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Applications of differential subordinations for certain classes of *p*-valent functions associated with generalized Srivastava-Attiya operator

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Abstract

The object of the present paper is to investigate some inclusion relations and other interesting properties for certain classes of *p*-valent functions involving generalized Srivastava-Attiya operator by using the principle of differential subordination.

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1 Introduction

Let $A(p)$ be the class of functions which are analytic and p-valent in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad \big(p \in \mathbb{N} = \{1, 2, \ldots\}\big). \tag{1.1}$$

Let also $A(1) = A_1$. For $g(z) \in A(p)$, given by $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$, the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z). \tag{1.2}$$

Next, in the usual notation, let $\Phi(z, s, a)$ denote the Hurwitz-Lerch Zeta function defined as follows:

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(k + a)^s} \tag{1.3}$$
$$\big(a \in \mathbb{C} \backslash \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}; s \in \mathbb{C} \text{ when } |z| < 1; \operatorname{Re}\{s\} > 1 \text{ when } |z| = 1\big).$$

For further interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ see [2, 5, 8, 9, 11], and [21].

Recently, Srivastava and Attiya [20] have introduced the linear operator $L_{s,b}: A_1 \to A_1$, defined in terms of the Hadamard product by

$$L_{s,b}(f)(z) = G_{s,b}(z) * f(z) \quad \left(z \in U; b \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C}\right), \tag{1.4}$$

where

$$G_{s,b} = (1+b)^s\left[\Phi(z,s,b) - b^{-s}\right] \quad (z \in U). \tag{1.5}$$

The Srivastava-Attiya operator $L_{s,b}$ contains, among its special cases, the integral operators introduced and investigated by Alexander [1], Libera [7] and Jung et al. [6].

Analogous to $L_{s,b}$, Liu [10] defined the operator $J_{p,s,b}: A(p) \to A(p)$ by

$$J_{p,s,b}(f)(z) = G_{p,s,b}(z) * f(z) \quad \left(z \in U; b \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C}; p \in \mathbb{N}\right), \tag{1.6}$$

where

$$G_{p,s,b} = (1+b)^s\left[\Phi_p(z,s,b) - b^{-s}\right]$$

and

$$\Phi_p(z,s,b) = \frac{1}{b^s} + \sum_{k=0}^{\infty} \frac{z^{k+p}}{(k+1+b)^s}. \tag{1.7}$$

It is easy to observe from (1.6) and (1.7) that

$$J_{p,s,b}(f)(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{1+b}{k+1+b}\right)^s a_{k+p} z^{k+p}. \tag{1.8}$$

We note that

(i) $J_{p,0,b}(f)(z) = f(z)$;

(ii) $J_{1,1,0}(f)(z) = Lf(z) = \int_0^z \frac{f(t)}{t}\,dt$ $(f \in A_1)$, where the operator L was introduced by Alexander [1];

(iii) $J_{1,s,b}(f)(z) = L_{s,b}f(z)$ $(s \in \mathbb{C}, b \in \mathbb{C}\backslash\mathbb{Z}_0^-)$, where the operator $L_{s,b}$ was introduced by Srivastava-Attiya [20];

(iv) $J_{p,1,\mu+p-1}(f)(z) = F_{\mu,p}(f)(z)$ $(\mu > -p, p \in \mathbb{N})$, where the operator $F_{\mu,p}$ was introduced by Choi et al. [3];

(v) $J_{p,\alpha,p}(f)(z) = I_p^\alpha f(z)$ $(\alpha > 0, p \in \mathbb{N})$, where the operator I_p^α was introduced by Shams et al. [18];

(vi) $J_{p,\gamma,p-1}(f)(z) = J_p^\gamma f(z)$ $(\gamma \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, p \in \mathbb{N})$, where the operator J_p^γ was introduced by El-Ashwah and Aouf [4];

(vii) $J_{p,\gamma,p+l-1}(f)(z) = J_p^\gamma(l)f(z)$ $(\gamma \in \mathbb{N}_0, p \in \mathbb{N}, l \geq 0)$, where the operator $J_p^\gamma(l)$ was introduced by El-Ashwah and Aouf [4].

It follows from (1.8) that

$$z\left(J_{p,s,b}(f)(z)\right)' = (b+1)J_{p,s-1,b}(f)(z) - (b+1-p)J_{p,s,b}(f)(z). \tag{1.9}$$

For two analytic functions $f, g \in A(p)$, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence (see [14]):

$$f(z) \prec g(z) \quad \Leftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Definition 1 For fixed parameters A and B, with $-1 \leq B < A \leq 1$, we say that $f \in A(p)$ is in the class $S_p^{s,b}(A,B)$ if it satisfies the following subordination condition:

$$\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz} \quad (p \in \mathbb{N}). \tag{1.10}$$

In view of the definition of subordination (1.10) is equivalent to the following condition:

$$\left| \frac{\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} - 1}{B\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} - A} \right| < 1 \quad (z \in U).$$

For convenience, we write $S_p^{s,b}(1-\frac{2\eta}{p}, -1) = S_p^{s,b}(\eta)$, where $S_p^{s,b}(\eta)$ denotes the class of functions in $A(p)$ satisfying the inequality

$$\operatorname{Re}\left(\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}}\right) > \eta \quad (0 \leq \eta < 1; p \in \mathbb{N}; z \in U).$$

In the present paper, we investigate some inclusion relations and other interesting properties for certain classes of p-valent functions involving an integral operator.

2 Preliminaries

To establish our main results, we need the following lemmas.

Lemma 1 ([13, 14]) *Let h be analytic and convex (univalent) in U with $h(0) = 1$. Suppose also that the function φ given by*

$$\varphi(z) = 1 + c_m z^m + c_{m+1} z^{m+1} + \cdots, \tag{2.1}$$

is analytic in U, where m is a positive integer. If

$$\varphi(z) + \frac{z\varphi'(z)}{\varrho} \prec h(z) \quad \left(\operatorname{Re}\{\varrho\} \geq 0; \varrho \neq 0\right), \tag{2.2}$$

then

$$\varphi(z) \prec \psi(z) = \frac{\varrho}{m} z^{-\frac{\varrho}{m}} \int_0^z t^{\frac{\varrho}{m}-1} h(t)\, dt \prec h(z) \tag{2.3}$$

and $\psi(z)$ is the best dominant of (2.2).

We denote by $H(\varrho)$ the class of functions $\Phi(z)$ given by

$$\Phi(z) = 1 + c_1 z + c_2 z^2 + \cdots, \tag{2.4}$$

which are analytic in U and satisfy the following inequality:

$$\operatorname{Re}\{\Phi(z)\} > \varrho \quad (0 \le \varrho < 1; z \in U).$$

Lemma 2 ([17]) *Let the function $\Phi(z) \in H(\varrho)$, where $\Phi(z)$ given by (2.4). Then*

$$\operatorname{Re}\{\Phi(\varrho)\} \ge 2\varrho - 1 + \frac{2(1-\varrho)}{1+|z|} \quad (0 \le \varrho < 1; z \in U).$$

Lemma 3 ([22]) *For $0 \le \varrho_1, \varrho_2 < 1$,*

$$H(\varrho_1) * H(\varrho_2) \subset H(\varrho_3), \quad \varrho_3 = 1 - 2(1-\varrho_1)(1-\varrho_2).$$

The result is best possible.

Lemma 4 ([24]) *Let μ be a positive measure on the unit interval $[0,1]$. Let $g(z,t)$ be a complex valued function defined on $U \times [0,1]$ such that $g(0,t)$ is analytic in U for each $t \in [0,1]$ and such that $g(z,0)$ is μ integrable on $[0,1]$ for all $z \in U$. In addition, suppose that $\operatorname{Re}\{g(z,t)\} > 0$, $g(-r,t)$ is real and*

$$\operatorname{Re}\left\{\frac{1}{g(z,t)}\right\} \ge \frac{1}{g(-r,t)} \quad \bigl(|z| \le r < 1; t \in [0,1]\bigr).$$

If G is defined by

$$G(z) = \int_0^1 g(z,t)\, d\mu(t),$$

then

$$\operatorname{Re}\left\{\frac{1}{G(z)}\right\} \ge \frac{1}{G(-r)} \quad \bigl(|z| \le r < 1\bigr).$$

Lemma 5 ([19]) *Let the function g be analytic in U with $g(0) = 1$ and $\operatorname{Re}\{g(z)\} > \frac{1}{2}$ $(z \in U)$. Then, for any function F analytic in U, $(g * F)(U)$ is contained in the convex hull of $F(U)$.*

Lemma 6 ([16]) *Let φ be analytic in U with $\varphi(0) = 1$ and $\varphi(z) = 0$for $0 < |z| < 1$ and let $A, B \in \mathbb{C}$ with $A \ne B$, $|B| \le 1$.*

(i) *Let $B \ne 0$ and $\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ satisfy either $|\frac{\gamma(A-B)}{B} - 1| \le 1$ or $|\frac{\gamma(A-B)}{B} + 1| \le 1$. If φ satisfies*

$$1 + \frac{z\varphi'(z)}{\gamma\varphi(z)} \prec \frac{1+Az}{1+Bz}, \tag{2.5}$$

then

$$\varphi(z) \prec (1+Bz)^{\gamma(\frac{A-B}{B})},$$

and this is best dominant.

(ii) *Let $B = 0$ and $\gamma \in \mathbb{C}^*$ be such that $|\gamma A| < \pi$. If φ satisfies (2.5), then*

$$\varphi(z) \prec e^{\gamma A z}$$

and this is the best dominant.

For real or complex numbers a, n and c ($c \notin \mathbb{Z}_0^-$) and $z \in U$, the Gaussian hypergeometric function defined by

$$\begin{aligned} {}_2F_1(a,n;c;z) &= 1 + \frac{an}{c} \cdot \frac{z}{1!} + \frac{a(a+1)n(n+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(a)_k (n)_k}{(c)_k} \frac{z^k}{k!}, \end{aligned} \tag{2.6}$$

where $(d)_k = d(d+1)\cdots(d+k-1)$ and $(d)_0 = 1$. We note that the series defined by (2.6) converges absolutely for $z \in U$, and hence, ${}_2F_1$ represents an analytic function in U (see, for details, [23, Ch.14]).

Lemma 7 ([23]) *For real or complex numbers a, n and c ($c \notin \mathbb{Z}_0^-$)*

$$\int_0^1 t^{n-1}(1-t)^{c-n-1}(1-tz)^{-a}\,dt = \frac{\Gamma(n)\Gamma(c-n)}{\Gamma(c)} {}_2F_1(a,n;c;z) \quad \bigl(\mathrm{Re}\{n\}, \mathrm{Re}\{c\} > 0\bigr), \tag{2.7}$$

$${}_2F_1(a,n;c;z) = (1-z)^{-a} {}_2F_1\left(a, c-n; c; \frac{z}{z-1}\right) \tag{2.8}$$

and

$${}_2F_1(a,n;c;z) = {}_2F_1(n,a;c;z). \tag{2.9}$$

3 Main results

Unless otherwise mentioned, we assume throughout this paper that $-1 \le B < A \le 1$, $s \in \mathbb{C}$, $b \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $p \in \mathbb{N}\backslash\{1\}$, $0 < \alpha \le 1$, m is a positive integer and the powers are understood as principle values.

Theorem 1 *Let f given by (1.1) satisfy the following subordination condition:*

$$(1-\alpha)\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} + \alpha \frac{(J_{p,s,b}(f)(z))''}{p(p-1)z^{p-2}} \prec \frac{1+Az}{1+Bz}. \tag{3.1}$$

Then

$$\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} \prec \Psi(z) \prec \frac{1+Az}{1+Bz}, \tag{3.2}$$

where

$$\Psi(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1} {}_2F_1\left(1,1;\frac{p-1}{m\alpha}+1;\frac{Bz}{1+Bz}\right) & \text{for } B \neq 0, \\ 1 + \frac{p-1}{m\alpha+p-1}Az & \text{for } B = 0, \end{cases} \tag{3.3}$$

is the best dominant of (3.2). Furthermore,

$$f \in S_p^{s,b}(\beta), \tag{3.4}$$

where

$$\beta = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1} {}_2F_1\left(1,1;\frac{p-1}{m\alpha}+1;\frac{B}{B-1}\right) & \text{for } B \neq 0, \\ 1 - \frac{p-1}{m\alpha + p - 1}A & \text{for } B = 0. \end{cases} \tag{3.5}$$

The estimate (3.4) is best possible.

Proof Let

$$\theta(z) = \frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} \quad (z \in U), \tag{3.6}$$

where θ is of the form (2.1) and is analytic in U. Differentiating (3.6) with respect to z, we get

$$(1-\alpha)\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} + \alpha\frac{(J_{p,s,b}(f)(z))''}{p(p-1)z^{p-2}} = \theta(z) + \frac{\alpha}{p-1}z\theta'(z) \prec \frac{1+Az}{1+Bz}.$$

Applying Lemma 1 for $\varrho = \frac{p-1}{\alpha}$ and Lemma 7, we have

$$\begin{aligned} \frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} \prec \Psi(z) &= \frac{p-1}{m\alpha} z^{-\frac{p-1}{m\alpha}} \int_0^z t^{\frac{p-1}{m\alpha}-1}\left(\frac{1+At}{1+Bt}\right) dt \\ &= \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1} {}_2F_1\left(1,1;\frac{p-1}{m\alpha}+1;\frac{Bz}{1+Bz}\right) & \text{for } B \neq 0, \\ 1 + \frac{p-1}{m\alpha+p-1}Az & \text{for } B = 0. \end{cases} \end{aligned}$$

This proves the assertion (3.2) of Theorem 1. Next, in order to prove the assertion (3.4) of Theorem 1, it suffices to show that

$$\inf_{|z|<1}\left\{\mathrm{Re}\big(\Psi(z)\big)\right\} = \Psi(-1).$$

Indeed, we have

$$\mathrm{Re}\left\{\frac{1+Az}{1+Bz}\right\} \geq \frac{1-Ar}{1-Br} \quad \big(|z| \leq r < 1\big).$$

Setting

$$G(z,\zeta) = \frac{1+A\zeta z}{1+B\zeta z} \quad \text{and} \quad d\nu(\zeta) = \frac{p-1}{m\alpha}\zeta^{\frac{p-1}{m\alpha}-1}\,d\zeta \quad (0 \leq \zeta \leq 1),$$

which is a positive measure on the closed interval $[0,1]$, we get

$$\Psi(z) = \int_0^1 G(z,\zeta)\,d\nu(\zeta).$$

Then

$$\operatorname{Re}\{\Psi(z)\} \geq \int_0^1 \frac{1 - A\zeta r}{1 - B\zeta r}\,d\nu(\zeta) = \Psi(-r) \quad (|z| \leq r < 1).$$

Letting $r \to 1^-$ in the above inequality, we obtain the assertion (3.4). Finally, the estimate (3.4) is best possible as Ψ is the best dominant of (3.2). This completes the proof of Theorem 1. □

Theorem 2 *If* $f \in S_p^{s,b}(\eta)$ $(0 \leq \eta < 1)$, *then*

$$\operatorname{Re}\left\{(1-\alpha)\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} + \alpha\frac{(J_{p,s,b}(f)(z))''}{p(p-1)z^{p-2}}\right\} > \eta \quad (|z| < R),$$

where

$$R = \left\{\frac{\sqrt{(p-1)^2 + (m\alpha)^2} - m\alpha}{p-1}\right\}^{\frac{1}{m}}. \tag{3.7}$$

The result is best possible.

Proof Let $f \in S_p^{s,b}(\eta)$, then we write

$$\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} = \eta + (1-\eta)u(z) \quad (z \in U), \tag{3.8}$$

where u is of the form (2.1), is analytic in U and has a positive real part in U. Differentiating (3.8) with respect to z, we have

$$\frac{1}{1-\eta}\left\{(1-\alpha)\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} + \alpha\frac{(J_{p,s,b}(f)(z))''}{p(p-1)z^{p-2}} - \eta\right\} = u(z) + \frac{\alpha}{p-1}zu'(z). \tag{3.9}$$

Applying the following well-known estimate [12]:

$$\frac{|zu'(z)|}{\operatorname{Re}\{u(z)\}} \leq \frac{2mr^m}{1-r^{2m}} \quad (|z| = r < 1),$$

in (3.9), we have

$$\begin{aligned} &\frac{1}{1-\eta}\operatorname{Re}\left\{(1-\alpha)\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} + \alpha\frac{(J_{p,s,b}(f)(z))''}{p(p-1)z^{p-2}} - \eta\right\} \\ &\quad \geq \operatorname{Re}\{u(z)\}\left(1 - \frac{2\alpha m r^m}{(p-1)[1-r^{2m}]}\right), \end{aligned} \tag{3.10}$$

such that the right-hand side of (3.10) is positive, if $r < R$, where R is given by (3.7).

In order to show that the bound R is best possible, we consider the function $f \in A(p)$ defined by

$$\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} = \eta + (1-\eta)\frac{1+z^m}{1-z^m} \quad (0 \leq \eta < 1; z \in U).$$

Note that

$$\frac{1}{1-\eta}\left\{(1-\alpha)\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}}+\alpha\frac{(J_{p,s,b}(f)(z))''}{p(p-1)z^{p-2}}-\eta\right\}=\frac{(p-1)(1-z^{2m})-2\alpha mz^{m}}{(p-1)(1-z^{m})^{2}}=0,$$

for $z = R^{\cdot}\exp\{\frac{\mathbf{i}\pi}{\mathbf{m}}\}$. This completes the proof of Theorem 2. □

For a function $f \in A(p)$, the generalized Bernardi-Libera-Livingston integral operator $F_{\mu,p}$ is defined by

$$\begin{aligned}F_{\mu,p}(f)(z)&=\frac{\mu+p}{z^{\mu}}\int_0^z t^{\mu-1}f(t)\,dt=\left(z^p+\sum_{k=1}^{\infty}\frac{\mu+p}{\mu+p+k}z^{k+p}\right)*f(z)\\&=z^p{}_2F_1(1,\mu+p;\mu+p+1;z)*f(z)\quad(\mu>-p;z\in U).\end{aligned}\tag{3.11}$$

From (1.8) and (3.11), we have

$$z\big(J_{p,s,b}\mathrm{F}_{\mu,p}\big(f(z)\big)\big)'=(\mu+p)J_{p,s,b}(f)(z)-\mu J_{p,s,b}\mathrm{F}_{\mu,p}\big(f(z)\big)\quad(\mu>-p;z\in U)\tag{3.12}$$

and

$$J_{p,s,b}\mathrm{F}_{\mu,p}\big(f(z)\big)=\mathrm{F}_{\mu,p}\big(J_{p,s,b}(f)(z)\big).$$

Theorem 3 *Let $f \in S_p^{s,b}(A,B)$ and $F_{\mu,p}$ be defined by (3.11). Then*

$$\frac{(J_{p,s,b}\mathrm{F}_{\mu,p}(f(z)))'}{pz^{p-1}}\prec\Phi(z)\prec\frac{1+Az}{1+Bz},\tag{3.13}$$

where

$$\Phi(z)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1,1;\frac{\mu+p}{m}+1;\frac{Bz}{1+Bz}\right) & \text{for } B\neq 0,\\ 1+\frac{\mu+p}{\mu+p+m}Az & \text{for } B=0,\end{cases}\tag{3.14}$$

is the best dominant of (3.13). Furthermore,

$$\operatorname{Re}\left\{\frac{(J_{p,s,b}\mathrm{F}_{\mu,p}(f(z)))'}{pz^{p-1}}\right\}>\psi\quad(z\in U),$$

where

$$\psi=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1,1;\frac{\mu+p}{m}+1;\frac{B}{B-1}\right) & \text{for } B\neq 0,\\ 1-\frac{\mu+p}{\mu+p+m}A & \text{for } B=0.\end{cases}$$

The result is best possible.

Proof Let

$$K(z)=\frac{(J_{p,s,b}\mathrm{F}_{\mu,p}(f(z)))'}{pz^{p-1}}\quad(z\in U),\tag{3.15}$$

where K is of the form (2.1) and is analytic in U. Using (3.12) in (3.15) and differentiating the resulting equation with respect to z, we have

$$\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} = K(z) + \frac{zK'(z)}{p+\mu} \prec \frac{1+Az}{1+Bz}.$$

The remaining part of the proof is similar to that of Theorem 1, and so we omit it. □

We note that

$$\frac{(J_{p,s,b}\mathrm{F}_{\mu,p}(f(z)))'}{pz^{p-1}} = \frac{p+\mu}{pz^{p+\mu}} \int_0^z t^{\mu} \left(J_{p,s,b}(f)(t)\right)' \, dt \quad \left(f \in A(p); z \in U\right). \tag{3.16}$$

Putting $A = 1 - \frac{2\delta}{p}$ $(0 \leq \delta < 1)$ and $B = -1$ in Theorem 3 and using (3.16), we obtain the following corollary.

Corollary 1 *If $f \in A(p)$ satisfies the following inequality:*

$$\mathrm{Re}\left\{\frac{(J_{p,s,b}f(z))'}{pz^{p-1}}\right\} > \delta \quad (0 \leq \delta < 1; z \in U),$$

then

$$\mathrm{Re}\left\{\frac{p+\mu}{pz^{p+\mu}} \int_0^z t^{\mu} \left(J_{p,s,b}(f)(t)\right)' \, dt\right\} > \frac{\delta}{p} + \left(1 - \frac{\delta}{p}\right)\left[{}_2F_1\left(1,1;\frac{\mu+p}{m}+1;\frac{1}{2}\right) - 1\right] \quad (z \in U).$$

The result is best possible.

Theorem 4 *Let $f, g \in A(p)$ satisfy the following inequality:*

$$\mathrm{Re}\left\{\frac{J_{p,s,b}(g)(z)}{z^p}\right\} > 0 \quad (z \in U).$$

If

$$\left|\frac{J_{p,s,b}(f)(z)}{J_{p,s,b}(g)(z)} - 1\right| < 1 \quad (z \in U),$$

then

$$\mathrm{Re}\left\{\frac{z(J_{p,s,b}(f)(z))'}{J_{p,s,b}(f)(z)}\right\} > 0 \quad \left(|z| < \mathring{R}\right),$$

where

$$\mathring{R} = \left(\frac{-3m + \sqrt{9m^2 + 4p(p+m)}}{2(p+m)}\right)^{\frac{1}{m}}. \tag{3.17}$$

Proof Let

$$q(z) = \frac{J_{p,s,b}(f)(z)}{J_{p,s,b}(g)(z)} - 1 = c_m z^m + c_{m+1} z^{m+1} + \cdots, \tag{3.18}$$

where $q(z)$ is analytic in U with $q(0) = 0$ and $|q(z)| \leq |z|^m$. Then, by applying the familiar Schwartz Lemma [15], we have $q(z) = z^m \mathcal{X}(z)$, where $\mathcal{X}$ is analytic in U and $|\mathcal{X}(z)| \leq 1$. Therefore (3.18) leads to

$$J_{p,s,b}(f)(z) = J_{p,s,b}(g)(z)\bigl(1 + z^m \mathcal{X}(z)\bigr) \quad (z \in U). \tag{3.19}$$

Differentiating (3.19) logarithmically with respect to z, we have

$$\frac{z(J_{p,s,b}(f)(z))'}{J_{p,s,b}(f)(z)} = \frac{z(J_{p,s,b}(g)(z))'}{J_{p,s,b}(g)(z)} + \frac{z^m\{m\mathcal{X}(z) + z\mathcal{X}'(z)\}}{1 + z^m \mathcal{X}(z)}. \tag{3.20}$$

Letting

$$\omega(z) = \frac{J_{p,s,b}(g)(z)}{z^p} \quad (z \in U),$$

where ω is in the form (2.1), is analytic in U, $\mathrm{Re}\{\omega(z)\} > 0$ and

$$\frac{z(J_{p,s,b}(g)(z))'}{J_{p,s,b}(g)(z)} = \frac{z\omega'(z)}{\omega(z)} + p,$$

then we have

$$\mathrm{Re}\left\{\frac{z(J_{p,s,b}(f)(z))'}{J_{p,s,b}(f)(z)}\right\} \geq p - \left|\frac{z\omega'(z)}{\omega(z)}\right| - \left|\frac{z^m\{m\mathcal{X}(z) + z\mathcal{X}'(z)\}}{1 + z^m \mathcal{X}(z)}\right|. \tag{3.21}$$

Using the following known estimates [12] (see also [15]):

$$\left|\frac{\omega'(z)}{\omega(z)}\right| \leq \frac{2mr^{m-1}}{1 - r^{2m}} \quad \text{and} \quad \left|\frac{m\mathcal{X}(z) + z\mathcal{X}'(z)}{1 + z^m \mathcal{X}(z)}\right| \leq \frac{m}{1 - r^m} \quad \bigl(|z| = r < 1\bigr),$$

in (3.21), we have

$$\mathrm{Re}\left\{\frac{z(J_{p,s,b}(f)(z))'}{J_{p,s,b}(f)(z)}\right\} \geq \frac{p - 3mr^m - (p + m)r^{2m}}{1 - r^{2m}} \quad \bigl(|z| = r < 1\bigr),$$

which is certainly positive, provided that $r < \mathring{R}$, where $\mathring{R}$ is given by (3.17). This completes the proof of Theorem 4. □

Theorem 5 *Let* $-1 \leq B_i < A_i \leq 1$ $(i = 1, 2)$ *and* $\tau < p$. *If each of the functions* $f_i \in A(p)$ *satisfies the following subordination condition:*

$$(1 - \alpha)\frac{(J_{p,s,b}(f_i)(z))'}{pz^{p-1}} + \alpha\frac{(J_{p,s,b}(f_i)(z))''}{p(p-1)z^{p-2}} \prec \frac{1 + A_i z}{1 + B_i z} \quad (i = 1, 2), \tag{3.22}$$

then

$$(1 - \alpha)\frac{(J_{p,s,b}(F)(z))'}{pz^{p-1}} + \alpha\frac{(J_{p,s,b}(F)(z))''}{p(p-1)z^{p-2}} \prec \frac{1 + (1 - \frac{2\tau}{p})z}{1 - z}, \tag{3.23}$$

where

$$F(z) = J_{p,s,b}(f_1 * f_2)(z) \tag{3.24}$$

and

$$\tau = p - 4p\frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\left[1 - \frac{1}{2}{}_2F_1\left(1, 1; \frac{p-1}{\alpha} + 1; \frac{1}{2}\right)\right]. \tag{3.25}$$

The result is best possible when $B_1 = B_2 = -1$.

Proof Suppose that the functions $f_i \in A(p)$ $(i = 1, 2)$ satisfy the condition (3.22). Then by setting

$$h_i(z) = (1-\alpha)\frac{(J_{p,s,b}(f_i)(z))'}{pz^{p-1}} + \alpha\frac{(J_{p,s,b}(f_i)(z))''}{p(p-1)z^{p-2}} \quad (i = 1, 2), \tag{3.26}$$

we have

$$h_i \in H(\varrho_i), \quad \varrho_i = \frac{1 - A_{\mathbf{i}}}{1 - B_{\mathbf{i}}} \ (i = 1, 2).$$

And

$$\left(J_{p,s,b}(f_{\mathbf{i}})(z)\right)' = \frac{p(p-1)}{\alpha}z^{\frac{(1-p)(1-\alpha)}{\alpha}}\int_0^z t^{\frac{p-1}{\alpha}-1}h_i(t)\,dt \quad (i = 1, 2), \tag{3.27}$$

from (3.24), (3.26) and (3.27), we have

$$\left(J_{p,s,b}(F)(z)\right)' = \frac{p(p-1)}{\alpha}\mathbf{z}^{\frac{(1-\mathbf{p})(1-\alpha)}{\alpha}}\int_0^z t^{\frac{p-1}{\alpha}-1}H(t)\,dt \quad (i = 1, 2). \tag{3.28}$$

For convenience,

$$\begin{aligned} H(z) &= (1-\alpha)\frac{(J_{p,s,b}(F)(z))'}{pz^{p-1}} + \alpha\frac{(J_{p,s,b}(F)(z))''}{p(p-1)z^{p-2}} \\ &= \frac{p(p-1)}{\alpha}z^{\frac{(1-p)}{\alpha}}\int_0^z t^{\frac{p-1}{\alpha}-1}(h_1 * h_2)(t)\,dt. \end{aligned} \tag{3.29}$$

Since $h_i \in H(\varrho_i)$ $(i = 1, 2)$, it follows from Lemma 3 that

$$(h_1 * h_2)(z) \in H(\varrho_3), \quad \varrho_3 = 1 - 2(1 - \varrho_1)(1 - \varrho_2). \tag{3.30}$$

By using (3.30) in (3.29) and applying Lemmas 2 and 3, we have

$$\begin{aligned} \operatorname{Re}\{H(z)\} &= \frac{p(p-1)}{\alpha}\int_0^1 s^{\frac{p-1}{\alpha}-1}\operatorname{Re}\{(h_1 * h_2)(sz)\}\,ds \\ &\geq \frac{p(p-1)}{\alpha}\int_0^1 s^{\frac{p-1}{\alpha}-1}\left(2\varrho_3 - 1 + \frac{2(1-\varrho_3)}{1+s|z|}\right)ds \\ &> \frac{p(p-1)}{\alpha}\int_0^1 s^{\frac{p-1}{\alpha}-1}\left(2\varrho_3 - 1 + \frac{2(1-\varrho_3)}{1+s}\right)ds \end{aligned}$$

$$\begin{aligned}
&= p - 4p\frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\left[1 - \frac{p-1}{\alpha}\int_0^1 s^{\frac{p-1}{\alpha}-1}(1+s)^{-1}\,ds\right] \\
&= p - 4p\frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\left[1 - \frac{1}{2}{}_2F_1\left(1,1;\frac{p-1}{\alpha}+1;\frac{1}{2}\right)\right] \quad (\mathbf{z} \to -\mathbf{1}) \\
&= \tau \quad (z \in U).
\end{aligned}$$

When $B_1 = B_2 = -1$, we consider $f_i \in A(p)$ $(i = 1, 2)$ satisfy the condition (3.22) and are defined by

$$\big(J_{p,s,b}(f_i)(z)\big)' = \frac{p(p-1)}{\alpha} z^{\frac{(1-p)(1-\alpha)}{\alpha}} \int_0^z t^{\frac{p-1}{\alpha}-1}\left(\frac{1+A_i t}{1-t}\right) dt \quad (i = 1, 2).$$

By using (3.29) and applying Lemma 3, we have

$$\begin{aligned}
H(z) &= \frac{p(p-1)}{\alpha}\int_0^1 s^{\frac{p-1}{\alpha}-1}\left[1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-sz}\right] ds \\
&= p - p(1+A_1)(1+A_2) + p(1+A_1)(1+A_2)(1-z)^{-1}{}_2F_1\left(1,1;\frac{p-1}{\alpha}+1;\frac{z}{z-1}\right) \\
&\to p - p(1+A_1)(1+A_2) + \frac{p}{2}(1+A_1)(1+A_2){}_2F_1\left(1,1;\frac{p-1}{\alpha}+1;\frac{1}{2}\right) \quad (z \to -1).
\end{aligned}$$

This completes the proof of Theorem 5. □

Remark 1 Putting $A_i = 1 - 2\theta_i$ $(0 \le \theta_i < 1)$ and $B_i = -1$ $(i = 1, 2)$ in Theorem 5, we obtain the result obtained by Liu [10, Theorem 5].

Putting $A_i = 1 - 2\theta_i$ $(0 \le \theta_i < 1)$, $B_i = -1$ $(i = 1, 2)$ and $s = 0$ in Theorem 5, we obtain the following corollary.

Corollary 2 *Let $\chi < p$ and $f_i \in A(p)$ satisfy the following inequality:*

$$\operatorname{Re}\left\{(1-\alpha)\frac{f_i'(z)}{pz^{p-1}} + \alpha\frac{f_i''(z)}{p(p-1)z^{p-2}}\right\} > \theta_i \quad (0 \le \theta_i < 1; i = 1, 2),$$

then

$$\operatorname{Re}\left\{(1-\alpha)\frac{(f_1 * f_2)'(z)}{pz^{p-1}} + \alpha\frac{(f_1 * f_2)''(z)}{p(p-1)z^{p-2}}\right\} > \frac{\chi}{p},$$

where

$$\chi = p - 4p(1-\theta_1)(1-\theta_2)\left[1 - \frac{1}{2}{}_2F_1\left(1,1;\frac{p-1}{\alpha}+1;\frac{1}{2}\right)\right].$$

The result is best possible.

Theorem 6 *Let $f \in S_p^{s,b}(A, B)$ and $g \in A(p)$ satisfy the following inequality:*

$$\operatorname{Re}\left\{\frac{g(z)}{z^p}\right\} > \frac{1}{2} \quad (z \in U), \tag{3.31}$$

then

$$(f * g)(z) \in S_p^{s,b}(A,B).$$

Proof We have

$$\frac{(J_{p,s,b}(f * g)(z))'}{pz^{p-1}} = \frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} * \frac{g(z)}{z^p} \quad (z \in U),$$

where $g(z)$ satisfies (3.31) and $\frac{1+Az}{1+Bz}$ is convex (univalent) in U. By using (1.10) and applying Lemma 5, we complete the proof of Theorem 6. □

Theorem 7 *Let $\sigma > 0$ and $f \in A(p)$ satisfy the following subordination condition:*

$$(1-\alpha)\frac{J_{p,s,b}(f)(z)}{z^p} + \alpha\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}. \tag{3.32}$$

Then

$$\operatorname{Re}\left\{\frac{J_{p,s,b}(f)(z)}{z^p}\right\}^{\frac{1}{\sigma}} > \gamma^{\frac{1}{\sigma}},$$

where

$$\gamma = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1,1;\frac{p}{m\alpha}+1;\frac{B}{B-1}\right) & \text{for } B \neq 0, \\ 1 - \frac{p}{m\alpha + p}A & \text{for } B = 0. \end{cases}$$

The result is best possible.

Proof Let

$$M(z) = \frac{J_{p,s,b}(f)(z)}{z^p} \quad (z \in U), \tag{3.33}$$

where M is of the form (2.1) and is analytic in U. Differentiating (3.33) with respect to z, we have

$$(1-\alpha)\frac{J_{p,s,b}(f)(z)}{z^p} + \alpha\frac{(J_{p,s,b}(f)(z))'}{pz^{p-1}} = M(z) + \frac{\alpha}{p}zM'(z) \prec \frac{1+Az}{1+Bz}.$$

Now, by following steps similar to the proof of Theorem 1 and using the elementary inequality

$$\operatorname{Re}\{\Upsilon^{1/\varkappa}\} \geq \{\operatorname{Re}\Upsilon\}^{1/\varkappa} \quad (\operatorname{Re}\{\Upsilon\} > 0; \varkappa \in \mathbb{N}),$$

we obtain the result asserted by Theorem 7. □

Theorem 8 *Let $\nu \in \mathbb{C}^*$ and $A, B \in \mathbb{C}$ with $A \neq B$ and $|B| \leq 1$. Suppose that*

$$\left|\frac{\nu(b+1)(A-B)}{B} - 1\right| \leq 1 \quad \textit{or} \quad \left|\frac{\nu(b+1)(A-B)}{B} + 1\right| \leq 1, \quad \textit{if } B \neq 0,$$

$$\left|(b+1)\nu A\right| \leq \boldsymbol{\pi}, \quad \textit{if } B = 0.$$

If $f \in A(p)$ with $J_{p,s,b}(f)(z) \neq 0$ for all $z \in U^* = U \setminus \{0\}$, then

$$\frac{J_{p,s-1,b}(f)(z)}{J_{p,s,b}(f)(z)} \prec \frac{1+Az}{1+Bz},$$

implies

$$\left(\frac{J_{p,s,b}(f)(z)}{z^p}\right)^{\nu} \prec q_1(z),$$

where

$$q_1(z) = \begin{cases} (1+Bz)^{\nu(b+1)(A-B)/B}, & \textit{if } B \neq 0, \\ e^{\nu(b+1)Az}, & \textit{if } B = 0, \end{cases}$$

is the best dominant.

Proof Let us put

$$\varphi(z) = \left(\frac{J_{p,s,b}(f)(z)}{z^p}\right)^{\nu} \quad (z \in U). \tag{3.34}$$

Then φ is analytic in U, $\varphi(0) = 1$ and $\varphi(z) \neq 0$ for all $z \in U$. Taking the logarithmic derivatives in both sides of (3.34) and using the identity (1.9), we have

$$1 + \frac{z\varphi'(z)}{\nu(b+1)\varphi(z)} = \frac{J_{p,s-1,b}(f)(z)}{J_{p,s,b}(f)(z)} \prec \frac{1+Az}{1+Bz}.$$

Now the assertions of Theorem 8 follow by using Lemma 6 for $\gamma = \nu(b+1)$. □

Putting $B = -1$ and $A = 1 - 2\sigma$, $0 \leq \sigma < 1$, in Theorem 8, we obtain the following corollary.

Corollary 3 *Assume that $\nu \in \mathbb{C}^*$ satisfies either $|2\nu(b+1)(1-\sigma) - 1| \leq 1$ or $|2\nu(b+1)(1-\sigma) + 1| \leq 1$. If $f \in A(p)$ with $J_{p,s,b}(f)(z) \neq 0$ for $z \in U^*$, then*

$$\operatorname{Re}\left\{\frac{J_{p,s-1,b}(f)(z)}{J_{p,s,b}(f)(z)}\right\} > \sigma \quad (z \in U),$$

implies

$$\left(\frac{J_{p,s,b}(f)(z)}{z^p}\right)^{\nu} \prec q_2(z) = (1-z)^{-2\nu(b+1)(1-\sigma)},$$

and q_2 is the best dominant.

Remark 2 Specializing the parameters s and b in the above results of this paper, we obtain the results for the corresponding operators $F_{\mu,p}$, I_p^{α}, J_p^{γ} and $J_p^{\gamma}(l)$ which are defined in the introduction.

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The authors declare that they have no competing interests.

Authors' contributions
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