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The asymptotic behavior of the solution of a doubly degenerate parabolic equation with the convection term

Huashui Zhan[1*] and Bifen Xu[2]

* Correspondence: huashuizhan@163.com
[1]School of Sciences, Jimei University, Xiamen 361021, People's Republic of China
Full list of author information is available at the end of the article

Abstract

The objective of this article is to study the large time asymptotic behavior of the nonnegative weak solution of the following nonlinear parabolic equation

$$u_t = \operatorname{div}\left(|Du^m|^{p-2}Du^m\right) + \operatorname{div}\left(B(u^m)\right)$$

with initial condition $u(x, 0) = u_0(x)$. By using Moser iteration technique, assuming that the uniqueness of the Barenblatt-type solution E_c of the equation $u_t = \operatorname{div}(|Du^m|^{p-2}Du^m)$ is true, then the solution u may satisfy

$$t^{\frac{1}{\mu}}\left|u(x,\ t) - E_c(x,\ t)\right| \to 0, \quad \text{as } t \to \infty,$$

which is uniformly true on the sets $\left\{x \in R^N : \quad |x| < at^{\frac{1}{\mu N}}, \quad a > 0\right\}$. Here $B(u^m) = (b_1(u^m), b_2(u^m), \ldots, b_N(u^m))$ satisfies some growth order conditions, the exponents m and p satisfy $m(p - 1) > 1$.

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1. Introduction

The objective of this article is to study the large time asymptotic behavior of the nonnegative weak solution of the nonlinear parabolic equation with the following type

$$u_t = \operatorname{div}(|Du^m|^{p-2}Du^m) + \operatorname{div}(B(u^m)), \qquad \text{in } S = R^N \times (0,\ \infty), \tag{1.1}$$

$$u(x,\ 0) = u_0(x), \quad \text{on } R^N, \tag{1.2}$$

where $m(p - 1) > 1$, $N \geq 1$, $u_0(x) \in L^1(R^N)$, D is the spatial gradient operator, and the convection term $\operatorname{div}(B(u^m)) = \sum_{i=1}^{N} \frac{\partial(b_i(u^m))}{\partial x_i}$.

Equation (1.1) appears in a number of different physical situations [1].

For example, in the study of water infiltration through porous media, Darcy's linear relation

$$V = -K(\theta)\nabla\phi, \tag{1.3}$$

satisfactorily describes flow conditions provided the velocities are small. Here V represents the seepage velocity of water, θ is the volumetric moisture content, $K(\theta)$ is the hydraulic conductivity and φ is the total potential, which can be expressed as the sum of a hydrostatic potential $\psi(\theta)$ and a gravitational potential z

$$\phi = \psi(\theta) + z. \tag{1.4}$$

However, (1.3) fails to describe the flow for large velocities. To get a more accurate description of the flow in this case, several nonlinear versions of (1.3) have been proposed. One of these versions is

$$V^{\alpha} = -K(\theta)\nabla\phi, \tag{1.5}$$

where α ranges from 1 for laminar flow to 2 for completely turbulent flow (cf. [2-4] and references therein). If it is assumed that infiltration takes place in a horizontal column of the medium, according to the continuity equation

$$\frac{\partial\theta}{\partial t} + \frac{\partial V}{\partial x} = 0,$$

then (1.4) and (1.5) give

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(D(\theta)^{p}|\theta_x|^{p-1}\theta_x\right)$$

with $\frac{1}{p} = \alpha$ and $D(\theta) = K(\theta)\psi'(\theta)$. Choosing $D(\theta) = D_0\theta^{m-1}$ (cf. [5,6]), one obtains (1.1) with $B(s) \equiv 0$, u being the volumetric moisture content.

Another example where Equation (1.1) appears is the one-dimensional turbulent flow of gas in a porous medium (cf. [7]), where u stands for the density, and the pressure is proportional to u^{m-1} (see also [8]). Typical values of p are again 1 for laminar (non-turbulent) flow and $\frac{1}{2}$ for completely turbulent flow.

The existence of nonnegative solution of (1.1)-(1.2) without the convection term div $(B(u^m))$, defined in some weak sense, had been well established (see [9] etc.). Here we quote the following definition.

Definition 1.1. A nonnegative function $u(x, t)$ is called a weak solution of (1.1)-(1.2) if u satisfies

(i)

$$u \in C(0,\ T; L^1(R^N)) \cap L^{\infty}(R^N \times (\tau,\ T)), \tag{1.6}$$

$$u^m \in L^p_{1oc}(0,\ T; W^{1,p}(R^N)), \quad u_t \in L^1(R^N \times\ (\tau,\ T)), \quad \forall\tau > 0; \tag{1.7}$$

(ii)

$$\iint_S [u\varphi_t - \left|Du^m\right|^{p-2} Du^m \cdot D\varphi - B(u^m)\cdot D\varphi]\, dxdt = 0, \quad \forall\varphi \in C_0^1(S); \tag{1.8}$$

(iii)

$$\lim_{t\to 0}\int_{R^N}\left|u(x,\ t)-u_0(x)\right|dx = 0. \tag{1.9}$$

If there exist the positive constants k_1, α such that

$$\left|B(s)\right| \ \le\ k_1|s|^{1+\alpha},\ \left|B'(s)\right|\ \le k_1|s|^{\alpha}, \forall s\in R^1 = (-\infty,\ +\infty), \tag{1.10}$$

Chen-Wang [10] had proved the existence and the uniqueness of the weak solutions of (1.1) and (1.2) in the sense of Definition 1.1.

As we have said before, we are mainly interested in the behavior of solution of (1.1) and (1.2) as $t \to \infty$. According to the different properties of the initial function $u_0(x)$, the corresponding nonnegative solutions may have different large time asymptotic behaviors, one can refer to the references [11-17]. In our article, we are going to study the large time asymptotic behavior for the solution of (1.1) and (1.2) by comparing it to the Barenblatt-type solution, let us give some details.

It is not difficult to verify that

$$E_c = t^{\frac{-1}{\mu}}\left\{\left[b-\frac{m(p-1)-1}{mp}(N\mu)^{\frac{-1}{p-1}}\left(|x|\,t^{\frac{-1}{N\mu}}\right)^{\frac{p}{p-1}}\right]_+\right\}^{\frac{p-1}{m(p-1)-1}}$$

is the Barenblatt-type solution of the Cauchy problem

$$u_t = \operatorname{div}\left(|Du^m|^{p-2}Du^m\right),\quad \text{in } S = R^N\times\ (0,\ \infty), \tag{1.11}$$

$$u(x,\ 0) = c\delta(x),\quad \text{on}\quad R^N, \tag{1.12}$$

where $\mu = m(p-1)-1+\frac{p}{N}$, $c = \int_{R^N} u_0(x)dx$, b is a constant such that $b = \int_{R^N} E_c(x,t)dx$, and δ denotes the Dirac mass centered at the origin.

By using some ideas of [9,14], we have the following

Theorem 1.2. Suppose $m(p - 1) >1$, B satisfies (1.10) with $\alpha < p - 1$ and $m(1+\alpha)\ge 1+\mu = m(p-1)+\frac{p}{N}$. If E_c is a unique solution of (1.11) and (1.12), then the solution u of (1.1) and (1.2) satisfies

$$t^{\frac{1}{\mu}}|u(x,\ t)-E_c(x,\ t)|\to 0,\quad \text{as } t\to\infty,$$

uniformly on the sets $\left\{x\in R^N:\quad |x|\ <\ at^{\frac{1}{\mu N}},\ a>0\right\}$, where $c = \int_{R^N} u_0 dx$ as before.

Remark 1.3. For $m = 1$, the uniqueness of solutions of (1.11) and (1.12) is known (see [18]).

By assuming that the uniqueness of the Barenblatt-type solution of (1.11) is true, Yang and Zhao [14] had established the similar large time behavior of solution of the Cauchy problem of the following equation

$$u_t = \operatorname{div}\left(|Du^m|^{p-2}Du^m\right) - u^q, \qquad \text{in } S = R^N \times (0, \infty), \tag{1.13}$$

While Zhan [17] had considered the Cauchy problem of the following equation

$$u_t = \operatorname{div}\left(\left|Du^m\right|^{p-2}Du^m\right) - \left|Du^m\right|^{p_1} - u^q, \qquad \text{in } S = R^N \times (0, \infty), \tag{1.14}$$

and also had got the similar result as Theorem 1.2. Comparing (1.1) with (1.13) or (1.14), the most difficulty comes from that the convection term $\operatorname{div}(B(u^m))$. The absorption term $-u^q$ in (1.13), or $-|Du^m|^{p_1} - u^q$ in (1.14), is always less than 0. This fact made us be able to draw it away in many estimates in [14] or [17]. But the convection term $\operatorname{div}(B(u^m))$ plays important role in this article, and it can not be drawn away randomly in the estimates we needed, we have to deal with it by some special techniques.

At the end of this introduction section, we would like to point that the condition $m(p - 1) > 1$ in Theorem 1.2, which means that the Equation (1.1) or (1.11) is a doubly degenerate parabolic equation, plays an important role in the proof of the theorem. In other words, if it is not true, (1.1) is in singular case, then the large time behavior of the solution in this case is still an open problem.

2. Some important lemmas

Let u be a nonnegative solution of (1.1) and (1.2). We define the family of functions

$$u_k = k^N u(kx, k^{N\mu}t), \quad k > 0.$$

It is easy to see that they are the solutions of the problems

$$u_t = \operatorname{div}\left(\left|Du^m\right|^{p-2}Du^m\right) + k^{N(1+\mu)}\operatorname{div}(B(k^{-Nm}u^m)), \qquad \text{in } S = R^N \times (0, \infty), \tag{2.1}$$

$$u(x, 0) = u_{0k}(x), \qquad \text{on } R^N, \tag{2.2}$$

where $\mu = m(p - 1) + \frac{p}{N} - 1$ as before and $u_{0k}(x) = k^N u_0(kx)$.

Lemma 2.1 For any $s \in (m\alpha, m(p - 1))$, the nonnegative solution u_k satisfies

$$\int_0^T \int_{B_R} \frac{u_k^{s-m}}{\left(1 + u_k^s\right)^2} \left|Du_k^m\right|^2 dxdt \le c(s, R, |u_0|_{L^1}), \tag{2.3}$$

$$\int_0^T \int_{B_R} u^{m(p-1)+\frac{p}{N}-s} dxdt \le c(s, R, |u_0|_{L^1}). \tag{2.4}$$

Proof. From Definition 1.1, we are able to deduce that (see [19]): for $\forall \varphi \in C^1(\bar{S})$, $\phi = 0$ when $|x|$ is large enough, for any $t \in [0, T]$, $0 < h < t$,

$$\int_{R^N} u_k\varphi(x, t)dx - \int_h^t \int_{R^N} \left[u_k\varphi_t - \left|Du_k^m\right|^{p-2}Du_k^m \cdot D\varphi - k^{N(1+\mu)}B\left(k^{-Nm}u_k^m\right)D\varphi\right] dxdt = \int_{R^N} u_k\varphi(x, h)dx. \tag{2.5}$$

Let

$$\psi_R \in C_0^\infty(B_{2R}), \quad 0 \le \psi_R \le 1, \quad \psi_R = 1 \text{ on } B_R, \quad |D\psi_R| \le cR^{-1}. \tag{2.6}$$

By an approximate procedure, we can choose $\varphi = \frac{u_k^s}{1+u_k^s}\psi_R^p$ in (2.5), then

$$\begin{aligned}
&\int_{R^N}\int_0^{u_k(x,t)} \frac{z^s}{1+z^s}dz\psi_R^p dx + s\int_h^t\int_{R^N} \frac{u_k^{s-m}}{(1+u_k^s)^2}|Du_k^m|^p\psi_R^p dxd\tau \\
&\le -p\int_h^t\int_{R^N}\frac{u_k^s}{1+u_k^s}|Du_k^m|^{p-2}\psi_R^{p-1}Du_k\cdot D\psi_R dxd\tau + \int_{R^N}\int_0^{u_k(x,t)}\frac{z^s}{1+z^s}dz\psi_R^p dx \\
&\quad + k^{N(1+\mu)}\left|\int_h^t\int_{R^N} B(k^{-Nm}u_k^m)D\left(\frac{u_k^s}{1+u_k^s}\psi_R^p\right)dxd\tau\right|.
\end{aligned}\tag{2.7}$$

Noticing

$$\begin{aligned}
&\left|\int_h^t\int_{R^N}\frac{u_k^s}{1+u_k^s}|Du_k^m|^{p-2}\psi_R^{p-1}(x)Du_k^m\cdot D\psi_R dxd\tau\right| \\
&\le \int_h^t\int_{R^N}\left\{\varepsilon\left[\frac{u_k^{(s-m)\frac{p-1}{p}}}{(1+u_k^s)^{2\frac{p-1}{p}}}|Du_k^m|^{p-1}\psi_R^{p-1}\right]^{\frac{p}{p-1}} + c(\varepsilon)\left[\frac{u_k^{s-\frac{(s-m)(p-1)}{p}}}{(1+u_k^s)^{1-2\frac{p-1}{p}}}|D\psi_R|\right]^p\right\}dxd\tau \\
&= \varepsilon\int_h^t\int_{R^N}\frac{u_k^{s-m}}{(1+u_k^s)^2}|Du_k^m|^p\psi_R^p dxd\tau + c(\varepsilon)\int_h^t\int_{R^N}u_k^{m(p-1)+s}|D\psi_R|^p dxd\tau,
\end{aligned}\tag{2.8}$$

$$\begin{aligned}
&k^{N(1+\mu)}\int_h^t\int_{R^N}B(k^{-Nm}u_k^m)D\left(\frac{u_k^s}{1+u_k^s}\psi_R^p\right)dxd\tau \\
&= k^{N(1+\mu)}\int_h^t\int_{R^N}B(k^{-Nm}u_k^m)\left[-\frac{Du^s}{(1+u^s)^2}\psi_R^p + \frac{pu^s}{1+u^s}\psi_R^{p-1}D\psi_R\right]dxd\tau.
\end{aligned}\tag{2.9}$$

Since $s > \alpha m$, $|B(k^{-Nm}u_k^m)| \le k_1 k^{-Nm(1+\alpha)}|u_k^m|^{1+\alpha}$,

$$\left|\frac{\psi_R^p u_k^{s-m+(1+\alpha)mp'}}{(1+u_k^s)^2}\right| \le 1$$

is always true, we have

$$\begin{aligned}
&\left|-\int_h^t\int_{R^N}\frac{B(k^{-Nm}u_k^m)}{(1+u_k^s)^2}\psi_R^p Du_k^s dxd\tau\right| = \left|\frac{s}{m}\int_h^t\int_{R^N}\frac{B(k^{-Nm}u_k^m)u_k^{s-m}}{(1+u_k^s)^2}\psi_R^p Du_k^m dxd\tau\right| \\
&\le \varepsilon\int_h^t\int_{R^N}\frac{s\psi_R^p u_k^{s-m}}{(1+u_k^s)^2}|Du_k^m|^p dxd\tau + c(\varepsilon)s\int_h^t\int_{R^N}\frac{s\psi_R^p u_k^{s-m}}{(1+u_k^s)^2}|B(k^{-Nm}u_k^m)|^{p'}dxd\tau, \\
&\le \varepsilon\int_h^t\int_{R^N}\frac{s\psi_R^p u_k^{s-m}}{(1+u_k^s)^2}|Du_k^m|^p dxd\tau + c(\varepsilon,\ R)k^{-Nm(1+\alpha)p'}, p' = \frac{p}{p-1},
\end{aligned}\tag{2.10}$$

and

$$\int_{R^N}\int_0^{u_k(x,h)} \frac{z^s}{1+z^s}dz\psi_R^p dx \le \int_{R^N} u(x,\ k^{N\mu}h)dx. \tag{2.11}$$

Noticing that the condition $m(p - 1) >1$ and

$$m(1+\alpha) \ge 1+\mu = m(p-1)+\frac{p}{N},$$

then by (2.7)-(2.11), we obtain

$$\begin{aligned}&\sup_{0<t<T}\int_{R^N}\int_0^{u_k(x,t)} \frac{z^s}{1+z^s}dzdx + \int_h^T\int_{R^N} \frac{u_k^{s-m}}{\left(1+u_k^s\right)^2}\left|Du_k^m\right|^p\psi_R^p dxd\tau \\ &\le c\int_{R^N} u(x,\ k^{N\mu}h)dx + c\int_h^T\int_{R^N} u_k^{m(p-1)+s}|D\psi_R|^p dxd\tau + c.\end{aligned} \tag{2.12}$$

Since $u_k \in L^\infty(R^N \times (h,\ T)) \cap L^1(S_T)$,

$$\lim_{R\to\infty}\int_h^T\int_{R^N} u_k^{m(p-1)+s}|D\psi_R|^p dxd\tau = 0. \tag{2.13}$$

Let $h \to 0$ in (2.12). Then

$$\sup_{0<t<T}\int_{B_{2R}}\int_0^{u_k(x,t)} \frac{z^s}{1+z^s}dzdx + \iint_{S_T} \frac{u_k^{s-m}}{\left(1+u_k^s\right)^2}\left|Du_k^m\right|^p dxd\tau \le c\int_{R^N} u_0 dx. \tag{2.14}$$

From this inequality, it is clear of that

$$\sup_{0<t<T}\int_{B_{2R}} u_k(x,\ t)dx + \int_0^T\int_{B_{2R}} \frac{u_k^{s-m}}{\left(1+u_k^s\right)^2}\left|Du_k^m\right|^p dxd\tau \le c(R). \tag{2.15}$$

So (2.3) is true.

Let

$$u_1 = \max\left\{u_k(x,\ t),\ 1\right\}, \quad w = u_1^{\frac{m(p-1)-s}{p}}.$$

By Sobolev's imbedding inequality (see [19]), for $\xi \in C_0^1(B_{2R})$, $\xi \ge 0$, we have

$$\left(\int_{B_{2R}} \xi^p w^r dx\right)^{\frac{1}{r}} \le c\left[\int_{B_{2R}} \left|D(\xi w)\right|^p dx\right]^{\frac{s}{p}}\left[\int_{B_{2R}} \frac{p}{w^{m(p-1)-s}}dx\right]^{\frac{(1-\theta)[m(p-1)-s]}{p}},$$

where

$$\theta = \left[\frac{m(p-1)-s}{p}-\frac{1}{r}\right]\left[\frac{1}{N}-\frac{1}{p}+\frac{m(p-1)-s}{p}\right]^{-1}, \quad r = \frac{p[m(p-1)+\frac{p}{N}-s]}{m(p-1)-s}.$$

It follows that

$$\iint\limits_{S_T} \xi^p w^r dxdt \le c \iint\limits_{S_T} \left|D(\xi w)\right|^p dxdt \sup_{t\in(0,T)} \left(\int\limits_{B_{2R}} \frac{p}{w^{m(p-1)-s}} dx \right)^{\frac{(r-p)[m(p-1)-s]}{p}}, \tag{2.16}$$

where we denote $S_T = R^N \times (0, T)$. Since

$$|Dw|^p \le c\frac{u_k^{s-m}}{\left(1+u_k^s\right)^2}\left|Du_k^m\right|^p \text{a.e. on } \{u_k \ge 1\} \text{ and } |Dw| = 0 \text{ on } \{u_k \le 1\},$$

we have

$$\begin{aligned} \iint\limits_{S_T}\int |D(\xi w)|^p dxdt &\le c \iint\limits_{S_T} (\xi^p |Dw|^p + w^p |D\xi|^p) dxdt \\ &\le c\left[\iint\limits_{S_T} |D\xi|^p u_1^{m(p-1)-s} dxdt + \int_0^T \int\limits_{B_{2R}} \frac{u_k^{s-m}}{(1+u_k^s)^2} |Du_k^m|^p dxdt \right]. \end{aligned} \tag{2.17}$$

Hence, by (2.16), (2.(17) and (2.15), we get

$$\iint\limits_{S_T} \xi^p u_1^{m(p-1)+\frac{p}{N}-s} dxdt \le c(s,\ R,\ |u_0|_{L^1}) \left(1 + \iint\limits_{S_T} |D\xi|^p u_1^{m(p-1)-s} dxdt \right).$$

Let $\xi = \psi_R^b$, ψ_R. be the function satisfying (2.6) and $b = \frac{N[m(p-1)+\frac{p}{N}-s]}{p}$. Then

$$\iint\limits_{s_T} \psi_R^{pb} u_1^{m(p-1)+\frac{p}{N}-s} dxdt \le c(s,\ R,\ |u_0|_{L^1}) \left(1 + \iint\limits_{s_T} \psi_R^{pb} u_1^{m(p-1)+\frac{p}{N}-s} dxdt \right)^{\dfrac{m(p-1)-s}{m(p-1)-s+\frac{p}{N}}},$$

by Moser iteration technique, the above inequality implies (2.4) is true.

Let $Q_\rho = B_\rho(x_0) \times (t_0 - \rho^p, t_0)$ with $t_0 > (2\rho)^p$ and $u_{k1} = \max\{u_k, 1\}$. Also by Moser iteration technique, we have

Lemma 2.2 The nonnegative solution u_k satisfies

$$\sup_{Q_\rho} u_k \le c(\rho,\ s_1) \left(\iint\limits_{Q_{2\rho}} u_{k1}^{m(p-1)-1+s_1} dxdt \right)^{1/s_1}, \tag{2.18}$$

where $c(\rho, s_1)$ depends on ρ and s_1, and s_1 can be any number satisfying $0 < s_1 < 1 + \frac{p}{N}$.

Proof. For $\forall\varphi \in C^1(\bar{S})$, $\phi = 0$ when $|x|$ is large enough, we have

$$\begin{aligned} \int\limits_{R^N} u_k(x,\ t)\varphi dx - \int_0^T \int\limits_{R^N} [u_k\varphi_t - \left|Du_k^m\right|^{p-2} Du_k^m \cdot D\varphi - k^{N(1+\mu)} B(k^{-Nm} u_k^m) D\varphi] dxdt \\ = \int\limits_{R^N} u_{0k}(x)\varphi(x,\ 0) dx. \end{aligned} \tag{2.19}$$

Let ξ be the cut function on Q_ρ, i.e.

$$0 \le \xi \le 1,\ \xi|_{Q_\rho} = 1,\ \xi|_{R^N \setminus Q_{2\rho}} = 0.$$

We choose the testing function in (2.19) as $\varphi = \xi^p u_k^{2\gamma-1}$, where $\gamma > \frac{1}{2}$ is a constant. Then

$$\begin{aligned} &\frac{1}{2\gamma}\int_{B_{2\rho}} \xi^p u_k^{2\gamma}(x,\ t)dx + \frac{2\gamma-1}{m}\int_0^t\int_{B_{2\rho}} \xi^p u_k^{2\gamma-1-m}\left|Du_k^m\right|^p dxds \\ &= p\int_0^t\int_{B_{2\rho}} \xi^{p-1}\,|D\xi|\,u_k^{2\gamma-1}\left|Du_k^m\right|^{p-1}dxds + \frac{p}{2\gamma}\int_0^t\int_{B_{2\rho}} \xi^{p-1}\,|\xi_t|\,u_k^{2\gamma}dxds \\ &\quad +pk^{N(1+\mu)}\int_0^t\int_{B_{2\rho}} \xi^{p-1}B(k^{-Nm}u_k^m)u_k^{2\gamma-1}D\xi dxdt \\ &\quad +k^{N(1+\mu)}\frac{2\gamma-1}{m}\int_0^t\int_{B_{2\rho}} \xi^p u_k^{2\gamma-m-1}B(k^{-Nm}u_k^m)Du_k^m dxds. \end{aligned} \tag{2.20}$$

Using Young inequality, by (1.10),

$$\begin{aligned} \xi^{p-1}\,|D\xi|\,u_k^{2\gamma-1}\left|Du_k^m\right|^{p-1} &= u_k^{2\gamma-1-m}\xi^{p-1}\left|Du_k^m\right|^{p-1}|D\xi|\,u_k^m \\ &\le u_k^{2\gamma-1-m}(\varepsilon\xi^p\left|Du_k^m\right|^p + c(\varepsilon)u_k^{mp}|D\xi|^p), \\ \left|\xi^p u_k^{2\gamma-m-1}B(k^{-Nm}u_k^m)Du_k^m\right| &\le k^{-Nm(1+\alpha)}\left|\xi^p u_k^{m\alpha+2\gamma-1}Du_k^m\right| \\ &\le k^{-Nm(1+\alpha)}\left|\xi^p u_k^{2\gamma-1-m}u_k^{m\alpha+m}Du_k^m\right| \\ &\le k^{-Nm(1+\alpha)}\left|\xi^p u_k^{2\gamma-1-m}\right|(c(\varepsilon)u_k^{m(1+\alpha)p'} + \varepsilon\left|Du_k^m\right|^p), \end{aligned}$$

from (2.20), we have

$$\begin{aligned} &\frac{1}{2\gamma}\int_{B_{2\rho}} \xi^p u_k^{2\gamma}(x,\ t)dx + \left[\frac{2\gamma-1}{m} - \varepsilon\left(1 + \frac{2\gamma-1}{m}\right)\right]\int_0^t\int_{B_{2\rho}} \xi^p u_k^{2\gamma-1-m}\left|\nabla u_k^m\right|^p dxds \\ &\le c\int_0^t\int_{B_{2\rho}} u_k^{2\gamma-1+m(p-1)}|D\xi|^p dxds + \frac{p}{2\gamma}\int_0^t\int_{B_{2\rho}} \xi^{p-1}\,|\xi_t|\,u_k^{2\gamma}dxds \\ &\quad +c(\varepsilon)k^{-Nm(1+\alpha)+N(1+\mu)}\int_0^t\int_{B_{2\rho}} |u_k|^{2\gamma-1-m+m(1+\alpha)p'}dxds. \end{aligned} \tag{2.21}$$

By the fact of that

$$\begin{aligned} \left|D\left(\xi u^{\frac{2\gamma-1+m(p-1)}{p}}\right)\right|^p &= \left|u_k^{\frac{2\gamma-1+m(p-1)}{p}}D\xi + \frac{2\gamma-1+m(p-1)}{mp}\xi u_k^{\frac{2\gamma-1-m}{p}}Du_k^m\right|^p \\ &\le c|D\xi|^p u_k^{2\gamma-1+m(p-1)} + c\xi^p\left|Du_k^m\right|^p u_k^{2\gamma-1-m}, \end{aligned}$$

from (2.21), we have

$$\begin{aligned}
&\sup_{t_0-2\rho^p<t<t_0}\int_{B_{2\rho}} \xi^p u_k^{2\gamma}\,dxds+\int\int_{Q_{2\rho}} \left| D\left(\xi u_k^{\frac{2\gamma-1+m(p-1)}{p}}\right)\right|^p dxds \\
&\le c\int_0^t\int_{B_{2\rho}} u_k^{2\gamma-1+m(p-1)}|D\xi|^p dxds \;+\; c\int_0^t\int_{B_{2\rho}} \xi^{p-1}\,|\xi_t|\,u_k^{2\gamma}\,dxds \\
&+ c(\varepsilon)k^{-Nm(1+\alpha)+N(1+\mu)}\,\frac{2\gamma-1}{m}\int_0^t\int_{B_{2\rho}} |u_k|^{2\gamma-1-m+m(1+\alpha)p'}\,dxds.
\end{aligned} \tag{2.22}$$

Let

$$\beta=\max\left\{1,\;\frac{2\gamma-1+m(p-1)}{\gamma}\right\},$$

and

$$w=\xi^\beta u_k^{\frac{2\gamma-1+m(p-1)}{p}}.$$

By the embedding theorem, from (2.22), we have

$$\int\int_{Q_{2\rho}} w^h dxdt\le c\left\{\sup_{t_0-2\rho^p<t<t_0}\int_{B_{2\rho}} w^{\frac{2\gamma p}{2\gamma-1+m(p-1)}}dx\right\}^{\frac{2\gamma-1+m(p-1)}{2\gamma p}(1-\delta)h}\cdot\int_{t_0-(2\rho^p)}\left(\int_{B_{2\rho}}|Dw|^p dx\right)^{\frac{\delta h}{p}}dx, \tag{2.23}$$

where

$$\delta=\left(\frac{2\gamma-1+m(p-1)}{2\gamma p}-\frac{1}{h}\right)\cdot\left(\frac{1}{N}-\frac{1}{p}+\frac{2\gamma-1+m(p-1)}{2\gamma p}\right)^{-1}.$$

In particular, we choose

$$h=p[1+\frac{2\gamma p}{N(2\gamma-1+m(p-1))}],$$

then from (2.23), we have

$$\begin{aligned}
&\int\int_{Q_{2\rho}} \xi^{\beta h}u_k^{2\gamma-1+m(p-1)+\frac{2\gamma p}{N}}dxdt \\
&\le c_1\left(\sup_{t_0-2\rho^p<t<t_0}\int_{B_{2\rho}} \xi^{\frac{2\gamma p\beta}{2\gamma-1+m(p-1)}}u_k^{2\gamma}dx\right)^{\frac{p}{N}}\cdot\int\int_{Q_{2\rho}}\left|D\left(\xi^\beta u_k^{\frac{2\gamma-1+m(p-1)}{p}}\right)\right|^p dxdt \\
&\le c_2\left\{\sup_{t_0-2\rho^p<t<t_0}\int_{B_{2\rho}} \xi^{\frac{2\gamma p\beta}{2\gamma-1+m(p-1)}}u_k^{2\gamma}dx+\int\int_{Q_{2\rho}}\left|D\left(\xi^\beta u_k^{\frac{2\gamma-1+m(p-1)}{p}}\right)\right|^p dxdt\right\}^{1+\frac{p}{N}}.
\end{aligned} \tag{2.24}$$

Now, for $\tau \in [\frac{1}{2}, 1]$, we denote that

$$\rho^l = 2\rho\left(\tau + \frac{1-\tau}{2^l}\right),\ l = 1, 2, \ \dots\ ,$$

and choose the cut functions $\xi_l(x, t)$ of $Q_{\rho l}$, such that on $Q_{\rho(l+1)}$, $\xi_l = 1$.
Denote

$$K = 1 + \frac{p}{N},\ 2\gamma = K^l.$$

and let

$$u_{1k} = \max\{1,\ u_k\}.$$

Then, by (2.23) and (2.24) and the assumption of that $a < p - 1$, which implies

$$2\gamma - 1 + m(1+\alpha)p' - m \le 2\gamma - 1 + m(p-1),$$

we have

$$\begin{aligned}\int\int_{Q_{\rho(l+1)}} u_k^{m(p-1)-1+K^{l+1}} dxdt &\le \int\int_{Q_{\rho(l+1)}} u_{1k}^{m(p-1)-1+K^{l+1}} dxdt \\ &\le \int\int_{Q_{\rho(l+1)}} u_k^{m(p-1)-1+K^{l+1}} dxdt + \text{mes}Q_{\rho(l+1)} \\ &\le \left\{\frac{cc_1^l}{[(1-\tau)\rho]^p}\int\int_{Q_{\rho l}} u_{1k}^{m(p-1)-1+K^l} dxdt\right\}^K.\end{aligned}$$

Using Moser iteration technique, we have

$$\sup_{2\tau\rho} u_{1k} \le \left\{\frac{1}{\left[(1-\tau)\rho\right]^{N+p}}\int\int_{Q_{2\rho}} u_{1k}^{m(p-1)-1+K} dxdt\right\}^{\frac{1}{K}}.$$

Then, we have

$$\sup_{2\tau\rho} u_{1k} \le (\sup_{Q_{2\rho}} u_{1k})^{\frac{K-r}{K}} \cdot \left\{\frac{1}{\left[(1-\tau)\rho\right]^{N+p}}\int\int_{Q_{2\rho}} u_{1k}^{m(p-1)-1+r} dxdt\right\}^{\frac{1}{K}}.$$

By Schwarz inequality,

$$\sup_{2\tau\rho} u_{1k} \le \frac{1}{2}\sup_{2\rho} u_{1k} + c(r)\left\{\frac{1}{\left[(1-\tau)\rho\right]^{N+p}}\int\int_{Q_{2\rho}} u_{1k}^{m(p-1)-1+r} dxdt\right\}^{\frac{1}{r}}.$$

By the Lemma 3.1 in [19], for any $\tau \in [\frac{1}{2}, 1)$, we have

$$\sup_{2\tau\rho} u_{1k} \le c(r,p)\left[\int\int_{Q_{2\rho}} u_{1k}^{m(p-1)-1+r} dxdt\right]^{\frac{1}{r}},$$

and from this inequality, we get the conclusion of the lemma.

Lemma 2.3 The nonnegative solution u_k satisfies

$$\int_{\tau}^{T}\int_{B_R} \left|Du_k^m\right|^p dxdt \le c(\tau,\ R). \tag{2.25}$$

$$\int_{\tau}^{T}\int_{B_R} |u_{kt}|^p dxdt \ \le\ c(\tau,\ R). \tag{2.26}$$

Proof. By Lemmas 2.1 and 2.2, $\{u_k\}$ are uniformly bounded on every compact set $K \subset S_T$. Let ψ_R be a function satisfying (2.6) and $\xi \in C_0^1(0,T)$ with $0 \le \xi \le 1$, $\xi = 1$ if $t \in (\tau, T)$. We choose $\eta = \psi_R^p \xi u_k^m$ in (2.5) to obtain

$$\begin{aligned} &\frac{1}{m+1}\int_{R^N} u_k^{m+1}(x,\ T)\psi_R^p dx + \iint_{S_T} \left|Du_k^m\right|^p \psi_R^p \xi\, dxdt \\ = &\frac{1}{m+1}\iint_{S_T} u_k^{m+1}\xi' \psi_R^p dxdt - p\iint_{S_T} u_k^m \left|Du_k^m\right|^{p-2} Du_k^m \cdot D\psi_R \psi_R^{p-1}\xi\, dxdt \\ &+k^{N(1+\mu)}\iint_{S_T} B(k^{-Nm}u_k^m)\xi(p\psi^{p-1}D\psi_R u_k^m + \psi_R^p Du_k^m)dxdt. \end{aligned} \tag{2.27}$$

Noticing

$$\begin{aligned} &\iint_{S_T} u_k^m \left|Du_k^m\right|^{p-1} |D\psi_R|\, \psi_R^{p-1}\xi\, dxdt \\ &\le \varepsilon \iint_{S_T} \left|Du_k^m\right|^p \psi_R^p \xi\, dxdt + c(\varepsilon)\iint_{S_T} u_k^{pm}|D\psi_R|^p \xi\, dxdt, \end{aligned}$$

$$\left|\iint_{S_T} B(k^{-Nm}u_k^m)\xi\psi^{p-1}D\psi_R u_k^m dxdt\right| \le \frac{c}{R}k^{-Nm(1+\alpha)}\int_0^T\int_{B_r} \xi u_k^{m(1+\alpha)}dxd\tau,$$

$$\begin{aligned} \left|\iint_{S_T} B(k^{-mN}u_k^m)\xi\psi_R^p Du_k^m)dxdt\right| &\le c(\varepsilon)\iint_{S_T} \left|B(k^{-Nm}u_k^m)\right|^{p'}\xi\psi_R^p dxd\tau + \varepsilon\iint_{S_T} \left|Du_k^m\right|^p \psi_R^p \xi\, dxdt \\ &\le c(\varepsilon)k^{-Nm(1+\alpha)}\iint_{S_T} u_k^{mp'(1+\alpha)}\xi\psi_R^p dxd\tau + \varepsilon\iint_{S_T} \left|Du_k^m\right|^p \psi_R^p \xi\, dxdt, \end{aligned}$$

from these inequalities, by (2.18) and (2.27), one knows that (2.25) is true. (2.26) is to be proved in what follows.

Let

$$v(x,\ t) = u_{kr}(x,\ t) = ru_k(x,\ r^{m(p-1)-1}t), \quad r \in (0, 1). \tag{2.28}$$

Then

$$v_t(x,\ t) = \mathrm{div}(|Dv^m|^{p-2}Dv^m) + r^{m(p-1)}k^{N(1+\mu)}\mathrm{div}(B(k^{-Nm}r^{-m}v^m)), \tag{2.29}$$

$$v(x,0) = ru_k(x,0). \tag{2.30}$$

By (2.1) and (2.29), for any $\varphi \in C_0^1(S_T)$, we have

$$\begin{aligned}&\iint\limits_{S_T} \varphi \frac{\partial}{\partial t}(u_k - v)dxdt + \iint\limits_{S_T} \left[|Du_k^m|^{p-2}Du_k^m - |Dv^m|^{p-2}Dv^m\right]D\varphi dxdt\\&+k^{N(1+\mu)}\iint\limits_{S_T}[B(k^{-Nm}u_k^m) - r^{m(p-1)}B(k^{-Nm}r^{-m}v^m)]D\varphi dxdt = 0.\end{aligned} \tag{2.31}$$

Let $g_n(s) = 1$ when $s > \frac{1}{n}$; $g_n(s) = ns$ when $0 \le s \le \frac{1}{n}$; $g_n(s) = 0$ when $s < 0$, and let ϕ in (2.31) be substituted by $\varphi g_n(u_k^m - v^m)$. Then

$$\begin{aligned}&\iint\limits_{S_T} \varphi g_n(u_k^m - v^m)\frac{\partial}{\partial t}(u_k - v)dxdt + \iint\limits_{S_T}[|Du_k^m|^{p-2}Du_k^m - |Dv^m|^{p-2}Dv^m][g_n' D(u_k^m - v^m)\varphi + g_n D\varphi]dxdt\\&+k^{N(1+\mu)}\iint\limits_{S_T}[B(k^{-Nm}u_k^m) - r^{m(p-1)}B(k^{-Nm}r^{-m}v^m)][g_n' D(u_k^m - v^m)\varphi + g_n D\varphi]dxdt = 0.\end{aligned} \tag{2.32}$$

Let $\varphi(x, t) = \theta(\frac{x}{k})\eta_j(t)$. Where $\theta \in C_0^1(R^N)$, $0 \le \theta \le 1$, $\theta(x) = 1$ when $x \in B_1$, and $\eta_j(t) \in C_0^1(0, T)$, $0 \le \eta_j \le 1$, which satisfies that $\eta_j \to \eta$ when $j \to \infty$, and η is the characteristic function of (s_1, s_2), $s_1 < s_2$.

Since $u_k, v \in L^\infty(R^N \times (\tau, T))$, $Du_k^m, Dv^m \in L^p(R^N \times (\tau, T))$, we have

$$\iint\limits_{S_T}[|Du_k^m|^{p-2}Du_k^m - |Dv^m|^{p-2}Dv^m]g_n'\theta(x)\eta_j(t)dxdt \ge 0,$$

$$\begin{aligned}&\left|\iint\limits_{S_T}[|Du_k^m|^{p-2}Du_k^m - |Dv^m|^{p-2}Dv^m]g_n D\theta\left(\frac{x}{k}\right)\eta_j(t)dxdt\right|\\&\le \frac{1}{k}\iint\limits_{S_T}[|Du_k^m|^{p-1} + |Dv^m|^{p-1}]|D_y\theta(y)|_{y=\frac{x}{k}}\eta_j(t)dxdt \to 0,\end{aligned}$$

as $k \to \infty$.

If we notice that, for any $i \in \{1,2, ..., N\}$,

$$\begin{aligned}k^{N(1+\mu)}\lim_{r\to 1}&\left|b_i(k^{-Nm}u_k^m) - r^{m(p-1)}b_i(k^{-Nm}r^{-m}v^m)\right|\\&= k^{N(1+\mu)}\left|b_i(k^{-Nm}u_k^m) - b_i(k^{-Nm}v^m)\right|\\&= k^{N(1+\mu)}\left|\int_{k^{-Nm}v^m}^{k^{-Nm}u_k^m} b_i'(s)ds\right|\\&\le k^{N(1+\mu)}\int_{k^{-Nm}v^m}^{k^{-Nm}u_k^m} s^\alpha ds = k_1 k^{-Nm(\alpha+1)+N(1+\mu)}\frac{k_1}{\alpha+1}\left|u_k^{m(\alpha+1)} - v^{m(\alpha+1)}\right|.\end{aligned}$$

then it is easy to show that

$$k^{N(1+\mu)}\lim_{r\to 1}\iint\limits_{S_T}[B(k^{-Nm}u_k^m) - r^{m(p-1)}B(k^{-Nm}r^{-m}v^m)]g_n'\varphi(Du_k^m - Dv^m)dxdt = 0$$

At the same time,

$$\begin{aligned}
&\lim_{k\to\infty} k^{N(1+\mu)} \left| \iint_{S_T} [B(k^{-Nm}u_k^m) - r^{m(p-1)}B(k^{-Nm}r^{-m}v^m)]g_n D\varphi dxdt \right| \\
&= k^{N(1+\mu)} \lim_{k\to\infty} \left| \iint_{S_T} [B(k^{-Nm}u_k^m) - r^{m(p-1)}B(k^{-Nm}r^{-m}v^m)]g_n D\theta(\frac{x}{k})\eta_j(t)dxdt \right| \\
&\le \lim_{k\to\infty} k^{N(1+\mu)-1} \iint_{S_T} \left| B(k^{-Nm}u_k^m) - r^{m(p-1)}B(k^{-Nm}r^{-m}v^m) \right| \left| D_y\theta(y) \right|_{y=\frac{x}{k}} \eta(t)dxdt \\
&\le \lim_{k\to\infty} k^{N(1+\mu)-1-Nm(1+\alpha)} \iint_{S_T} \left| u_k^{m(1+\alpha)} + r^{m(p-2-\alpha)}v^{m(1+\alpha)} \right| \left| D_y\theta(y) \right|_{y=\frac{x}{k}} \eta(t)dxdt = 0.
\end{aligned}$$

Then, if we let $k \to \infty$, $n \to \infty$ and let $r \to 1$ in (2.32), since $\mu < \alpha$, we have

$$\lim_{r\to 1} \iint_{S_T} \eta_j(t)\text{sgn}_+(u_k - v)\frac{\partial(u_k - v)}{\partial t}dxdt \le 0,$$

in other words,

$$\lim_{r\to 1} \iint_{S_T} \eta_j'(t)(u_k - v)_+ dxdt \ge 0.$$

Let $j \to \infty$. Then

$$\lim_{r\to 1} \int_{R^N} (u_k(x, s_2) - v(x, s_2))_+ dx \le \lim_{r\to 1} \int_{R^N} (u_k(x, s_1) - v(x, s_1))_+ dx.$$

Similarly, we have

$$\lim_{r\to 1} \int_{R^N} (v(x, s_2) - u_k(x, s_2))_+ dx \le \lim_{r\to 1} \int_{R^N} (v(x, s_1) - u_k(x, s_1))_+ dx. \tag{2.33}$$

Let $s_1 \to 0$. Then

$$u_k \ge \lim_{r\to 1} u_{kr}.$$

It follows that

$$\lim_{r\to 1} \frac{u_k(x, r^{m(p-1)-1}t) - u_k(x,t)}{(r^{m(p-1)-1} - 1)t} \ge \lim_{r\to 1} \frac{r-1}{(1 - r^{m(p-1)-1})t} u_k(x, r^{m(p-1)-1}t),$$

which implies that

$$u_{kt} \ge -\frac{u_k}{[m(p-1)-1]t}. \tag{2.34}$$

Denote $w = t^{\gamma} u_k(x, t)$, $\gamma = \frac{1}{m(p-1)-1}$. By (2.34), $w_t \geq 0$. By (2.1),

$$\begin{aligned}
&\int_\tau^T \int_{B_{2R}} t^{-\gamma} w_t \psi_R dxdt = -\int_\tau^T \int_{B_{2R}} \left|Du_k^m\right|^{p-2} Du_k^m \cdot D\psi_R dxdt \\
&-k^{N(1+\mu)} \int_\tau^T B(k^{-Nm} u_k^m) D\psi_R dxdt + \gamma \int_\tau^T \int_{B_{2R}} t^{-1} u_k(x) \psi_R dxdt \\
&\leq \frac{\beta}{\tau} \int_\tau^T \int_{B_{2R}} u_k dxdt + \left(\int_\tau^T \int_{B_{2R}} \left|Du_k^m\right|^p dxdt \right)^{\frac{p-1}{p}} \left(\int_\tau^T \int_{B_{2R}} |D\psi_R|^p dxdt \right)^{\frac{1}{p}} \\
&+k^{-Nm(\alpha+1)+N(1+\mu)} \int_\tau^T \int_{B_{2R}} u_k^{m(\alpha+1)} |D\psi_R| dxdt.
\end{aligned} \tag{2.35}$$

From (2.15), (2.18) and (2.35), we obtain (2.26).

3. Proof of Theorem 1.2

Proof of Theorem 1.2. By Lemmas 2.1-2.3, there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ and a function v such that on every compact set $K \subset S$

$$u_{k_j} \to v \text{ in } C(K),\ Du_k^m \rightharpoonup Dv^m \text{ in } L^p_{1\mathrm{oc}}(S_T),\ |u_{kt}|_{L^1_{1\mathrm{oc}}(S_T)} \leq c.$$

Similar to what was done in the proof of Theorem 2 in [9], we can prove v satisfies (1.11) in the sense of distribution.

We now prove $v(x, 0) = c\delta(x)$. Let $\chi \in C_0^1(B_R)$. Then we have

$$\begin{aligned}
&\int_{R^N} u_k(x, t) \chi dx - \int_{R^N} \varphi_k \chi dx \\
&= -\int_0^t \int_{R^N} \left|Du_k^m\right|^{p-2} Du_k^m \cdot D\chi dxds - k^{N(1+\mu)} \int_0^t \int_{R^N} B(k^{-Nm} u_k^m) D\chi dxds.
\end{aligned} \tag{3.1}$$

To estimate $\int_0^t \int_{R^N} \left|Du_k^m\right|^{p-2} Du_k^m \cdot D\chi dxds$, without loss of the generality, one can assume that $u_k > 0$. By Hölder inequality and Lemma 2.1,

$$\begin{aligned}
&\left| \int_0^t \int_{R^N} \left|Du_k^m\right|^{p-2} Du_k^m \cdot D\chi dxdt \right| \\
&\leq c \left[\int_0^t \int_{B_{2R}} \frac{u_k^{s-m}}{\left(1+u_k^s\right)^2} \left|Du_k^m\right|^p dxd\tau \right]^{\frac{p-1}{p}} \cdot \left[\int_0^T \int_{B_R} \left(1+u_k^s\right)^{2(p-1)} u_k^{(p-1)(m-s)} dxd\tau \right]^{\frac{1}{p}} \\
&\leq c \left[\int_0^t \int_{B_R} \left(u_{k1}^{(p-1)(m-s)} + u_{k1}^{(p-1)(s+m)} dxd\tau \right]^{\frac{1}{p}} \\
&\leq c \left[\int_0^t \int_{B_R} u_{k1}^{m(p-1)+\frac{p}{N}-s} dxd\tau \right]^{\frac{1}{p}-d} t^d,
\end{aligned} \tag{3.2}$$

where $s \in (0, \frac{1}{N})$, $d = \frac{1-Ns}{m(p-1)N+p-sN} < \frac{1}{P}$, $u_{k1} = \max(u_k, 1)$.

Hence from (3.1), we get

$$\left|\int_{R^N} u_k \chi\, dx - \int_{R^N} u_{0k} \chi\, dx\right| \le ct^d + k^{-Nm(\alpha+1)+N(1+\mu)} \int_0^t \int_{R^N} u_k^{m(\alpha+1)} |D\chi|\, d\tau\, ds. \qquad (3.3)$$

Letting $k \to \infty$, $t \to 0$ in turn, we obtain

$$\lim_{t\to 0_R} \int_{R^N} v\chi\, dx = \chi(0) \int_{R^N} \varphi dx.$$

Thus

$$v(x, 0) = c\delta(x), \; c = \int_{R^N} \varphi dx,$$

$v(x, t)$ is a solution of (1.11) and (1.12). By the assumption on uniqueness of solution, we have $v(x, t) = E_c(x, t)$ and the entire sequence $\{u_k\}$ converges to E_c as $k \to \infty$. Set $t = 1$.

Then

$$u_k(x, 1) = k^N u(kx, k^{N\mu}) \to E_c(x, 1)$$

uniformly on every compact subset of R^N. Thus by writing $kx = k'$, $k^{N\mu} = t'$, and dropping the prime again, we see that

$$t^{\frac{1}{\mu}} u(x, t) \to E_c(xt^{\frac{1}{N\mu}}, 1) = t^{\frac{1}{\mu}} E_c(x, t)$$

uniformly on the sets $\left\{x \in R^N : |x| \le at^{\frac{1}{N\mu}}\right\}$, $a > 0$. Thus Theorem 1.2 is true.

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Author details
[1]School of Sciences, Jimei University, Xiamen 361021, People's Republic of China [2]College of Teacher Education, Jimei University, Xiamen 361021, People's Republic of China

Authors' contributions
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