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A class of retarded nonlinear integral inequalities and its application in nonlinear differential-integral equation

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Abstract

In this paper, we discuss a class of retarded nonlinear integral inequalities and give an upper bound estimation of an unknown function by the integral inequality technique. This estimation can be used as a tool in the study of differential-integral equations with the initial conditions.

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1 Introduction

Gronwall-Bellman inequalities [1, 2] can be used as important tools in the study of existence, uniqueness, boundedness, stability, and other qualitative properties of solutions of differential equations, integral equations, and integral-differential equations. There can be found a lot of generalizations of Gronwall-Bellman inequalities in various cases from literature (e.g., [3–13]).

Lemma 1 (Abdeldaim and Yakout [4]) We assume that u(t) and f(t) are nonnegative real-valued continuous functions defined on $I = [0, \infty)$ and they satisfy the inequality

$$u^{p+1}(t) \le u_0 + \left(\int_0^t f(s)u^p(s) \, ds\right)^2 + 2\int_0^t f(s)u^p(s) \left[u^p(s) + \int_0^s f(\lambda)u^p(\lambda) \, d\lambda\right] ds, \quad (1.1)$$

for all $t \in I$, where $u_0 > 0$ and $p \in (0,1)$ are constants. Then

$$u(t) \le u_0^{\frac{1}{p+1}} + \frac{2}{p+1} \int_0^t f(s) D_2(s) \, ds, \quad \forall t \in I,$$
 (1.2)

where

$$D_2(t) = \beta(t) \left[u_0^{\frac{1-p}{1+p}} + 2\frac{1-p}{1+p} \int_0^t f(s) \exp\left(-2\frac{1-p}{p} \int_0^s f(\lambda) \, d\lambda\right) ds \right]^{\frac{p}{1-p}},\tag{1.3}$$

and $\beta(t) = \exp(2\int_0^t f(s) ds)$, for all $t \in I$.



In this paper, we discuss a class of retarded nonlinear integral inequalities and give an upper bound estimation of an unknown function by the integral inequality technique.

2 Main result

In this section, we discuss some retarded integral inequalities of Gronwall-Bellman type. Throughout this paper, let $I = [0, \infty)$.

Theorem 1 Suppose $\alpha \in C^1(I,I)$ is increasing function with $\alpha(t) \leq t$, $\alpha(0) = 0$, $\forall t \in I$. We assume that u(t) and f(t) are nonnegative real-valued continuous functions defined on I, and they satisfy the inequality

$$u^{p+1}(t) \le u_0 + \left(\int_0^{\alpha(t)} f(s) u^p(s) \, ds\right)^2$$

$$+ 2 \int_0^{\alpha(t)} f(s) u^p(s) \left[u^p(s) + \int_0^s f(\lambda) u^p(\lambda) \, d\lambda \right] ds, \quad \forall t \in I,$$

$$(2.1)$$

where $u_0 > 0$ and $p \in (0,1)$ are constants. Then

$$u(t) \le u_0^{\frac{1}{p+1}} + \frac{2}{p+1} \int_0^{\alpha(t)} f(s)\theta_1(\alpha^{-1}(s)) ds, \quad \forall t \in I,$$
 (2.2)

where

$$\theta_1(t) = \beta_1(t) \left[u_0^{\frac{1-p}{1+p}} + 2\frac{1-p}{1+p} \int_0^{\alpha(t)} f(s) \exp\left(-2\frac{1-p}{p} \int_0^s f(\lambda) \, d\lambda\right) ds \right]^{\frac{p}{1-p}},\tag{2.3}$$

and $\beta_1(t) = \exp(2\int_0^{\alpha(t)} f(s) ds)$, for all $t \in I$.

Remark 1 *If* $\alpha(t) = t$, then Theorem 1 reduces Lemma 1.

Proof Let $z_1^{p+1}(t)$ denote the function on the right-hand side of (2.1), which is a positive and nondecreasing function on I with $z_1(0) = u_0^{\frac{1}{p+1}}$. Then (2.1) is equivalent to

$$u(t) \le z_1(t), \qquad u(\alpha(t)) \le z_1(\alpha(t)), \quad \forall t \in I.$$
 (2.4)

Differentiating $z_1^{p+1}(t)$ with respect to t, using (2.4) we have

$$(p+1)z_{1}^{p}(t)\frac{dz_{1}(t)}{dt} = 2\alpha'(t)f(\alpha(t))u^{p}(\alpha(t))\int_{0}^{\alpha(t)}f(s)u^{p}(s)ds$$

$$+2\alpha'(t)f(\alpha(t))u^{p}(\alpha(t))\left[u^{p}(\alpha(t)) + \int_{0}^{\alpha(t)}f(\lambda)u^{p}(\lambda)d\lambda\right] \qquad (2.5)$$

$$\leq 2\alpha'(t)f(\alpha(t))z_{1}^{p}(t)\left[z_{1}^{p}(t) + 2\int_{0}^{\alpha(t)}f(\lambda)z_{1}^{p}(\lambda)d\lambda\right], \quad \forall t \in I.$$

Since $z_1^p(t) > 0$, from (2.5) we have

$$\frac{dz_1(t)}{dt} \le \frac{2}{p+1} \alpha'(t) f(\alpha(t)) Y_1(t), \quad \forall t \in I,$$
(2.6)

where

$$Y_1(t) := z_1^p(t) + 2 \int_0^{\alpha(t)} f(\lambda) z_1^p(\lambda) d\lambda, \quad \forall t \in I.$$

$$(2.7)$$

Then $Y_1(t)$ is a positive and nondecreasing function on I with $Y_1(0) = u_0^{p/(p+1)}$ and

$$z_1(t) \le Y_1(t)^{1/p}. \tag{2.8}$$

Differentiating $Y_1(t)$ with respect to t, and using (2.6), (2.7) and (2.8), we get

$$\frac{dY_1(t)}{dt} \leq \frac{2p}{p+1}\alpha'(t)f(\alpha(t))z_1^{p-1}(t)Y_1(t) + 2\alpha'(t)f(\alpha(t))z_1^p(\alpha(t))$$

$$\leq \frac{2p}{p+1}\alpha'(t)f(\alpha(t))Y_1^{\frac{2p-1}{p}}(t) + 2\alpha'(t)f(\alpha(t))Y_1(t), \quad \forall t \in I.$$
(2.9)

From (2.9), we have

$$Y_{1}^{\frac{1-2p}{p}}(t)\frac{dY_{1}(t)}{dt} - 2\alpha'(t)f(\alpha(t))Y_{1}^{\frac{1-p}{p}}(t) \le \frac{2p}{p+1}\alpha'(t)f(\alpha(t)), \quad \forall t \in I.$$
 (2.10)

Let $S_1(t) = Y_1^{\frac{1-p}{p}}(t)$, then $S_1(0) = u_0^{\frac{1-p}{p+1}}$, from (2.10) we obtain

$$\frac{dS_1(t)}{dt} - 2\frac{1-p}{p}\alpha'(t)f(\alpha(t))S_1(t) \le 2\frac{1-p}{p+1}\alpha'(t)f(\alpha(t)), \quad \forall t \in I.$$
(2.11)

Consider the ordinary differential equation

$$\begin{cases} \frac{dS_{2}(t)}{dt} - 2\frac{1-p}{p}\alpha'(t)f(\alpha(t))S_{2}(t) = 2\frac{1-p}{p+1}\alpha'(t)f(\alpha(t)), & \forall t \in I, \\ S_{2}(0) = u_{0}^{\frac{1-p}{p+1}}. \end{cases}$$
(2.12)

The solution of Equation (2.12) is

$$S_{2}(t) = \exp\left(\int_{0}^{\alpha(t)} 2\frac{1-p}{p} f(s) \, ds\right) \left(u_{0}^{\frac{1-p}{p+1}} + \int_{0}^{\alpha(t)} 2\frac{1-p}{p+1} f(s) \exp\left(-\int_{0}^{s} 2\frac{1-p}{p} f(\tau) \, d\tau\right) ds\right), \tag{2.13}$$

for all $t \in I$. By (2.11), (2.12) and (2.13), we obtain

$$Y_1(t) = S_1^{\frac{p}{1-p}}(t) \le S_2^{\frac{p}{1-p}}(t) = \theta_1(t), \quad \forall t \in I,$$
 (2.14)

where $\theta_1(t)$ as defined in (2.3). From (2.6) and (2.14), we have

$$\frac{dz_1(t)}{dt} \le \frac{2}{p+1} \alpha'(t) f(\alpha(t)) \theta_1(t), \quad \forall t \in I.$$

By taking t = s in the above inequality and integrating it from 0 to t, we get

$$u(t) \le z_1(t) \le u_0^{\frac{1}{p+1}} + \frac{2}{p+1} \int_0^{\alpha(t)} f(s)\theta_1(\alpha^{-1}(s)) ds, \quad \forall t \in I.$$

The estimation (2.2) of the unknown function in the inequality (2.1) is obtained. \Box

Theorem 2 Suppose $\alpha \in C^1(I,I)$ is increasing function with $\alpha(t) \leq t$, $\alpha(0) = 0$, $\forall t \in I$. We assume that u(t) and f(t) are nonnegative real-valued continuous functions defined on I and satisfy the inequality

$$u^{p+1}(t) \le u_0 + \left(\int_0^{\alpha(t)} f(s) u^p(s) \, ds\right)^2$$

$$+ 2 \int_0^{\alpha(t)} f(s) u^p(s) \left[u(s) + \int_0^s f(\lambda) u(\lambda) \, d\lambda \right] ds, \quad \forall t \in I,$$

$$(2.15)$$

where $u_0 > 0$ and $p \in (0,1)$ are constants. Then

$$u(t) \le u_0^{\frac{1}{p+1}} + \frac{2}{p+1} \int_0^{\alpha(t)} f(s)\theta_2(\alpha^{-1}(s)) ds, \quad \forall t \in I,$$
 (2.16)

where

$$\theta_2(t) = \beta_2(t) \left[u_0^{\frac{1-p}{p+1}} + \int_0^{\alpha(t)} (1-p)f(s) \exp\left(-\int_0^s \frac{(1-p)(p+3)}{p+1} f(\tau) d\tau\right) ds \right]^{\frac{1}{1-p}}, \quad (2.17)$$

and
$$\beta_2(t) = \exp(\int_0^{\alpha(t)} \frac{p+3}{p+1} f(s) ds)$$
, for all $t \in I$.

Proof Let $z_2^{p+1}(t)$ denote the function on the right-hand side of (2.15), which is a positive and nondecreasing function on I with $z_2(0) = u_0^{\frac{1}{p+1}}$. Then (2.15) is equivalent to

$$u(t) < z_2(t), \qquad u(\alpha(t)) < z_2(\alpha(t)), \quad \forall t \in I.$$
 (2.18)

Differentiating $z_2^{p+1}(t)$ with respect to t, using (2.18) we have

$$(p+1)z_{2}^{p}(t)\frac{dz_{2}(t)}{dt} = 2\alpha'(t)f(\alpha(t))u^{p}(\alpha(t))\int_{0}^{\alpha(t)}f(s)u^{p}(s)ds$$

$$+2\alpha'(t)f(\alpha(t))u^{p}(\alpha(t))\left[u(\alpha(t))+\int_{0}^{\alpha(t)}f(\lambda)u(\lambda)d\lambda\right]$$

$$\leq 2\alpha'(t)f(\alpha(t))z_{2}^{p}(t)\left[z_{2}(t)+\int_{0}^{\alpha(t)}f(\lambda)z_{2}(\lambda)d\lambda\right]$$

$$+\int_{0}^{\alpha(t)}f(\lambda)z_{2}^{p}(\lambda)d\lambda\right], \quad \forall t \in I.$$

$$(2.19)$$

Since $z_2^p(t) > 0$, we have

$$\frac{dz_2(t)}{dt} \le \frac{2}{p+1} \alpha'(t) f(\alpha(t)) Y_2(t), \quad \forall t \in I,$$
(2.20)

where

$$Y_2(t) := z_2(t) + \int_0^{\alpha(t)} f(\lambda) z_2(\lambda) d\lambda + \int_0^{\alpha(t)} f(\lambda) z_2^p(\lambda) d\lambda, \quad \forall t \in I.$$
 (2.21)

Then $Y_2(t)$ is a positive and nondecreasing function on I with $Y_2(0) = z_2(0) = u_0^{1/(p+1)}$ and

$$z_2(t) \le Y_2(t). \tag{2.22}$$

Differentiating $Y_2(t)$ with respect to t, and using (2.20), (2.21) and (2.22), we get

$$\frac{dY_{2}(t)}{dt} \leq \frac{2}{p+1}\alpha'(t)f(\alpha(t))Y_{2}(t) + \alpha'(t)f(\alpha(t))z_{2}(\alpha(t)) + \alpha'(t)f(\alpha(t))z_{2}^{p}(\alpha(t))$$

$$\leq \frac{p+3}{p+1}\alpha'(t)f(\alpha(t))Y_{2}(t) + \alpha'(t)f(\alpha(t))Y_{2}^{p}(t), \quad \forall t \in I.$$
(2.23)

From (2.23), we have

$$Y_{2}^{-p}(t)\frac{dY_{2}(t)}{dt} - \frac{p+3}{p+1}\alpha'(t)f(\alpha(t))Y_{2}^{1-p}(t) \le \alpha'(t)f(\alpha(t)), \quad \forall t \in I.$$
 (2.24)

Let $S_3(t) = Y_2^{1-p}(t)$, then $S_3(0) = u_0^{\frac{1-p}{p+1}}$, from (2.24) we obtain

$$\frac{dS_3(t)}{dt} - \frac{(1-p)(p+3)}{p+1}\alpha'(t)f(\alpha(t))S_3(t) \le (1-p)\alpha'(t)f(\alpha(t)), \quad \forall t \in I.$$
 (2.25)

Consider the ordinary differential equation

$$\begin{cases} \frac{dS_4(t)}{dt} - \frac{(1-p)(p+3)}{p+1} \alpha'(t) f(\alpha(t)) S_4(t) = (1-p)\alpha'(t) f(\alpha(t)), & \forall t \in I, \\ S_4(0) = u_0^{\frac{1-p}{p+1}}. \end{cases}$$
(2.26)

The solution of Equation (2.26) is

$$S_4(t) = \exp\left(\int_0^{\alpha(t)} \frac{(1-p)(p+3)}{p+1} f(s) \, ds\right) \left(u_0^{\frac{1-p}{p+1}} + \int_0^{\alpha(t)} (1-p)f(s) \exp\left(-\int_0^s \frac{(1-p)(p+3)}{p+1} f(\tau) \, d\tau\right) ds\right), \tag{2.27}$$

for all $t \in I$. By (2.25), (2.26) and (2.27), we obtain

$$Y_2(t) = S_3^{\frac{1}{1-p}}(t) \le S_4^{\frac{1}{1-p}}(t) = \theta_2(t), \quad \forall t \in I,$$
 (2.28)

where $\theta_2(t)$ as defined in (2.17). From (2.20) and (2.28), we have

$$\frac{dz_2(t)}{dt} \le \frac{2}{p+1} \alpha'(t) f(\alpha(t)) \theta_2(t), \quad \forall t \in I.$$

By taking t = s in the above inequality and integrating it from 0 to t, we get

$$u(t) \le z_2(t) \le u_0^{\frac{1}{p+1}} + \frac{2}{p+1} \int_0^{\alpha(t)} f(s)\theta_2(\alpha^{-1}(s)) ds, \quad \forall t \in I.$$

The estimation (2.16) of the unknown function in the inequality (2.15) is obtained. \Box

Theorem 3 Suppose $\phi_1, \phi_2, \phi_2/\phi_1, \alpha \in C^1(I,I)$ are increasing functions with $\alpha(t) \leq t$, $\phi_i(t) > 0$, $\forall t > 0$, i = 1, 2, $\alpha(0) = 0$. We assume that u(t) and f(t) are nonnegative real-valued continuous functions defined on I and satisfy the inequality

$$u(t) \leq u_0 + \left(\int_0^{\alpha(t)} f(s)\phi_1(u(s)) ds\right)^2$$

$$+ 2\int_0^{\alpha(t)} f(s)\phi_1(u(s)) \left[u(s) + \int_0^s g(\lambda)\phi_2(u(\lambda)) d\lambda\right] ds, \quad \forall t \in I,$$

$$(2.29)$$

where $u_0 > 0$ is a constant. Then

$$u(t) \le \Phi_1^{-1} \left[\Phi_2^{-1} \left(\Phi_2 \left(\Phi_1(u_0) + \int_0^{\alpha(t)} g(s) \, ds \right) + \int_0^{\alpha(t)} 2f(s) \, ds \right) \right], \quad \forall t < T_1,$$
 (2.30)

where

$$\Phi_1(r) := \int_1^r \frac{dt}{\phi_2(t)}, \quad r > 0, \tag{2.31}$$

$$\Phi_2(r) := \int_1^r \frac{\phi_2(\Phi_1^{-1}(s)) \, ds}{\phi_1(\Phi_1^{-1}(s))(\Phi_1^{-1}(s) + 1)}, \quad r > 0, \tag{2.32}$$

and T_1 is the largest number such that

$$\Phi_{2}\left(\Phi_{1}(u_{0}) + \int_{0}^{\alpha(t)} g(s) \, ds\right) + \int_{0}^{\alpha(t)} 2f(s) \, ds \leq \int_{1}^{\infty} \frac{\phi_{2}(\Phi_{1}^{-1}(s)) \, ds}{\phi_{1}(\Phi_{1}^{-1}(s))(\Phi_{1}^{-1}(s) + 1)},$$

$$\Phi_{2}^{-1}\left(\Phi_{2}\left(\Phi_{1}(u_{0}) + \int_{0}^{\alpha(t)} g(s) \, ds\right) + \int_{0}^{\alpha(t)} 2f(s) \, ds\right) \leq \int_{1}^{\infty} \frac{dt}{\phi_{2}(t)}$$

for all $t \leq T_1$.

Proof Let $z_3(t)$ denote the function on the right-hand side of (2.29), which is a positive and nondecreasing function on I with $z_3(0) = u_0$. Then (2.29) is equivalent to

$$u(t) \le z_3(t), \qquad u(\alpha(t)) \le z_3(t), \quad \forall t \in I.$$
 (2.33)

Differentiating $z_3(t)$ with respect to t, using (2.33) we have

$$\frac{dz_{3}(t)}{dt} \leq 2\alpha'(t)f(\alpha(t))\phi_{1}(z_{3}(t)) \int_{0}^{\alpha(t)} f(s)\phi_{1}(z_{3}(s)) ds
+ 2\alpha'(t)f(\alpha(t))\phi_{1}(z_{3}(t)) \left[z_{3}(t) + \int_{0}^{\alpha(t)} g(\lambda)\phi_{2}(z_{3}(\lambda)) d\lambda \right] ds
\leq 2\alpha'(t)f(\alpha(t))\phi_{1}(z_{3}(t)) \left[z_{3}(t) + \int_{0}^{\alpha(t)} f(s)\phi_{1}(z_{3}(s)) ds \right]
+ \int_{0}^{\alpha(t)} g(\lambda)\phi_{2}(z_{3}(\lambda)) d\lambda , \quad \forall t \in I.$$
(2.34)

Let

$$Y_3(t) := z_3(t) + \int_0^{\alpha(t)} f(s)\phi_1(z_3(s)) ds + \int_0^{\alpha(t)} g(\lambda)\phi_2(z_3(\lambda)) d\lambda, \quad \forall t \in I.$$
 (2.35)

Then $Y_3(t)$ is a positive and nondecreasing function on I with $Y_3(0) = z_3(0) = u_0$ and

$$z_3(t) \le Y_3(t). \tag{2.36}$$

Differentiating $Y_3(t)$ with respect to t, and using (2.34), (2.35) and (2.36), we get

$$\frac{dY_3(t)}{dt} \leq 2\alpha'(t)f(\alpha(t))\phi_1(z_3(t))Y_3(t) + \alpha'(t)f(\alpha(t))\phi_1(z_3(t))
+ \alpha'(t)g(\alpha(t))\phi_2(z_3(t))
\leq 2\alpha'(t)f(\alpha(t))\phi_1(Y_3(t))(Y_3(t) + 1) + \alpha'(t)g(\alpha(t))\phi_2(Y_3(t)),$$
(2.37)

for all $t \in I$. Since $\phi_2(Y_3(t)) > 0$, $\forall t > 0$, from (2.37) we have

$$\frac{dY_3(t)}{\phi_2(Y_3(t))} \leq 2\alpha'(t)f\left(\alpha(t)\right)\frac{\phi_1(Y_3(t))(Y_3(t)+1)}{\phi_2(Y_3(t))} + \alpha'(t)g\left(\alpha(t)\right), \quad \forall t \in I.$$

By taking t = s in the above inequality and integrating it from 0 to t, we get

$$\Phi_{1}(Y_{3}(t)) \leq \Phi_{1}(Y_{3}(0)) + \int_{0}^{t} 2\alpha'(s)f(\alpha(s)) \frac{\phi_{1}(Y_{3}(s))(Y_{3}(s)+1)}{\phi_{2}(Y_{3}(s))} ds
+ \int_{0}^{t} \alpha'(s)g(\alpha(s)) ds,$$
(2.38)

for all $t \in I$, where Φ_1 is defined by (2.31). From (2.38), we have

$$\Phi_{1}(Y_{3}(t)) \leq \Phi_{1}(Y_{3}(0)) + \int_{0}^{T} \alpha'(s)g(\alpha(s)) ds
+ \int_{0}^{t} 2\alpha'(s)f(\alpha(s)) \frac{\phi_{1}(Y_{3}(s))(Y_{3}(s) + 1)}{\phi_{2}(Y_{3}(s))} ds,$$
(2.39)

for all t < T, where $0 < T < T_1$ is chosen arbitrarily. Let $Y_4(t)$ denote the function on the right-hand side of (2.39), which is a positive and nondecreasing function on I with $Y_4(0) = T_1(t)$

 $\Phi_1(u_0) + \int_0^T \alpha'(s)g(\alpha(s)) ds$ and

$$Y_3(t) \le \Phi_1^{-1}(Y_4(t)), \quad \forall t < T.$$
 (2.40)

Differentiating $Y_4(t)$ with respect to t, using the hypothesis on ϕ_2/ϕ_1 , from (2.40) we have

$$\frac{dY_4(t)}{dt} \le 2\alpha'(t)f(\alpha(t))\frac{\phi_1(Y_3(t))(Y_3(t)+1)}{\phi_2(Y_3(t))} \\
\le 2\alpha'(t)f(\alpha(t))\frac{\phi_1(\Phi_1^{-1}(Y_4(t)))(\Phi_1^{-1}(Y_4(t))+1)}{\phi_2(\Phi_1^{-1}(Y_4(t)))}, \quad \forall t < T.$$
(2.41)

By the definition of Φ_2 in (2.32), from (2.41) we obtain

$$\Phi_{2}(Y_{4}(t)) \leq \Phi_{2}(Y_{4}(0)) + \int_{0}^{t} 2\alpha'(s)f(\alpha(s)) ds
\leq \Phi_{2}(\Phi_{1}(u_{0}) + \int_{0}^{\alpha(T)} g(s) ds) + \int_{0}^{\alpha(t)} 2f(s) ds, \quad \forall t < T.$$
(2.42)

Let t = T, from (2.42) we have

$$\Phi_2(Y_4(T)) \le \Phi_2\left(\Phi_1(u_0) + \int_0^{\alpha(T)} g(s) \, ds\right) + \int_0^{\alpha(T)} 2f(s) \, ds. \tag{2.43}$$

Since $0 < T < T_1$ is chosen arbitrarily, from (2.33), (2.36), (2.40) and (2.43), we have

$$u(t) \leq \Phi_1^{-1} \left[\Phi_2^{-1} \left(\Phi_2 \left(\Phi_1(u_0) + \int_0^{\alpha(t)} g(s) \, ds \right) + \int_0^{\alpha(t)} 2f(s) \, ds \right) \right], \quad \forall t < T_1.$$

This proves
$$(2.30)$$
.

3 Application

In this section, we apply our Theorem 3 to the following differential-integral equation

$$\begin{cases} \frac{dx(t)}{dt} = F(t, x(\alpha(t))) + H(t, x(\alpha(t))), & \forall t \in I, \\ x(0) = x_0, \end{cases}$$
(3.1)

where $F \in C(I \times I, \mathbf{R}), H \in C(I^3, \mathbf{R}), |x_0| > 0$ is a constant satisfying the following conditions

$$\left| F(t, x(t)) \right| \le f^2(t)\phi_1^2(\left| x(t) \right|),\tag{3.2}$$

$$\left|H(t,x(t))\right| \le 2f(t)\phi_1(\left|x(t)\right|)\left(\left|x(t)\right| + \int_0^t g(s)\phi_2(\left|x(s)\right|) ds\right),\tag{3.3}$$

where f, g is nonnegative real-valued continuous function defined on I.

Corollary 1 Consider the nonlinear system (3.1) and suppose that F, H satisfy the conditions (3.2) and (3.3), and $\phi_1, \phi_2, \phi_2/\phi_1, \alpha \in C^1(I,I)$ are increasing functions with $\alpha(t) \leq t$,

 $\phi_i(t) > 0$, $\forall t > 0$, i = 1, 2, $\alpha(0) = 0$. Then all solutions of Equation (3.1) exist on I and satisfy the following estimation

$$|x(t)| \le \Phi_1^{-1} \left[\Phi_2^{-1} \left(\Phi_2 \left(\Phi_1 \left(|x_0| \right) + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \, ds \right) + \int_0^{\alpha(t)} 2 \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \, ds \right) \right], (3.4)$$

for all $t < T_2$, where

$$\begin{split} &\Phi_1(r) := \int_1^r \frac{dt}{\phi_2(t)}, \quad r > 0, \\ &\Phi_2(r) := \int_1^r \frac{\phi_2(\Phi_1^{-1}(s)) \, ds}{\phi_1(\Phi_1^{-1}(s))(\Phi_1^{-1}(s) + 1)}, \quad r > 0, \end{split}$$

and T_2 is the largest number such that

$$\begin{split} & \Phi_2\bigg(\Phi_1\big(|x_0|\big) + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \, ds\bigg) + \int_0^{\alpha(t)} 2\frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \, ds \\ & \leq \int_1^\infty \frac{\phi_2(\Phi_1^{-1}(s)) \, ds}{\phi_1(\Phi_1^{-1}(s))(\Phi_1^{-1}(s)+1)}, \\ & \Phi_2^{-1}\bigg(\Phi_2\bigg(\Phi_1\big(|x_0|\big) + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \, ds\bigg) + \int_0^{\alpha(t)} 2\frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \, ds\bigg) \leq \int_1^\infty \frac{dt}{\phi_2(t)} \end{split}$$

for all $t \leq T_2$.

Proof Integrating both sides of Equation (3.1) from 0 to t, we get

$$x(t) = x_0 + \int_0^t F(s, x(\alpha(s))) ds + \int_0^t H(s, x(\alpha(s))) ds, \quad \forall t \in I.$$
 (3.5)

Using the conditions (3.2) and (3.3), from (3.5) we obtain

$$|x(t)| \leq |x_{0}| + \int_{0}^{t} f^{2}(s)\phi_{1}^{2}(|x(\alpha(s))|) ds$$

$$+ 2 \int_{0}^{t} f(s)\phi_{1}(|x(\alpha(s))|) \left(|x(\alpha(s))| + \int_{0}^{s} g(\tau)\phi_{2}(|x(\alpha(\tau))|) d\tau\right) ds$$

$$\leq |x_{0}| + \left(\int_{0}^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \phi_{1}(|x(s)|) ds\right)^{2}$$

$$+ 2 \int_{0}^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \phi_{1}(|x(s)|) \left(|x(s)| + \int_{0}^{s} \frac{g(\alpha^{-1}(\tau))}{\alpha'(\alpha^{-1}(\tau))} \phi_{2}(|x(\tau)|) d\tau\right) ds,$$

$$(3.6)$$

for all $t \in I$. Applying Theorem 3 to (3.6), we get the estimation (3.4). This completes the proof of the Corollary 1.

Competing interests

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