha,\beta}$ *is bounded on* $M_s^{p,q}(\mathbb{R}^2)$ *for* $0 < p < 1$, $0 < q \le \infty$ *and* $s \in \mathbb{R}$.

In [5], the authors extended the results of [4] and [24] to high dimensions. They investigated the L^p-boundedness of the operator

$$T_{n,\alpha,\beta}f(x) = p.v. \int_{-1}^{1} f\bigl(x-\Gamma_\theta(t)\bigr)\frac{e^{-2\pi i|t|^{-\beta}}}{t|t|^{\alpha}}\,dt, \tag{1.4}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ and

$$\Gamma_\theta(t) = \bigl(\theta_1|t|^{p_1}, \theta_2|t|^{p_2}, \ldots, \theta_n|t|^{p_n}\bigr) \quad \text{or} \quad \operatorname{sgn} t\bigl(\theta_1|t|^{p_1}, \theta_2|t|^{p_2}, \ldots, \theta_n|t|^{p_n}\bigr).$$

They proved that $T_{n,\alpha,\beta}$ is bounded on $L^2(\mathbb{R}^n)$ if and only if $\beta \ge (n+1)\alpha$. While $\beta > (n+1)\alpha$, the operator $T_{n,\alpha,\beta}$ is bounded on $L^p(\mathbb{R}^n)$ for $\frac{2\beta}{2\beta-(n+1)\alpha} < p < \frac{2\beta}{(n+1)\alpha}$. Inspired by Chen, Fan, Wang and Zhu, the authors of [8] investigated the boundedness of the operator defined by (1.4) on modulation spaces $M_s^{p,q}(\mathbb{R}^n)$ for $1 \le p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. While for $0 < p < 1$, we prove the following result.

Theorem 1.3 *Suppose that* $p_1, p_2, \dots, p_n$ *are distinct positive numbers and the operator* $T_{n,\alpha,\beta}$ *is defined as (1.4). If* $\beta \geq (n+1)\alpha$*, then* $T_{n,\alpha,\beta}$ *is bounded on* $M_s^{p,q}(\mathbb{R}^n)$ *for* $0 < p < 1$, $0 < q \leq \infty$ *and* $s \in \mathbb{R}$.

Obviously, Theorem 1.1 is the extension of Theorem 1.2 in [7]. Theorems 1.2 and 1.3 are respectively the extensions of Theorems 1.1 and 1.2 in [8].

In what follows, we always denote C to be a positive constant that may be different at each place, but independent of the essential variables.

This paper is organized as follows. In Section 2, we give the definitions and basic properties of modulation spaces. Section 3 is devoted to the proofs of our main results.

2 Basic definition and important lemma

The following notations will be used throughout this paper. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ be the topological dual of $\mathcal{S}(\mathbb{R}^n)$. For a function f in $\mathcal{S}(\mathbb{R}^n)$, its Fourier transform is defined by $\hat{f}(\omega) = \int f(t)e^{-2\pi i\omega\cdot t}\,dt$, and its inverse Fourier transform is $\check{f}(t) = \hat{f}(-t)$. The translation and the modulation operators are defined by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i\omega\cdot t} f(t)$$

for every $x, \omega \in \mathbb{R}^n$. For $s \in \mathbb{R}$ and $x \in \mathbb{R}^n$, the weight function $\langle x\rangle^s = (1+|x|^2)^{\frac{s}{2}}$.

Let g be a non-zero Schwartz function and $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$, the weighted modulation space $M_s^{p,q}(\mathbb{R}^n)$ is defined as the closure of the Schwartz class with respect to the norm

$$\|f\|_{M_s^{p,q}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left|V_g f(x,w)\right|^p dx\right)^{\frac{q}{p}} \langle w\rangle^{sq}\, dw\right)^{\frac{1}{q}},$$

with obvious modifications for p or $q = \infty$, where $V_g f(x,w)$ is the so-called short time Fourier transform (STFT), which is defined by

$$V_g f(x,w) = \langle f, M_\xi T_x g\rangle = \int e^{-2\pi i w\cdot y} f(y)\overline{g(y-x)}\,dy,$$

i.e., the Fourier transform $\mathcal{F}$ applied to $f\overline{T_x g}$.

Recently, the above definition has been generated by Kobayashi in [15] to the case $0 < p, q \leq \infty$. In his definition, the function g is restricted to the space $\Phi^\delta(\mathbb{R}^n)$, which is defined by

Definition 2.1 For $\delta > 0$, we define $\Phi^\delta(\mathbb{R}^n)$ to be the spaces of all $g \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\operatorname{supp}\hat{g} \subset \{\xi : |\xi| \leq 1\} \quad \text{and} \quad \sum_{k\in\mathbb{Z}^n} \hat{g}(\xi - \delta k) = 1.$$

We may choose a sufficiently small δ, such that the function space $\Phi^\delta(\mathbb{R}^n)$ is not empty.

Definition 2.2 Given a $g \in \Phi^\delta(\mathbb{R}^n)$, and $0 < p, q \leq \infty$, $s \in \mathbb{R}$, we define the modulation space $M_s^{p,q}(\mathbb{R}^n)$to be the space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the quasinorm

$$\|f\|_{M_s^{p,q}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left|f * (M_w g)(x)\right|^p \, dx\right)^{\frac{q}{p}} \langle w\rangle^{sq} \, dw\right)^{\frac{1}{q}}$$

is finite, with obvious modifications for p or $q = \infty$.

The following basic properties of modulation spaces, which play important roles in this article, can be found in [14] and [15].

Lemma 2.1 *Let $0 < p, q \le \infty$ and $g \in \Phi^\delta(\mathbb{R}^n)$. Then*

(1) Different test functions $g_1, g_2 \in \Phi^\delta(\mathbb{R}^n)$ define the same space and equivalent quasi-norms on $M_s^{p,q}(\mathbb{R}^n)$.

(2) Let $0 < p_0 \le p_1 \le \infty$ and $0 < q_0 \le q_1 \le \infty$, then

$$M_s^{p_0,q_0}\left(\mathbb{R}^n\right) \hookrightarrow M_s^{p_1,q_1}\left(\mathbb{R}^n\right).$$

(3) If $0 < p, q < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $M_s^{p,q}(\mathbb{R}^n)$.

To prove our main results, we need the definition of the Wiener amalgam space $W(\mathcal{F}L^p, L^\infty)$.

Definition 2.3 Let $g \in \Phi^\delta(\mathbb{R}^n)$ and $0 < p < \infty$, we define $W(\mathcal{F}L^p, L^\infty)$ to be the space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{W(\mathcal{F}L^p, L^\infty)} = \sup_{x \in \mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left|(f \cdot T_x \hat{g})^\vee(w)\right|^p \, dw\right)^{\frac{1}{p}} < \infty.$$

Lemma 2.2 *Let $0 < p < \infty$ and $K = 2([\frac{n}{2p}] + 1)$. Define the space $\mathcal{B}^K$ by*

$$\mathcal{B}^K = \left\{ f \in C^K\left(\mathbb{R}^n\right) \Big| \sum_{|\alpha| \le K} \left\|\partial^\alpha f\right\|_{L^\infty} < \infty \right\}.$$

Then $\mathcal{B}^K \subset W(\mathcal{F}L^p, L^\infty)$.

The proof of Lemma 2.2 can be found in [16].

Lemma 2.3 *Let $0 < p < 1$, $0 < q \le \infty$ and $s \in \mathbb{R}$. If $\sigma \in W(\mathcal{F}L^p, L^\infty)$, then the multiplier operator $\sigma(D)$ defined by*

$$\sigma(D)f(x) = \int \sigma(\xi)\hat{f}(\xi)e^{2\pi i x \cdot \xi} \, d\xi$$

is bounded on weighted modulation spaces $M_s^{p,q}(\mathbb{R}^n)$.

The proof of Lemma 2.3 can be found in [3], we omit here.

3 Proof of the main results

In this section, we will prove our main results. Firstly, we come to the proof of Theorem 1.1.

Proof of Theorem 1.1 Our convolution operator T, which is defined by $Tf(x) = (K_\alpha * f)(x)$, may be realized on the transform side as a Fourier multiplier operator

$$\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi),$$

where

$$m(\xi) = \widehat{K_\alpha}(\xi) = \int_{\mathbb{R}^n} K_\alpha(x)e^{-2\pi i x\cdot\xi}\,dx.$$

By Lemma 2.3, it suffices to prove $m(\xi) \in W(\mathcal{F}L^p, L^\infty)$. In accordance with Lemma 2.2, if we can prove $\|\partial^\gamma m\|_{L^\infty} \le C$ for all $|\gamma| \le 2([\frac{n}{2p}] + 1)$, then we get the result. For $|\gamma| = 0$, in [23], Wainger proved that $\|m\|_{L^\infty} \le C$ if and only if $\alpha \le \frac{n\beta}{2}$. Thus when $\alpha \le \frac{\beta}{2}$, we also have $\|m\|_{L^\infty} \le C$. Therefore, in the following, we only need to prove $\|\partial^\gamma m\|_{L^\infty} \le C$ for all $\alpha \le \frac{\beta}{2}$ and $0 < |\gamma| \le 2([\frac{n}{2p}] + 1)$.

Differentiating the function $m(\xi)$, and using the polar coordinate transformation, we have

$$\begin{aligned}\partial^\gamma m(\xi) &= (-2\pi i)^{|\gamma|}\int_{\mathbb{R}^n} x^\gamma \cdot |x|^{-n-\alpha}\cdot e^{i|x|^{-\beta}}\eta(x)e^{-2\pi i x\cdot\xi}\,dx \\ &= (-2\pi i)^{|\gamma|}\int_{S^{n-1}} (x')^\gamma \int_0^\infty r^{|\gamma|-\alpha-1}\eta(r)e^{i[r^{-\beta}-2\pi r x'\cdot\xi]}\,dr\,dx'.\end{aligned}$$

Denote $I(x',\xi) = \int_0^\infty r^{|\gamma|-\alpha-1}\eta(r)e^{i[r^{-\beta}-2\pi r x'\cdot\xi]}\,dr$. Choose $\theta \in C_0^\infty(\mathbb{R})$ supported in $[\frac{1}{2}, 2]$ such that $\sum_{j=0}^\infty \theta(2^j r) = 1$ for all r. We can rewrite $I(x',\xi)$ as

$$I(x',\xi) = \sum_{j=0}^\infty \int_0^\infty r^{|\gamma|-\alpha-1}\eta(r)\theta(2^j r)e^{i[r^{-\beta}-2\pi r x'\cdot\xi]}\,dr.$$

Set $s = 2^j r$, then

$$\begin{aligned}I(x',\xi) &= \sum_{j=0}^\infty 2^{-j(|\gamma|-\alpha)}\int_{\frac{1}{2}}^2 r^{|\gamma|-\alpha-1}\eta(2^{-j}r)\theta(r)e^{i[2^{j\beta}r^{-\beta}-2^{-j+1}\pi r x'\cdot\xi]}\,dr \\ &= \sum_{j=0}^\infty 2^{-j(|\gamma|-\alpha)} I_j(x',\xi).\end{aligned}$$

Let $\phi(r) = 2^{j\beta}r^{-\beta} - 2^{-j+1}\pi r x'\cdot\xi$, then $\phi'(r) = -\beta 2^{j\beta}r^{-\beta-1} - 2^{-j+1}\pi x'\cdot\xi$ and $\phi''(r) = \beta(\beta + 1)2^{j\beta}r^{-\beta-2} \ge C\cdot 2^{j\beta}$ for all $r \in [\frac{1}{2}, 2]$. By Van der Corput lemma, we get

$$|I_j(x',\xi)| \le C\cdot 2^{-\frac{j\beta}{2}}.$$

Therefore,

$$|I(x',\xi)| \le C\cdot\sum_{j=0}^\infty 2^{-j(|\gamma|-\alpha)}\cdot 2^{-\frac{j\beta}{2}} = C\cdot\sum_{j=0}^\infty 2^{-j|\gamma|}\cdot 2^{-j(\frac{\beta}{2}-\alpha)} \le C.$$

Then for all $\alpha \le \frac{\beta}{2}$ and $0 < |\gamma| \le 2([\frac{n}{2p}] + 1)$, we have $\|\partial^\gamma m\|_{L^\infty} \le C$. The proof of Theorem 1.1 is completed. □

Next, we give the proof of Theorem 1.2.

Proof of Theorem 1.2 Using Fourier transformation, the operator $T_{\alpha,\beta}$ can be written as

$$\widehat{T_{\alpha,\beta}f}(\xi_1,\xi_2) = m_{\alpha,\beta}(\xi_1,\xi_2)\hat{f}(\xi_1,\xi_2),$$

where

$$\begin{aligned} m_{\alpha,\beta}(\xi_1,\xi_2) &= p.v.\int_{-1}^{1} e^{-2\pi i[\xi_1\cdot t+\xi_2\cdot\gamma(t)]}\cdot\frac{e^{-2\pi i|t|^{-\beta}}}{t|t|^\alpha}\,dt \\ &= \int_0^1 e^{-2\pi i[\xi_1\cdot t+\xi_2\cdot t^k]}\cdot\frac{e^{-2\pi i t^{-\beta}}}{t^{1+\alpha}}\,dt - \int_0^1 e^{2\pi i[\xi_1\cdot t-\xi_2\cdot t^k]}\cdot\frac{e^{-2\pi i t^{-\beta}}}{t^{1+\alpha}}\,dt \\ &= m^+_{\alpha,\beta}(\xi_1,\xi_2) - m^-_{\alpha,\beta}(\xi_1,\xi_2). \end{aligned}$$

In the following, we only need to consider $m^+_{\alpha,\beta}(\xi_1,\xi_2)$, the other half can be dealt with similarly. Take $l = (l_1,l_2) \in \mathbb{N}^2$, then, by Lemma 2.2 and Lemma 2.3, it suffices to prove

$$\left\|\partial^l m^+_{\alpha,\beta}\right\|_{L^\infty} \le C$$

for all $|l| \le 2([\frac{1}{p}] + 1)$. Firstly, we consider the case $0 < |l| \le 2([\frac{1}{p}] + 1)$. Differentiate $m^+_{\alpha,\beta}(\xi_1,\xi_2)$, and then

$$\begin{aligned} \partial^l_\xi m^+_{\alpha,\beta}(\xi_1,\xi_2) &= \sum_{l_1+l_2=|l|} C_{l_1,l_2}\partial^{l_1}_{\xi_1}\partial^{l_2}_{\xi_2} m^+_{\alpha,\beta}(\xi_1,\xi_2) \\ &= \sum_{l_1+l_2=|l|} C_{l_1,l_2}\int_0^1 (-2\pi it)^{l_1}\cdot\left(-2\pi it^k\right)^{l_2} e^{-2\pi i[\xi_1\cdot t+\xi_2\cdot t^k]}\cdot\frac{e^{-2\pi it^{-\beta}}}{t^{1+\alpha}}\,dt \\ &= \sum_{l_1+l_2=|l|} C_{l_1,l_2}(-2\pi i)^{l_1+l_2}\int_0^1 t^{l_1+kl_2-\alpha-1}e^{-2\pi i[\xi_1\cdot t+\xi_2\cdot t^k+t^{-\beta}]}\,dt. \end{aligned}$$

Denote

$$I = \int_0^1 t^{l_1+kl_2-\alpha-1}e^{-2\pi i[\xi_1\cdot t+\xi_2\cdot t^k+t^{-\beta}]}\,dt$$

and

$$\phi(s) = \xi_1\cdot s+\xi_2\cdot s^k+s^{-\beta}, \qquad G(t) = \int_0^t e^{-2\pi i\phi(s)}\,ds,$$

then $I = \int_0^1 t^{l_1+kl_2-\alpha-1}G'(t)\,dt$.

The following proof will be decomposed into two cases.

Case 1. $\xi_2 > 0$. Then the first and second derivative of ϕ about s is

$$\phi'(s) = \xi_1 + k\xi_2\cdot s^{k-1} - \beta s^{-\beta-1}$$

and

$$\phi''(s) = k(k-1)\xi_2 \cdot s^{k-2} + \beta(\beta+1)s^{-\beta-2} \geq \beta(\beta+1)s^{-\beta-2}$$

for $k \geq 2$ and $s \in (0,1)$.

By Van der Corput lemma, we get $|G(t)| \leq C \cdot t^{\frac{\beta+2}{2}}$ and

$$\begin{aligned} |I| &\leq C \cdot t^{l_1+kl_2-\alpha-1} \cdot t^{\frac{\beta+2}{2}}|_0^1 + C \cdot |l_1 + kl_2 - \alpha - 1| \cdot \int_0^1 t^{\frac{\beta+2}{2}} \cdot t^{l_1+kl_2-\alpha-2}\,dt \\ &= C \cdot t^{\frac{\beta-2\alpha}{2}+l_1+kl_2}|_0^1 + C \cdot \int_0^1 t^{\frac{\beta-2\alpha}{2}+l_1+kl_2-1}\,dt \\ &\leq C \end{aligned}$$

for all $\beta \geq 3\alpha$.

Case 2. $\xi_2 < 0$. In this case, the third derivative of ϕ satisfies

$$\phi'''(s) = k(k-1)(k-2) \cdot \xi_2 \cdot s^{k-3} - \beta(\beta+1)(\beta+2)s^{-\beta-3} \leq -\beta(\beta+1)(\beta+2)s^{-\beta-3}$$

for all $k \geq 2$. Van der Corput lemma indicates that $|G(t)| \leq C \cdot t^{\frac{\beta+3}{3}}$. Thus for $0 < |l| \leq 2([\frac{1}{p}]+1)$, we have

$$\begin{aligned} |I| &\leq C \cdot t^{\frac{\beta+3}{3}} \cdot t^{l_1+kl_2-\alpha-1}|_0^1 + C \cdot \int_0^1 t^{\frac{\beta+3}{3}} \cdot t^{l_1+kl_2-\alpha-2}\,dt \\ &= C \cdot t^{\frac{\beta-3\alpha}{3}+l_1+kl_2}|_0^1 + C \cdot \int_0^1 t^{\frac{\beta-3\alpha}{3}+l_1+kl_2-1}\,dt \\ &\leq C \end{aligned}$$

for $\beta \geq 3\alpha$.

Hence $|\partial_\xi^l m_{\alpha,\beta}^+(\xi_1,\xi_2)| \leq C$ for all $0 < |l| \leq 2([\frac{1}{p}]+1)$. Using the same methods, we can also prove $|\partial_\xi^l m_{\alpha,\beta}^-(\xi_1,\xi_2)| \leq C$ for all $0 < |l| \leq 2([\frac{1}{p}]+1)$. Thus for all $0 < |l| \leq 2([\frac{1}{p}]+1)$, we have $\|\partial_\xi^l m_{\alpha,\beta}\|_{L^\infty} \leq C$. While for $|l| = 0$, in [24], Zielinski proved that $\|m_{\alpha,\beta}\|_{L^\infty} \leq C$ if and only if $\beta \geq 3\alpha$. Therefore, for all $\beta \geq 3\alpha$ and $|l| \leq 2([\frac{1}{p}]+1)$, we have $\|\partial_\xi^l m_{\alpha,\beta}\|_{L^\infty} \leq C$, and then $m_{\alpha,\beta}(\xi_1,\xi_2) \in W(\mathcal{F}L^p, L^\infty)$.

By Lemma 2.3, we complete the proof of Theorem 1.2. □

Finally, we turn our attention to the proof of Theorem 1.3.

Proof of Theorem 1.3 By checking our proof in the following, we can assume $\Gamma(t) = (t^{p_1}, t^{p_2}, \ldots, t^{p_n})$ and consider the operator

$$T^*_{n,\alpha,\beta} = \int_0^1 f\bigl(x - \Gamma(t)\bigr)\frac{e^{-2\pi i t^{-\beta}}}{t^{1+\alpha}}\,dt.$$

Using Fourier transformation, we get the Fourier multiplier of the operator $T^*_{n,\alpha,\beta}$ is

$$m(\xi) = \int_0^1 e^{-2\pi i[\xi\cdot\Gamma(t)+t^{-\beta}]} \cdot \frac{dt}{t^{1+\alpha}}.$$

Denote $l = (l_1, l_2, \ldots, l_n) \in \mathbb{N}^n \setminus \{0\}$ such that $\sum_{j=1}^{n} l_j = |l|$, then

$$\begin{aligned}\partial_\xi^l m(\xi) &= \partial_{\xi_1}^{l_1} \partial_{\xi_2}^{l_2} \cdots \partial_{\xi_n}^{l_n} m(\xi_1, \xi_2, \ldots, \xi_n) \\ &= (-2\pi i)^{|l|} \int_0^1 t^{p_1 l_1 + p_2 l_2 + \cdots + p_n l_n} \cdot e^{-2\pi i[\xi_1 \cdot t^{p_1} + \xi_2 \cdot t^{p_2} + \cdots + \xi_n \cdot t^{p_n} + t^{-\beta}]} \cdot \frac{dt}{t^{1+\alpha}}.\end{aligned}$$

Set $\delta = \sum_{j=1}^{n} p_j l_j$, then

$$\partial_\xi^l m(\xi) = (-2\pi i)^{|l|} \int_0^1 e^{-2\pi i[\xi \cdot \Gamma(t) + t^{-\beta}]} \cdot \frac{dt}{t^{1+\alpha-\delta}}.$$

So $\partial_\xi^l m(\xi)$ is the Fourier multiplier of the operator $T_{n,\alpha-\delta,\beta}$. Using the same methods as in [5], when $\beta \geq (n+1)\alpha \geq (n+1)(\alpha - \delta)$, $\|\partial_\xi^l m(\xi)\|_{L^\infty} \leq C$ for all $|l| \leq 2([\frac{n}{2p}] + 1)$. By Lemma 2.2, we finish the proof of Theorem 1.3. □

Competing interests
The author declares that they have no competing interests.

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