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Strong convergence theorems for equilibrium problems and fixed point problem of multivalued nonexpansive mappings via hybrid projection method

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Abstract

In this paper, a new iterative process by the hybrid projection method is constructed. Strong convergence of the iterative process to a common element of the set of common fixed points of a finite family of generalized nonexpansive multivalued mappings and the solution set of two equilibrium problems in a Hilbert space is proved. Our results extend some important recent results.

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Keywords: equilibrium problem; hybrid projection method; strong convergence; common fixed point; generalized nonexpansive multivalued mapping

1 Introduction

Let C be a nonempty closed convex subset of a Hilbert space H. A subset $C \subset H$ is called proximal if for each $x \in H$ there exists an element $y \in C$ such that

$$\|x - y\| = \text{dist}(x, C) = \inf\{\|x - z\| : z \in C\}.$$

We denote by $CB(C)$ and $P(C)$ the collection of all nonempty closed bounded subsets and nonempty proximal bounded subsets of C, respectively. The Hausdorff metric H on $CB(H)$ is defined by

$$H(A, B) := \max\left\{\sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A)\right\},$$

for all $A, B \in CB(H)$.

Let $T : H \to 2^H$ be a multivalued mapping. An element $x \in H$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points of T will be denoted by $F(T)$.

Definition 1.1 A multivalued mapping $T : H \to CB(H)$ is called
  (i) nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in H.$$

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$ for all $x \in H$ and all $p \in F(T)$.

Recently, J. Garcia-Falset, E. Llorens-Fuster and T. Suzuki [1] introduced a new condition on singlevalued mappings, called condition (E), which is weaker than nonexpansiveness.

Definition 1.2 A mapping $T : H \to H$ is said to satisfy the condition (E_μ) provided that

$$\|x - Ty\| \leq \mu \|x - Tx\| + \|x - y\|, \quad x, y \in H.$$

We say that T satisfies the condition (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Now we modify this condition for multivalued mappings as follows (see also [2]):

Definition 1.3 A multivalued mapping $T : H \to CB(H)$ is said to satisfy the condition (P) provided that

$$H(Tx, Ty) \leq \mu \operatorname{dist}(x, Tx) + \eta \|x - y\|, \quad x, y \in H,$$

for some $\mu, \eta \geq 1$.

It is obvious that every nonexpansive multivalued mapping satisfies the condition (P). The theory of multivalued mappings has applications in control theory, convex optimization, differential equations and economics. Theory of nonexpansive multivalued mappings is harder than the corresponding theory of nonexpansive single valued mappings. Different iterative processes have been used to approximate fixed points of multivalued nonexpansive mappings (see [3–8]). Let Φ be a bifunction from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $\Phi : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$\Phi(x, y) \geq 0, \quad \forall y \in C.$$

The set of solutions is denoted by $EP(\Phi)$. It is well known that this problem is closely related to minimax inequalities (see [9] and [10]). The equilibrium problem includes fixed point problems, optimization problems and variational inequality problems as special cases. Some methods have been proposed to solve the equilibrium problem, see, for example, [11–14].

Recently, many authors have studied the problems of finding a common element of the set of fixed points of nonexpansive single valued mappings and the set of solutions of an equilibrium problem in the framework of Hilbert spaces: see, for instance, [15–29] and the references therein. In this paper, a new iterative process by the hybrid projection method is constructed. Strong convergence of the iterative process to a common element of a set of common fixed points of a finite family of multivalued mappings satisfying the condition (P) and the solution set of two equilibrium problems in a Hilbert space is proved. Our results generalize some results of Tada, Takahashi [15] and many others.

2 Preliminaries

Let us recall the following definitions and results which will be used in the sequel.

Lemma 2.1 ([6]) *Let H be a real Hilbert space. Then for $i, j = 1, 2, \ldots, k$ we have*

$$\|a_1 x_1 + a_2 x_2 + \cdots + a_k x_k\|^2 \le a_1 \|x_1\|^2 + a_2 \|x_2\|^2 + \cdots + a_k \|x_k\|^2 - a_i a_j \|x_i - x_j\|^2$$

for all $x_i, x_j \in H$ and $a_i, a_j \in [0, 1]$ with $\sum_{i=1}^{k} a_i = 1$.

Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping.

Lemma 2.2 ([16]) *Let C be a closed convex subset of H. Given $x \in H$ and a point $z \in C$. Then $z = P_C x$ if and only if*

$$\langle x - z, z - y \rangle \ge 0, \quad \forall y \in C.$$

Lemma 2.3 ([30]) *Let C be a closed convex subset of H. Then for all $x \in H$ and $y \in C$ we have*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \le \|x - y\|^2.$$

For solving the equilibrium problem, we assume that the bifunction Φ satisfies the following conditions:

(A1) $\Phi(x, x) = 0$ for any $x \in C$,

(A2) Φ is monotone, *i.e.*, $\Phi(x, y) + \Phi(y, x) \le 0$ for any $x, y \in C$,

(A3) Φ is upper-hemicontinuous, *i.e.*, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Phi\big(tz + (1 - t)x, y\big) \le \Phi(x, y),$$

(A4) $\Phi(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

The following lemma was proved in [11].

Lemma 2.4 *Let C be a nonempty closed convex subset of H and let Φ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\Phi(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \ge 0 \quad \forall y \in C.$$

The following lemma was given in [14].

Lemma 2.5 *Assume that* $\Phi : C \times C \to \mathbb{R}$ *satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping* $T_r : H \to C$ *as follows:*

$$T_r x = \left\{ z \in C : \Phi(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}.$$

Then, the following hold:

  (i) T_r *is single valued;*

  (ii) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\| T_r x - T_r y \|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

  (iii) $F(T_r) = EP(\Phi)$;

  (iv) $EP(\Phi)$ *is closed and convex.*

The following lemma was proved in [31] for nonexpansive multivalued mappings. The statement is true for quasi-nonexpansive multivalued mappings as well. To avoid repetition, we omit the details of the proof.

Lemma 2.6 *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to CB(C)$ be a quasi-nonexpansive multivalued mapping such that $T(p) = \{p\}$ for all $p \in F(T)$. Then $F(T)$ is closed and convex.*

Now, following Shahzad and Zegeye [3], we remove the restriction $T(p) = \{p\}$ for all $p \in F(T)$. Let $T : C \to P(C)$ be a multivalued mapping and

$$P_T(x) = \left\{ y \in Tx : \| x - y \| = \mathrm{dist}(x, Tx) \right\}.$$

We use a similar argument as in the proof of Lemma 3.1 in [31] to obtain the following lemma.

Lemma 2.7 *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to P(C)$ be a multivalued mapping such that P_T is quasi-nonexpansive. Then $F(T)$ is closed and convex.*

Note that for all $p \in F(T)$, $P_T(p) = \{p\}$. We remark that there exist some examples of multivalued mappings for which P_T is nonexpansive (see [3] for details), so that the assumption on T is not artificial.

3 The main result

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H, Φ_1 and Φ_2 be two bifunctions of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $T_i : C \to CB(C)$, $(i = 1, 2, \ldots, m)$, be a finite family of quasi-nonexpansive multivalued mappings, each satisfying the condition (P). Assume further that $\mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \cap EP(\Phi_1) \cap EP(\Phi_2) \neq \emptyset$ and $T_i(p) = \{p\}$, $(i = 1, 2, \ldots, m)$, for each $p \in \mathcal{F}$. For $C_0 = C$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

the following algorithm:

$$
\begin{cases}
x_0 \in C, \\
u_n \in C \text{ such that } \Phi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0; \quad \forall y \in C, \\
u'_n \in C \text{ such that } \Phi_2(u'_n, y) + \frac{1}{s_n}\langle y - u'_n, u'_n - x_n \rangle \geq 0; \quad \forall y \in C, \\
v_n = \delta_n u_n + (1 - \delta_n)u'_n, \\
w_n = a_{n,0}v_n + a_{n,1}z_{n,1} + \cdots + a_{n,m}z_{n,m}, \\
y_n = b_{n,0}v_n + b_{n,1}z'_{n,1} + \cdots + b_{n,m}z'_{n,m}, \\
C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}, \\
x_{n+1} = P_{C_{n+1}}x_0 : \quad \forall n \geq 0,
\end{cases}
$$

where $z_{n,i} \in T_i v_n$, $z'_{n,i} \in T_i w_n$ for $i = 1, 2, \ldots, m$. Assume that $\{a_{n,j}\}$, $\{b_{n,j}\}$, $\{\delta_n\}$ and $\{r_n\}$, $\{s_n\}$ satisfy the following conditions:

(i) $\{a_{n,j}\}, \{b_{n,j}\}, \{\delta_n\} \subset [a, b] \subset (0, 1)$ ($j = 0, 1, 2, \ldots, m$),

(ii) $\{r_n\}, \{s_n\} \subset (0, \infty)$, and $\liminf_{n \to \infty} r_n > 0$ and $\liminf_{n \to \infty} s_n > 0$.

Then, the sequences $\{x_n\}$ and $\{v_n\}$ converge strongly to $P_{\mathcal{F}}x_0$.

Proof First, we show that $\mathcal{F} \subset C_n$ for all $n \geq 0$. Fix $q \in \mathcal{F}$. We set

$$
T_r^{\Phi_1}(x) = \left\{ z \in C : \Phi_1(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0 \; \forall y \in C \right\}.
$$

Hence we have $u_n = T_{r_n}^{\Phi_1} x_n$ and $u'_n = T_{s_n}^{\Phi_2} x_n$. By Lemma 2.5, we have

$$
\|u_n - q\| = \left\| T_{r_n}^{\Phi_1} x_n - T_{r_n}^{\Phi_1} q \right\| \leq \|x_n - q\|
$$

and

$$
\left\| u'_n - q \right\| = \left\| T_{s_n}^{\Phi_2} x_n - T_{s_n}^{\Phi_2} q \right\| \leq \|x_n - q\|,
$$

which implies that

$$
\|v_n - q\| \leq \delta_n \|u_n - q\| + (1 - \delta_n)\left\| u'_n - q \right\| \leq \|x_n - q\|.
$$

Since T_i is quasi-nonexpansive, for $i = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
\|w_n - q\| &= \|a_{n,0}v_n + a_{n,1}z_{n,1} + \cdots + a_{n,m}z_{n,m} - q\| \\
&\leq a_{n,0}\|v_n - q\| + a_{n,1}\|z_{n,1} - q\| + \cdots + a_{n,m}\|z_{n,m} - q\| \\
&\leq a_{n,0}\|x_n - q\| + a_{n,1}\operatorname{dist}(z_{n,1}, T_1 q) + \cdots + a_{n,m}\operatorname{dist}(z_{n,m}, T_m q) \\
&\leq a_{n,0}\|x_n - q\| + a_{n,1}H(T_1 v_n, T_1 q) + \cdots + a_{n,m}H(T_m v_n, T_m q) \\
&\leq a_{n,0}\|x_n - q\| + a_{n,1}\|v_n - q\| + \cdots + a_{n,m}\|v_n - q\| \\
&\leq a_{n,0}\|x_n - q\| + a_{n,1}\|x_n - q\| + \cdots + a_{n,m}\|x_n - q\| \leq \|x_n - q\|,
\end{aligned}
$$

and also

$$
\begin{aligned}
\|y_n - q\| &= \left\| b_{n,0} v_n + b_{n,1} z'_{n,1} + \cdots + b_{n,m} z'_{n,m} - q \right\| \\
&\leq b_{n,0} \|v_n - q\| + b_{n,1} \left\| z'_{n,1} - q \right\| + \cdots + b_{n,m} \left\| z'_{n,m} - q \right\| \\
&\leq b_{n,0} \|x_n - q\| + b_{n,1} \operatorname{dist}\left(z'_{n,1}, T_1 q \right) + \cdots + b_{n,m} \operatorname{dist}\left(z'_{n,m}, T_m q \right) \\
&\leq b_{n,0} \|x_n - q\| + b_{n,1} H(T_1 w_n, T_1 q) + \cdots + b_{n,m} H(T_m w_n, T_m q) \\
&\leq b_{n,0} \|x_n - q\| + b_{n,1} \|w_n - q\| + \cdots + b_{n,m} \|w_n - q\| \\
&\leq b_{n,0} \|x_n - q\| + b_{n,1} \|x_n - q\| + \cdots + b_{n,m} \|x_n - q\| \leq \|x_n - q\|.
\end{aligned}
$$

Therefore $q \in C_n$, which implies that

$$
\mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \cap EP(\Phi_1) \cap EP(\Phi_2) \subset C_n, \quad \text{for all } n \geq 0.
$$

We observe that C_n is closed and convex (see [32]). Now we show that $\lim_{n \to \infty} \|x_n - x_0\|$ exists. By Lemma 2.6 we have $\mathcal{F}$ is closed and convex. Put $w = P_{\mathcal{F}} x_0$. From $x_n = P_{C_n} x_0$ and $x_{n+1} \in C_{n+1} \subset C_n$ we have

$$
\|x_n - x_0\| \leq \|x_{n+1} - x_0\|.
$$

Also from $w \in \mathcal{F} \subset C_n$ and $x_n = P_{C_n} x_0$ for all $n \geq 0$, we get that

$$
\|x_n - x_0\| \leq \|w - x_0\|.
$$

It follows that the sequence $\{x_n\}$ is bounded and nondecreasing. Hence the limit $\lim_{n \to \infty} \|x_n - x_0\|$ exists. We show that $\lim_{n \to \infty} x_n = u \in C$. For $k > n$ we have $x_k = P_{C_k} x_0 \in C_k \subset C_n$. Now by applying Lemma 2.3 we have

$$
\|x_k - x_n\|^2 \leq \|x_k - x_0\|^2 - \|x_n - x_0\|^2.
$$

Since $\lim_{n \to \infty} \|x_n - x_0\|$ exists, it follows that $\{x_n\}$ is a Cauchy sequence, and hence there exists $u \in C$ such that $\lim_{n \to \infty} x_n = u$. Putting $k = n + 1$, in the above inequality we have

$$
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
$$

From $x_{n+1} \in C_{n+1}$, we have

$$
\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|,
$$

so that $\lim_{n \to \infty} \|y_n - x_{n+1}\| = 0$. This implies that $\lim_{n \to \infty} y_n = u$. Take $q \in \mathcal{F}$. By Lemma 2.1, for each $1 \leq i \leq m$, we have

$$
\begin{aligned}
\|w_n - q\|^2 &= \|a_{n,0} v_n + a_{n,1} z_{n,1} + \cdots + a_{n,m} z_{n,m} - q\|^2 \\
&\leq a_{n,0} \|v_n - q\|^2 + a_{n,1} \|z_{n,1} - q\|^2
\end{aligned}
$$

$$+ \cdots + a_{n,m} \|z_{n,m} - q\|^2 - a_{n,i} a_{n,o} \|v_n - z_{n,i}\|^2$$
$$\leq a_{n,0} \|v_n - q\|^2 + a_{n,1} \operatorname{dist}(z_{n,1}, T_1 q)^2$$
$$+ \cdots + a_{n,m} \operatorname{dist}(z_{n,m}, T_m q)^2 - a_{n,i} a_{n,o} \|v_n - z_{n,i}\|^2$$
$$\leq a_{n,0} \|v_n - q\|^2 + a_{n,1} H(T_1 v_n, T_1 q)^2$$
$$+ \cdots + a_{n,m} H(T_m v_n, T_m q)^2 - a_{n,i} a_{n,o} \|v_n - z_{n,i}\|^2$$
$$\leq a_{n,0} \|v_n - q\|^2 + a_{n,1} \|v_n - q\|^2$$
$$+ \cdots + a_{n,m} \|v_n - q\|^2 - a_{n,i} a_{n,o} \|v_n - z_{n,i}\|^2$$
$$\leq \|v_n - q\|^2 - a_{n,i} a_{n,o} \|v_n - z_{n,i}\|^2,$$

and also

$$\|y_n - q\|^2 = \left\| b_{n,0} v_n + b_{n,1} z'_{n,1} + \cdots + b_{n,m} z'_{n,m} - q \right\|^2$$
$$\leq b_{n,0} \|v_n - q\|^2 + b_{n,1} \left\| z'_{n,1} - q \right\|^2 + \cdots + b_{n,m} \left\| z'_{n,m} - q \right\|^2$$
$$\leq b_{n,0} \|v_n - q\|^2 + b_{n,1} \operatorname{dist}\left(z'_{n,1}, T_1 q \right)^2 + \cdots + b_{n,m} \operatorname{dist}\left(z'_{n,m}, T_m q \right)^2$$
$$\leq b_{n,0} \|v_n - q\|^2 + b_{n,1} H(T_1 w_n, T_1 q)^2 + \cdots + b_{n,m} H(T_m w_n, T_m q)^2$$
$$\leq b_{n,0} \|v_n - q\|^2 + b_{n,1} \|w_n - q\|^2 + \cdots + b_{n,m} \|w_n - q\|^2$$
$$\leq b_{n,0} \|v_n - q\|^2 + b_{n,1} \|v_n - q\|^2 + \cdots + b_{n,m} \|v_n - q\|^2$$
$$- b_{n,i} a_{n,i} a_{n,o} \|v_n - z_{n,i}\|^2$$
$$\leq \|x_n - q\|^2 - b_{n,i} a_{n,i} a_{n,o} \|v_n - z_{n,i}\|^2. \tag{1}$$

So we have that

$$a^2 b \|v_n - z_{n,i}\|^2 \leq b_{n,i} a_{n,i} a_{n,o} \|v_n - z_{n,i}\|^2$$
$$\leq \|v_n - q\|^2 - \|y_n - q\|^2$$
$$\leq \|x_n - q\|^2 - \|y_n - q\|^2,$$

which implies that

$$\lim_{n \to \infty} \|v_n - z_{n,i}\| = 0, \quad \text{for } i = 1, 2, \ldots, m.$$

Hence

$$\lim_{n \to \infty} \operatorname{dist}(v_n, T_i v_n) \leq \lim_{n \to \infty} \|v_n - z_{n,i}\| = 0 \quad (i = 1, 2, \ldots, m).$$

As above $u_n = T_{r_n}^{\Phi_1} x_n$ so that

$$\|u_n - q\|^2 = \left\| T_{r_n}^{\Phi_1} x_n - T_{r_n}^{\Phi_1} q \right\|^2$$
$$\leq \left\langle T_{r_n}^{\Phi_1} x_n - T_{r_n}^{\Phi_1} q, x_n - q \right\rangle$$

$$= \langle u_n - q, x_n - q \rangle$$
$$= \frac{1}{2}\left(\|u_n - q\|^2 + \|x_n - q\|^2 - \|x_n - u_n\|^2\right)$$

and hence

$$\|u_n - q\|^2 \le \|x_n - q\|^2 - \|x_n - u_n\|^2. \tag{2}$$

And also by $u_n' = T_{s_n}^{\Phi_2} x_n$ we have

$$\begin{aligned}\left\|u_n' - q\right\|^2 &= \left\|T_{s_n}^{\Phi_2} x_n - T_{s_n}^{\Phi_2} q\right\|^2 \\ &\le \left\langle T_{s_n}^{\Phi_2} x_n - T_{s_n}^{\Phi_2} q, x_n - q\right\rangle \\ &= \left\langle u_n' - q, x_n - q\right\rangle \\ &= \frac{1}{2}\left(\left\|u_n' - q\right\|^2 + \|x_n - q\|^2 - \left\|x_n - u_n'\right\|^2\right)\end{aligned}$$

and hence

$$\left\|u_n' - q\right\|^2 \le \|x_n - q\|^2 - \left\|x_n - u_n'\right\|^2. \tag{3}$$

Now we use (2) and (3) to obtain

$$\begin{aligned}\|v_n - q\|^2 &\le \delta_n \|u_n - q\|^2 + (1 - \delta_n)\left\|u_n' - q\right\|^2 \\ &\le \|x_n - q\|^2 - \delta_n \|x_n - u_n\|^2 - (1 - \delta_n)\left\|x_n - u_n'\right\|^2.\end{aligned}$$

It follows from (1) and the last inequality that

$$\|y_n - q\|^2 \le \|v_n - q\|^2 \le \|x_n - q\|^2 - \delta_n \|x_n - u_n\|^2 - (1 - \delta_n)\left\|x_n - u_n'\right\|^2.$$

So we have

$$a\|x_n - u_n\|^2 \le \delta_n \|x_n - u_n\|^2 \le \|x_n - q\|^2 - \|y_n - q\|^2,$$

and

$$(1 - b)\left\|x_n - u_n'\right\|^2 \le (1 - \delta_n)\left\|x_n - u_n'\right\|^2 \le \|x_n - q\|^2 - \|y_n - q\|^2.$$

Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = u$, we obtain that

$$\lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \left\|u_n' - x_n\right\| = 0,$$

which implies that

$$\lim_{n\to\infty} \|v_n - x_n\| = 0.$$

Since $\lim_{n\to\infty} \|v_n - x_n\| = 0$, for $i = 1, 2, \ldots, m$, we obtain that

$$\mathrm{dist}(x_n, T_i x_n) \leq \|x_n - v_n\| + \mathrm{dist}(v_n, T_i v_n) + H(T_i v_n, T_i x_n)$$

$$\leq (\eta + 1)\|x_n - v_n\| + (\mu + 1)\,\mathrm{dist}(v_n, T_i v_n) \to 0 \quad \text{as } n \to \infty. \tag{4}$$

We observe that $u \in \bigcap_{i=1}^m F(T_i)$. Indeed,

$$\mathrm{dist}(u, T_i u) \leq \|u - x_n\| + \mathrm{dist}(x_n, T_i x_n) + H(T_i x_n, T_i u)$$

$$\leq (\eta + 1)\|u - x_n\| + (\mu + 1)\,\mathrm{dist}(x_n, T_i x_n) \to 0 \quad \text{as } n \to \infty,$$

which implies that $u \in \bigcap_{i=1}^m F(T_i)$. Let us show that $u \in EP(\Phi_1) \cap EP(\Phi_2)$. From $\lim_{n\to\infty} x_n = u$ and $\lim_{n\to\infty} \|x_n - u_n\| = 0$, we have $u_n \to u$ as $n \to \infty$. Since $u_n = T_{r_n}^{\Phi_1} x_n$ we obtain

$$\Phi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \forall y \in C.$$

From (A2), we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \Phi_1(y, u_n),$$

and hence

$$\left\langle y - u_n, \frac{u_n - x_n}{r_n}\right\rangle \geq \Phi_1(y, u_n).$$

Since

$$\frac{u_n - x_n}{r_n} \to 0,$$

and $u_n \to u$, from (A4) we have

$$0 \geq \Phi_1(y, u), \quad \forall y \in C.$$

For $t \in (0, 1]$ and $y \in C$, let $y_t = ty + (1 - t)u$. Since $y, u \in C$, and C is convex we have $y_t \in C$ and hence $\Phi_1(y_t, u) \leq 0$. So, from (A1) and (A4), we have

$$0 = \Phi_1(y_t, y_t) \leq t\Phi_1(y_t, y) + (1 - t)\Phi_1(y_t, u) \leq t\Phi_1(y_t, y),$$

which gives $\Phi_1(y_t, y) \geq 0$. From (A3) we have $0 \leq \Phi_1(u, y)$, $\forall y \in C$ and hence $u \in EP(\Phi_1)$. Similarly, we have $u \in EP(\Phi_2)$. Now we show that $u = P_{\mathcal{F}} x_0$. Since $x_n = P_{C_n} x_0$, by Lemma 2.2 we have

$$\langle z - x_n, x_0 - x_n\rangle \leq 0, \quad \forall z \in C_n.$$

Since $u \in \mathcal{F} \subset C_n$ we get

$$\langle z - u, x_0 - u\rangle \leq 0, \quad \forall z \in \mathcal{F}.$$

Now by Lemma 2.2 we obtain that $u = P_{\mathcal{F}} x_0$. □

By substituting P_{T_i} by T_i and using a similar argument as in Theorem 3.1, we obtain the following result.

Theorem 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H, Φ_1 and Φ_2 be two bifunctions of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $T_i : C \to P(C)$, ($i = 1, 2, \ldots, m$), be a finite family of multivalued mappings such that each P_{T_i} is quasi-nonexpansive and satisfies the condition (P). Assume further that $\mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \cap EP(\Phi_1) \cap EP(\Phi_2) \neq \emptyset$. For $C_0 = C$, let $\{x_n\}$ and $\{v_n\}$ be sequences generated by the following algorithm:*

$$
\begin{cases}
x_0 \in C, \\
u_n \in C \text{ such that } \Phi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0; \quad \forall y \in C, \\
u'_n \in C \text{ such that } \Phi_2(u'_n, y) + \frac{1}{s_n}\langle y - u'_n, u'_n - x_n \rangle \geq 0; \quad \forall y \in C, \\
v_n = \delta_n u_n + (1 - \delta_n)u'_n, \\
w_n = a_{n,0} v_n + a_{n,1} z_{n,1} + \cdots + a_{n,m} z_{n,m}, \\
y_n = b_{n,0} v_n + b_{n,1} z'_{n,1} + \cdots + b_{n,m} z'_{n,m}, \\
C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}, \\
x_{n+1} = P_{C_{n+1}} x_0 : \quad \forall n \geq 0,
\end{cases}
$$

where $z_{n,i} \in P_{T_i}(v_n)$, $z'_{n,i} \in P_{T_i}(w_n)$ for $i = 1, 2, \ldots, m$. Assume that $\{a_{n,j}\}$, $\{b_{n,j}\}$, $\{\delta_n\}$ and $\{r_n\}$, $\{s_n\}$ satisfy the following conditions:

(i) $\{a_{n,j}\}, \{b_{n,j}\}, \{\delta_n\} \subset [a, b] \subset (0, 1)$ ($j = 0, 1, 2, \ldots, m$),

(ii) $\{r_n\}, \{s_n\} \subset (0, \infty)$, and $\liminf_{n \to \infty} r_n > 0$ and $\liminf_{n \to \infty} s_n > 0$.

Then, the sequences $\{x_n\}$ and $\{v_n\}$ converge strongly to $P_{\mathcal{F}} x_0$.

As a result, for single valued mappings we obtain the following theorem.

Theorem 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H and Φ_1 and Φ_2 be two bifunctions of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $T_i : C \to C$ ($i = 1, 2, \ldots, m$), be a finite family of quasi-nonexpansive mappings, each satisfying the condition (P). Assume further that $\mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \cap EP(\Phi_1) \cap EP(\Phi_2) \neq \emptyset$. For $C_0 = C$, let $\{x_n\}$ and $\{v_n\}$ be sequences generated by the following algorithm:*

$$
\begin{cases}
x_0 \in C, \\
u_n \in C \text{ such that } \Phi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0; \quad \forall y \in C, \\
u'_n \in C \text{ such that } \Phi_2(u'_n, y) + \frac{1}{s_n}\langle y - u'_n, u'_n - x_n \rangle \geq 0; \quad \forall y \in C, \\
v_n = \delta_n u_n + (1 - \delta_n)u'_n, \\
w_n = a_{n,0} v_n + a_{n,1} T_1 v_n + \cdots + a_{n,m} T_m v_n, \\
y_n = b_{n,0} v_n + b_{n,1} T_1 w_n + \cdots + b_{n,m} T_m w_n, \\
C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}, \\
x_{n+1} = P_{C_{n+1}} x_0 : \quad \forall n \geq 0.
\end{cases}
$$

Assume that $\{a_{n,j}\}$, $\{b_{n,j}\}$, $\{\delta_n\}$ and $\{r_n\}$, $\{s_n\}$ satisfy the following conditions:

(i) $\{a_{n,j}\}, \{b_{n,j}\}, \{\delta_n\} \subset [a, b] \subset (0, 1)$ ($j = 0, 1, 2, \ldots, m$),

(ii) $\{r_n\}, \{s_n\} \subset (0, \infty)$, *and* $\liminf_{n \to \infty} r_n > 0$ *and* $\liminf_{n \to \infty} s_n > 0$.

Then, the sequences $\{x_n\}$ and $\{v_n\}$ converge strongly to $P_\mathcal{F} x_0$.

Theorem 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_i : C \to P(C)$ $(i = 1, 2, \ldots, m)$, be a finite family of multivalued mappings such that P_{T_i} is quasi-nonexpansive and satisfies the condition (P). Assume further that $\mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. For $C_0 = C$, let $\{x_n\}$ be the sequence generated by the following algorithm:*

$$\begin{cases} x_0 \in C, \\ w_n = a_{n,0} x_n + a_{n,1} z_{n,1} + \cdots + a_{n,m} z_{n,m}, \\ y_n = b_{n,0} x_n + b_{n,1} z'_{n,1} + \cdots + b_{n,m} z'_{n,m}, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0 : \quad \forall n \geq 0, \end{cases}$$

where $z_{n,i} \in P_{T_i}(x_n)$, $z'_{n,i} \in P_{T_i}(w_n)$ for $i = 1, 2, \ldots, m$. Assume that $\{a_{n,j}\}, \{b_{n,j}\} \subset [a, b] \subset (0, 1)$ $(j = 0, 1, 2, \ldots, m)$, Then, the sequence $\{x_n\}$ converges strongly to $P_\mathcal{F} x_0$.

Proof Putting $\Phi_1(x, y) = \Phi_2(x, y) = 0$ for all $x, y \in C$ and $r_n = s_n = 1$ in Theorem 3.2, we have $u_n = u'_n = x_n$ and hence $v_n = x_n$. Now, the desired conclusion follows directly from Theorem 3.2. $\qquad \square$

Now, we supply an example to illustrate the main result of this paper.

Example 3.5 We consider the nonempty closed convex subset $C = [0, 5]$ of the Hilbert space $\mathbb{R}$. Define two mappings T_1 and T_2 on C as follows:

$$T_1(x) = \left[\frac{x}{6}, \frac{x}{2}\right], \qquad T_2(x) = \begin{cases} [0, \frac{x}{5}], & x \neq 5, \\ \{1\}, & x = 5. \end{cases}$$

We note that T_1 and T_2 are quasi-nonexpansive mappings satisfying the condition (P), (for details, see [2]). Also we define two bifunctions Φ_1 and Φ_2 as follows:

$$\begin{cases} \Phi_1 : C \times C \to \mathbb{R}, \\ \Phi_1(x, y) = y - x, \end{cases} \qquad \begin{cases} \Phi_2 : C \times C \to \mathbb{R}, \\ \Phi_2(x, y) = y^2 + xy - 2x^2. \end{cases}$$

It is easy to see that Φ_1 and Φ_2 satisfy the conditions (A1)-(A4). If we put $r_n = 5$ and $s_n = 1$, then $u_n = T_{r_n}^{\Phi_1} x_n = 0$ and $u'_n = T_{s_n}^{\Phi_1} x_n = \frac{x_n}{3 s_n + 1} = \frac{x_n}{4}$ (for details, see [26]). Put $a_{n,i} = b_{n,i} = \frac{1}{3}$ for $i = 0, 1, 2$ and $\delta_n = \frac{1}{2}$. For any arbitrary $x_0 \in C$ we have

$$C_1 = \left\{v \in C : |y_0 - v| \leq |x_0 - v|\right\} = \left[0, \frac{x_0 + y_0}{2}\right].$$

Since $\frac{x_0 + y_0}{2} \leq x_0$, we obtain that

$$x_1 = P_{C_1} x_0 = \frac{x_0 + y_0}{2}.$$

By continuing this process we obtain

$$C_{n+1} = \left\{ v \in C_n : |y_n - v| \le |x_n - v| \right\} = \left[0, \frac{x_n + y_n}{2} \right],$$

and hence

$$x_{n+1} = P_{C_{n+1}} x_0 = \frac{x_n + y_n}{2}.$$

Now, we have the following algorithm:

$$
\begin{cases}
x_0 \in C, \\
v_n = \frac{x_n}{8}, \\
z_{n,i} \in T_i(v_n), \quad i = 1, 2, \\
w_n = \frac{1}{3} v_n + \frac{1}{3} z_{n,1} + \frac{1}{3} z_{n,2}, \\
z'_{n,i} \in T_i w_n, \quad i = 1, 2, \\
y_n = \frac{1}{3} v_n + \frac{1}{3} z'_{n,1} + \frac{1}{3} z'_{n,2}, \\
x_{n+1} = \frac{x_n + y_n}{2} : \quad \forall n \ge 0.
\end{cases}
$$

Putting $z_{n,1} = z_{n,2} = \frac{v_n}{6}$ and $z'_{n,1} = z'_{n,2} = \frac{w_n}{6}$ we get that

$$x_{n+1} = \frac{679}{1,296} x_n = \left(\frac{679}{1,296} \right)^{n+1} x_0, \quad \forall n \ge 0.$$

We observe that for an arbitrary $x_0 \in C$, x_n is convergent to zero. We note that $\mathcal{F} = F(T_1) \cap F(T_2) \cap EP(\Phi_1) \cap EP(\Phi_2) = \{0\}$.

Remark 3.6 Since every nonexpansive mapping is quasi-nonexpansive and satisfies the condition (P), our results hold for nonexpansive mappings.

Remark 3.7 Our results generalize the results of Tada and Takahashi [15], of a nonexpansive single valued mapping to a finite family of generalized nonexpansive multivalued mappings.

Authors' contributions
All the authors contributed equally to the writing of the present article. And they also read and approved the final manuscript.

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