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Limit theorems for delayed sums of random sequence

Ding Fang-qing[1] and Wang Zhong-zhi[2*]

* Correspondence: wzz30@ahut.edu.cn
[2]Faculty of Mathematics & Physics, AnHui University of Technology, Ma'anshan 243002, P. R. China
Full list of author information is available at the end of the article

Abstract

For a sequence of arbitrarily dependent random variables $(X_n)_{n\in \mathbf{N}}$ and Borel sets $(B_n)_{n\in \mathbf{N}}$, on real line the strong limit theorems, represented by inequalities, i.e. the strong deviation theorems of the delayed average $S_{n.k_n}(\omega)$ are investigated by using the notion of asymptotic delayed log-likelihood ratio. The results obtained popularizes the methods proposed by Liu.

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1. Introduction

Let $(a_n)_{n\in \mathbf{N}}$ be a sequence of real numbers and let $(k_n)_{n\in \mathbf{N}}$ be a sequence of positive integers. The numbers

$$\rho_{n,k_n} = \left\{ \sum_{j=1}^{k_n} a_{n+j-1} \right\} / k_n$$

are called the (forward) delayed first arithmetic means (See [1]). In [2], using the limiting behavior of delayed average, Chow found necessary and sufficient conditions for the Borel summability of i.i.d. random variables and also obtained very simple proofs of a number of well-known results such as the Hsu-Robbins-Spitzer-Katz theorem. In [3], Lai studied the analogues of the law of the iterated logarithim for delayed sums of independent random variables. Recently, Chen [4] has presented an accurate description the limiting behavior of delayed sums under a non-identically distribution setup, and has deduced Chover-type laws of the iterated logarithm for them.

Our aim in this article is to establish strong deviation theorems (limit theorem expressed by inequalities, see [5]) of delayed average for the dependent absolutely continuous random variables. By using the notion of asymptotic delayed log-likelihood ratio, we extend the analytic technique proposed by Liu [5] to the case of delayed sums. The crucial part of the proof is to construct a delayed likelihood ratio depending on a parameter, and then applies the Borel-Cantelli lemma.

Throughout, let $(X_n)_{n\in \mathbf{N}}$ be a sequence of absolutely continuous random variables on a fixed probability space $\{\Omega, \mathcal{F}, P\}$ with the joint density function $g^{1,\ n}(x_1,\ldots, x_n)$, $n \in \mathbf{N}$, and $f_j(x)$, $j = 1, 2,\ldots$ be the the marginal density function of random variable X_j. (k_n)

$_{n\in\mathbf{N}}$ be a subsequence of positive integers, such that, for every $\varepsilon > 0$, $\sum_{n=1}^{\infty}\exp(-k_n\varepsilon) < \infty$.

Definition 1. The delayed likelihood ratio is defined by

$$\mathcal{L}_n(\omega) = \begin{cases} \dfrac{\Pi_{j=n}^{n+k_n-1} f_j(X_j)}{g^{n,n+k_n-1}(X_n,\ldots,X_{n+k_n-1})}, & \text{if denominator } > 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

Let

$$\mathcal{L}(\omega) = -\liminf_{n} \frac{1}{k_n}\log\mathcal{L}_n(\omega) \tag{1.2}$$

$\mathcal{L}(\omega)$ is called asymptotic delayed log-likelihood ratio, where $g^{n,n+k_n-1}(x_n,\ldots,x_{n+k_n-1})$ denotes the joint density function of random vector $(X_n,\ldots,X_{n+k_n-1})$, ω is a sample point (with log 0 = -∞).

It will be shown in Lemma 1 that $\mathcal{L}(\omega) \geq 0\,a.e.$ in any case.

Remark 1. It will be seen below that $\mathcal{L}(\omega)$ has the analogous properties of the likelihood ratio in [5], Although $\mathcal{L}(\omega)$ is not a proper metric among probability measures, we nevertheless consider it as a measure of “discrimination” between the dependence (their joint distribution) and independence (the product of their marginals). Obviously, $\mathcal{L}_n(\omega) = 1$, a.e. $n\in\mathbf{N}$ if $(X_n)_{n\in\mathbf{N}}$ is independent. In view of the above discussion of the asymptotic logarithmic delayed likelihood ratio, it is natural for us to think of $\mathcal{L}(\omega)$ as a measure how far (the random deviation) of $(X_n)_{n\in\mathbf{N}}$ is from being independent and how dependent they are. The closer $\mathcal{L}(\omega)$ approaches to 0, the smaller the deviation is.

Lemma 1. *Let $\mathcal{L}_n(\omega)$ be define as above, then*

$$\limsup_{n} \frac{1}{k_n}\log\mathcal{L}_n(\omega) \leq 0,\ a.e. \tag{1.3}$$

Proof. Let $B = \left\{(x_n,\ldots,x_{n+k_n-1}) : g^{n,n+k_n-1}(x_n,\ldots,x_{n+k_n-1}) > 0\right\}$. Since

$$\begin{aligned}
&E[\mathcal{L}_n(\omega)] \\
&= \int\cdots\int_{(x_n,\ldots,x_{n+k_n-1})\in B} \frac{\prod_{j=n}^{n+k_n-1} f_j(x_j)}{g^{n,n+k_n-1}(x_n,\ldots,x_{n+k_n-1})} \\
&\quad .g^{n,n+k_n-1}(x_n,\ldots,x_{n+k_n-1})\,dx_n\ldots dx_{n+k_n-1} \\
&= \int\cdots\int_{(x_n,\ldots,x_{n+k_n-1})\in B} \prod_{j=n}^{n+k_n-1} f_j(x_j)\,dx_n\ldots dx_{n+k_n-1} \\
&\leq \int\cdots\int_{(x_n,\ldots,x_{n+k_n-1})\in \mathbf{R}^{k_n}} \prod_{j=n}^{n+k_n-1} f_j(x_j)\,dx_n\ldots dx_{n+k_n-1} = 1.
\end{aligned}$$

From Markov inequality, for every $\varepsilon > 0$, we have

$$P\left[\frac{1}{k_n}\log\mathcal{L}_n(\omega) \geq \varepsilon\right] = P\left[\mathcal{L}_n(\omega) \geq \exp(k_n\varepsilon)\right] \leq 1\cdot\exp(-k_n\varepsilon).$$

Hence

$$\sum_{n=1}^{\infty} P\left[\frac{1}{k_n}\log \mathcal{L}_n(\omega) \geq \varepsilon\right] \leq \sum_{n=1}^{\infty} \exp(-k_n\varepsilon) < \infty.$$

By Borel-Cantelli lemma, we have

$$P\left[\limsup_n \frac{1}{k_n}\log \mathcal{L}_n(\omega) \geq 2\varepsilon\right] = 0,$$

for any $\varepsilon > 0$, (1.3) follows immediately. □

2. Main results and proofs

Theorem 1. *Let* $(X_n)_{n\in \mathbf{N}}$, $\mathcal{L}_n(\omega)$, $\mathcal{L}(\omega)$ *be defined as above,* $(B_n)_{n\in \mathbf{N}}$ *be a sequence of Borel sets of the real line. Let* $S_{n,k_n}(\omega) = \frac{1}{k_n}\sum_{j=n}^{n+k_n-1} \mathbf{1}_{B_j}(X_j)$, *and assume*

$$c = \limsup_n \frac{1}{k_n}\sum_{j=n}^{n+k_n-1} P(X_j \in B_j), \tag{2.1}$$

then

$$\limsup_n S_{n,k_n}(\omega) \leq \left(\sqrt{\mathcal{L}(\omega)} + \sqrt{c}\right)^2, \ a.e. \tag{2.2}$$

where $\mathbf{1}_{B_n}(\cdot)$ *be the indicator function of* B_n.

Proof. Assume $s > 0$ to be a constant, and let

$$h_j(x_j) = \frac{s^{\mathbf{1}_{B_j}(x_j)} f_j(x_j)}{1 + (s-1)\int_{B_j} f_j(x_j)\,dx_j}, \quad j = 1, 2, \ldots \tag{2.3}$$

It is not difficult to see that $\int h_j(x_j)\,dx_j = 1$, $j = 1, 2,\ldots$ Let

$$\Lambda_n(s,\omega) = \begin{cases} \dfrac{\Pi_{j=n}^{n+k_n-1} h_j(X_j)}{g^{n,n+k_n-1}(X_{n,\ldots,}X_{n+k_n-1})}, & \text{if denominator } > 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

From Lemma 1, there exists $A(s) \in \mathcal{F}$, $P(A(s)) = 1$, such that

$$\limsup_n \frac{1}{k_n}\log \Lambda_n(s,\omega) \leq 0, \quad \omega \in A(s) \tag{2.5}$$

Since $\int_{B_j} f_j(x_j)\,dx_j = P(X_j \in B_j)$, by (2.3) we have

$$\begin{aligned} &\prod_{j=n}^{n+k_n-1} h_j(x_j) \\ &= \prod_{j=n}^{n+k_n-1} \frac{s^{\mathbf{1}_{B_j}(x_j)} f_j(x_j)}{1 + (s-1)\int_{B_j} f_j(x_j)\,dx_j} \\ &= s^{\sum_{j=n}^{n+k_n-1} \mathbf{1}_{B_j}(x_j)} \prod_{j=n}^{n+k_n-1} \frac{f_j(x_j)}{1 + (s-1)P(X_j \in B_j)} \end{aligned} \tag{2.6}$$

It follows from (1.1), (2.4) and (2.6) that

$$\log \Lambda_n (s, \omega) = \sum_{j=n}^{n+k_n-1} \mathbf{1}_{B_j}\left(X_j\right) \log s - \sum_{j=n}^{n+k_n-1} \log\left[1 + (s-1) P\left(X_j \in B_j\right)\right] + \log \mathcal{L}_n (\omega) \tag{2.7}$$

(2.5) and (2.7) yield

$$\limsup_n \frac{1}{k_n}\left(\log s \sum_{j=n}^{n+k_n-1} \mathbf{1}_{B_j}(X_j) - \sum_{j=n}^{n+k_n-1} \log\left[1+(s-1)P(X_j \in B_j)\right] + \log \mathcal{L}_n(\omega)\right) \le 0, \quad \omega \in A(s) \tag{2.8}$$

Let $s > 1$, dividing the two sides of (2.8) by $\log s$, we have

$$\limsup_n \frac{1}{k_n}\left(\sum_{j=n}^{n+k_n-1} \mathbf{1}_{B_j}(X_j) - \sum_{j=n}^{n+k_n-1} \frac{\log\left[1+(s-1)P\left(X_j \in B_j\right)\right]}{\log s} + \frac{\log \mathcal{L}_n(\omega)}{\log s}\right) \le 0, \quad \omega \in A(s) \tag{2.9}$$

By (1.2), (2.9) and the property $\limsup_n(a_n - b_n) \le d \Rightarrow \limsup_n(a_n - c_n) \le \limsup_n (b_n - c_n) + d$, one gets

$$\limsup_n \frac{1}{k_n}\left(\sum_{j=n}^{n+k_n-1} \mathbf{1}_{B_j}(X_j) - \sum_{j=n}^{n+k_n-1} \frac{\log\left[1+(s-1)P\left(X_j \in B_j\right)\right]}{\log s}\right) \le \frac{\mathcal{L}(\omega)}{\log s}, \quad \omega \in A(s) \tag{2.10}$$

By (2.10) and the property of the superior above and the inequality $0 < \log(1+x) \le x$ $(x > 0)$, we obtain

$$\begin{aligned}
&\limsup_n S_{n,k_n}(\omega) \\
&\le \limsup_n \frac{1}{k_n}\left(\sum_{j=n}^{n+k_n-1} \frac{\log\left[1+(s-1)P\left(X_j \in B_j\right)\right]}{\log s}\right) + \frac{\mathcal{L}(\omega)}{\log s} \\
&\le \limsup_n \frac{1}{k_n}\left(\sum_{j=n}^{n+k_n-1} \frac{(s-1)P\left(X_j \in B_j\right)}{\log s}\right) + \frac{\mathcal{L}(\omega)}{\log s} \\
&\le c\left(\frac{s-1}{\log s}\right) + \frac{\mathcal{L}(\omega)}{\log s}, \quad \omega \in A(s)
\end{aligned} \tag{2.11}$$

(2.11) and the inequality $1 - \frac{1}{s} < \log s \, (s > 1)$ imply

$$\limsup_n S_{n,k_n}(\omega) \le c \cdot s + \frac{s\mathcal{L}(\omega)}{s-1}, \quad \omega \in A(s) \tag{2.12}$$

Let D be a set of countable real numbers dense in the interval $(1, +\infty)$, and let $A^* = \cap_{s\in D} A(s)$, $g(s, x) = cs + sx/(s - 1)$, then we have by (2.12)

$$\limsup_n S_{n,k_n}(\omega) \le g(s, \mathcal{L}(\omega)), \quad \omega \in A^*, \quad s \in D \tag{2.13}$$

Let $c > 0$, it easy to see that if $\mathcal{L}(\omega) > 0$, *a.e.*, then, for fixed ω, $g(s, \mathcal{L}(\omega))$ as a function of s attains its smallest value $g\left(1 + \sqrt{\mathcal{L}(\omega)/c}, \mathcal{L}(\omega)\right) = 2\sqrt{c\mathcal{L}(\omega)} + \mathcal{L}(\omega) + c$ on the interval $(1, +\infty)$, and $g(s, 0)$ is increasing on the interval $(1, +\infty)$ and $\lim_{s\to 1^+} g(s, 0) = 0$. For each $\omega \in A^* \cap A(1)$, if $\mathcal{L}(\omega) \ne \infty$, take $\kappa_n(\omega) \in D$, $n = 1, 2,\ldots$, such that $\kappa_n(\omega) \to 1 + \sqrt{\mathcal{L}(\omega)/c}$. We have by the continuity of $g(s, \mathcal{L}(\omega))$ with respect to s,

$$\lim_{n\to+\infty} g\left(\kappa_n(\omega), \mathcal{L}(\omega)\right) = \left(\sqrt{\mathcal{L}(\omega)} + \sqrt{c}\right)^2, \tag{2.14}$$

By (2.13), we obtain

$$\limsup_n S_{n,k_n} \le g\left(\kappa_n(\omega),\ \mathcal{L}(\omega)\right), \quad n = 1, 2, \ldots \tag{2.15}$$

(2.14) and (2.15) imply

$$\limsup_n S_{n,k_n}(\omega) \le \left(\sqrt{\mathcal{L}(\omega)} + \sqrt{c}\right)^2, \quad \omega \in A^* \cap A(1) \tag{2.16}$$

If $\mathcal{L}(\omega) = \infty$, (2.16) holds trivially. Since $P(A^* \cap A(1)) = 1$, (2.2) holds by (2.16), when $c > 0$.

When $c = 0$, we have by letting $s = e$ in (2.11),

$$\limsup_n S_{n,k_n}(\omega) \le \mathcal{L}(\omega), \quad \omega \in A(e) \tag{2.17}$$

since $P(A(e)) = 1$, (2.2) also holds by (2.17) when $c = 0$. □

Theorem 2. *Let* $(X_n)_{n\in\mathbf{N}}$, $\mathcal{L}_n(\omega)$, $\mathcal{L}(\omega)$, $(B_n)_{n\in\mathbf{N}}$, $S_{n,k_n}(\omega)$ *be defined as in Theorem* 1 *and assume*

$$c' = \liminf_n \frac{1}{k_n} \sum_{j=n}^{n+k_n-1} P\left(X_j \in B_j\right), \tag{2.18}$$

then, if $0 \le \mathcal{L}(\omega) \le c'$ *a.e., then*

$$\liminf_n S_{n,k_n}(\omega) \ge c' - 2\sqrt{c'\mathcal{L}(\omega)}, \quad a.e. \tag{2.19}$$

Proof. Let $0 < s < 1$, dividing the two sides of (2.8) by log s, we have

$$\liminf_n \frac{1}{k_n}\left(\sum_{j=n}^{n+k_n-1} \mathbf{1}_{B_j}(X_j) - \sum_{j=n}^{n+k_n-1} \frac{\log\left[1 + (s-1)P\left(X_j \in B_j\right)\right]}{\log s} + \frac{\log \mathcal{L}_n(\omega)}{\log s}\right) \ge 0, \quad \omega \in A(s) \tag{2.20}$$

By (1.2), (2.20) and the property $\liminf_n(a_n - b_n) \ge d \Rightarrow \liminf_n(a_n - c_n) \ge \liminf_n(b_n - c_n) + d$, one gets

$$\liminf_n \frac{1}{k_n}\left(\sum_{j=n}^{n+k_n-1} \mathbf{1}_{B_j}(X_j) - \sum_{j=n}^{n+k_n-1} \frac{\log\left[1 + (s-1)P\left(X_j \in B_j\right)\right]}{\log s}\right) \ge \frac{\mathcal{L}(\omega)}{\log s}, \quad \omega \in A(s) \tag{2.21}$$

By (2.21) and the property of the inferior above and the inequality $\log(1+x) \le x(-1 < x \le 0)$, we obtain

$$\begin{aligned} &\liminf_n S_{n,k_n}(\omega) \\ &\ge \liminf_n \frac{1}{k_n}\left(\sum_{j=n}^{n+k_n-1} \frac{\log\left[1 + (s-1)P\left(X_j \in B_j\right)\right]}{\log s}\right) + \frac{\mathcal{L}(\omega)}{\log s} \\ &\ge \liminf_n \frac{1}{k_n}\left(\sum_{j=n}^{n+k_n-1} \frac{(s-1)P\left(X_j \in B_j\right)}{\log s}\right) + \frac{\mathcal{L}(\omega)}{\log s} \\ &\ge c'\left(\frac{s-1}{\log s}\right) + \frac{\mathcal{L}(\omega)}{\log s}, \quad \omega \in A(s) \end{aligned} \tag{2.22}$$

(2.22) and the inequality $1 - \frac{1}{s} < \log s$ and $\log s < s - 1$ $(0 < s < 1)$ imply

$$\liminf_{n} S_{n,k_n}(\omega) \geq c' \cdot s + \frac{\mathcal{L}(\omega)}{s-1}, \quad \omega \in A(s) \cap A(1) \tag{2.23}$$

Let D' be a set of countable real numbers dense in the interval (0, 1), and let $A_* = \cap_{s \in D'} A(s)$, $h(s, x) = c's + x/(s - 1)$, then we have by (2.23)

$$\liminf_{n} S_{n,k_n}(\omega) \geq h(s, \mathcal{L}(\omega)), \quad \omega \in A_*, \quad s \in D' \tag{2.24}$$

Let $c' > 0$, it easy to see that if $0 < \mathcal{L}(\omega) < c'$, *a.e.*, then, for fixed ω, $h(s, \mathcal{L}(\omega))$ as a function of s attains its maximum value $h\left(1 - \sqrt{\mathcal{L}(\omega)/c'}, \mathcal{L}(\omega)\right) = c' - 2\sqrt{c'\mathcal{L}(\omega)}$, on the interval (0, 1), and $h(s, 0)$ is increasing on the interval (0, 1) and $\lim_{s \to 1^+} h(s, 0) = c'$. For each $\omega \in A_* \cap A(1)$, if $\mathcal{L}(\omega) \neq \infty$, take $l_n(\omega) \in D'$, $n = 1, 2, \ldots$, such that $l_n(\omega) \to 1 - \sqrt{\mathcal{L}(\omega)/c'}$. We have by the continuity of $h(s, \mathcal{L}(\omega))$ with respect to s,

$$\lim_{n \to +\infty} h(l_n(\omega), \mathcal{L}(\omega)) = c' - 2\sqrt{c'\mathcal{L}(\omega)}, \tag{2.25}$$

By (2.24), we obtain

$$\liminf_{n} S_{n,k_n} \geq h(l_n(\omega), \mathcal{L}(\omega)), \quad n = 1, 2, \ldots \tag{2.26}$$

(2.25) and (2.26) imply

$$\liminf_{n} S_{n,k_n}(\omega) \geq c' - 2\sqrt{c'\mathcal{L}(\omega)}, \quad \omega \in A_* \cap A(1) \tag{2.27}$$

If $\mathcal{L}(\omega) = \infty$, (2.27) holds trivially. Since $P(A_* \cap A(1)) = 1$, (2.19) holds by (2.27), when $c' > 0$. (2.19) also holds trivially when $c' = 0$. □

Remark 2. In case $\mathcal{L}(\omega) > c' \geq 0$, *a.e.*, we cannot get a better lower bound of $\liminf_n S_{n,k_n}(\omega)$. This motivates the following problem: under the conditions of Theorem 2, how to get a better lower bound of $\liminf_n S_{n,k_n}(\omega)$ in case of $\mathcal{L}(\omega) > c' \geq 0$, *a.e.*?

Definition 2. (Generalized empirical distribution function) Let $(X_n)_{n \in \mathbf{N}}$ be identically distribution with common distribution function F, for each $m, n \in \mathbf{N}$, let

$$F_{m,n}(x) = \frac{1}{n} \sum_{k=m}^{m+n-1} \mathbf{1}_{(X_k \leq x)}.$$

$F_{m,n}$ = the observed frequency of values that are $\leq x$ from time m to $m + n - 1$. The $F_{1,n}$ is the usual empirical distribution function, hence the name given above.

In particular, let $B = (-\infty, x]$, $x \in R$ in Theorems 1 and 2, we can get a strong limit theorem for the generalized empirical distribution function.

Corollary 1. *Let* $(X_n)_{n \in \mathbf{N}}$ *be i.i.d. random variables with common distribution function F, let* $B_n = (-\infty, x]$, $n = 1, 2, \ldots$, *then*

$$\lim_{n} F_{n,n+k_n-1}(x) = F(x), \ a.e.$$

Corollary 2. *Let* $(X_n)_{n \in \mathbf{N}}$ *be independent random variables and* $(B_n)_{n \in \mathbf{N}}$ *be as Theorem* 1, *then*

$$\lim_n \frac{1}{k_n} \sum_{j=n}^{n+k_n-1} \left[\mathbf{1}_{B_j}(X_j) - P(X_j \in B_j) \right] = 0, \ a.e. \tag{2.28}$$

Proof. Note that $P(X_j \in B_j) \le 1$, $j = 1, 2,...$ and in this case, $0 \le c, c' \le 1$, $\mathcal{L}(\omega) = 0$ *a. e.*, we have by (2.11)

$$\limsup_n \frac{1}{k_n} \left(\sum_{j=n}^{n+k_n-1} \mathbf{1}_{B_j}(X_j) - \sum_{j=n}^{n+k_n-1} \frac{\log\left[1 + (s-1) P(X_j \in B_j)\right]}{\log s} \right) \le 0, \quad \omega \in A(s) \tag{2.29}$$

by (2.29) and the property of the superior above and the inequality $0 \le \log(1+x) \le x$ $(x > 0)$, we obtain

$$\begin{aligned} &\limsup_n \frac{1}{k_n} \sum_{j=n}^{n+k_n-1} \left[\mathbf{1}_{B_j}(X_j) - P(X_j \in B_j) \right] \\ &\le \limsup_n \frac{1}{k_n} \left(\sum_{j=n}^{n+k_n-1} \frac{\log\left[1 + (s-1) P(X_j \in B_j)\right]}{\log s} - P(X_j \in B_j) \right) \\ &\le \limsup_n \frac{1}{k_n} \left(\sum_{j=n}^{n+k_n-1} \frac{(s-1) P(X_j \in B_j)}{\log s} - P(X_j \in B_j) \right) \\ &\le \left(\frac{s-1}{\log s} - 1 \right), \quad \omega \in A(s) \end{aligned} \tag{2.30}$$

Analogously as in the proof of Theorem 1, we obtain

$$\limsup_n \frac{1}{k_n} \sum_{j=n}^{n+k_n-1} \left[\mathbf{1}_{B_j}(X_j) - P(X_j \in B_j) \right] \le 0, \ a.e. \tag{2.31}$$

Similarly, we have $\liminf_n \frac{1}{k_n} \sum_{j=n}^{n+k_n-1} \left[\mathbf{1}_{B_j}(X_j) - P(X_j \in B_j) \right] \ge 0$, *a.e.* hence (2.28) follows immediately. □

Remark 3. Let $B_n = B$, Corollary 2 implies that $\frac{\lim S_{n,k_n}}{k_n} = P(X_1 \in B)$ which gives the strong law of large numbers for the delayed arithmatic means.

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Author details
[1]Department of Mathematics & Physics, HeFei University, HeFei 230601, P. R. China [2]Faculty of Mathematics & Physics, AnHui University of Technology, Ma'anshan 243002, P. R. China

Authors' contributions
WZ and DF carried out the design of the study and performed the analysis. DF drafted the manuscript. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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