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Sufficient conditions for global optimality of semidefinite optimization

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Abstract

In this article, by using the Lagrangian function, we investigate the sufficient global optimality conditions for a class of semi-definite optimization problems, where the objective function are general nonlinear, the variables are mixed integers subject to linear matrix inequalities (LMIs) constraints as well as bounded constraints. In addition, the sufficient global optimality conditions for general nonlinear programming problems are derived, where the variables satisfy LMIs constraints and box constraints or bivalent constraints. Furthermore, we give the sufficient global optimality conditions for standard semi-definite programming problem, where the objective function is linear, the variables satisfy linear inequalities constraints and box constraints.

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1 Introduction

As we know semi-definite programming (SDP) can be viewed as a natural extension of linear programming where the componentwise inequalities between vectors are replaced by matrix inequalities. The SDP has many important applications in systems and control theory [1] and combinatorial optimization [2-4]. Many survey articles such as [1,5-7] featured various applications of SDP and algorithmic aspects. With the development of optimization software, more and more problems are modeled as SDP problems. SDP became one of the basic modeling and optimization tools along with linear and quadratic programming.

Recently, many researchers focused on characterizing the global minimizer of many mathematical programming problems. Beck and Teboulle [8] have established a necessary global optimality condition for nonconvex quadratic optimization problems with binary constrains. Jeyakumar et al. [9] have given Lagrange multiplier conditions for global optimality of general quadratic minimization problems with quadratic constraints. Jeyakumar et al. [10] have obtained sufficient global optimality conditions for a quadratic minimization problem subject to box constraints or binary constraints. Jeyakumar et al. [11] have established some necessary and sufficient conditions for a given feasible point to be a global minimizer of some minimization problems with mixed variables. Wu and Bai [12] have given some global optimality conditions for mixed quadratic programming problems, their approach is based on L-subdifferential and L-normal cone. Especially Jeyakumar and Wu [13] have presented sufficient conditions for global optimality of non-convex quadratic programs involving linear matrix

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inequality (LMI) constraints by using Lagrangian function and by examining conditions which minimizes a quadratic subgradient of the Lagrangian function over simple bounding constraints. Jeyakumar [14] have obtained some necessary and sufficient constraint qualifications (CQs) for the strong duality in convex semidefinite optimization.

In this article, we consider the following semidefinite optimization model problem:

$$\begin{aligned} (\mathrm{SDP}_f) \qquad & \min_{x \in R^n} f(x) \\ s.t.\ & F_0 + \sum_{k=1}^{n} F_k x_k \succcurlyeq O, \\ & x_i \in [u_i, v_i], \quad i \in M, \\ & x_j \in \{p_j, p_j + 1, \ldots, q_j\}, \quad j \in N \end{aligned}$$

where $M \cap N = \emptyset$, $M \cup N = \{1,\ldots, n\}$, $u_i, v_i \in$ R and $u_i < v_i$ for any $i \in M$, p_j, q_j are integers and $p_j < q_j$ for all $j \in N$; $f : R^n \to R$ are twice continuously differentiable functions on an open subset of R^n containing set $\{x \in R^n | x_i \in [u_i, v_i], i \in M; x_j \in [p_j, q_j], j \in N\}$. For $k = 0, 1, 2,\ldots, n$, $F_k \in S_m$, the space of symmetric $(m \times m)$ matrices with the trace inner product and $\succcurlyeq$ denotes the *Löwner* partial order of S_m, that is, for $A, B \in S_m$, $A \succcurlyeq B$ if and only if $(A - B)$ is positive semidefinite. Such semidefinite optimization model problem has been intensely studied in the last 10 years since it arose from control system analysis and design. Interested reader may refer to [15,16]. The convex semidefinite optimization model problem has been studied in [14,16,17] for it's valuable numerical and modeling tool for system and control theory.

The purpose of this article is to present some sufficient global optimality condition for a given feasible point to be a global minimizer of programming problems (SDP_f) with nonlinear objects. We develop the sufficient global optimality conditions for nonlinear programming problem (SDP_f) with LMI and bounded constraints of mixed integer variables by using the Lagrangian function. We also deduce the sufficient global optimality conditions for nonlinear programs with LMI and box constraints or bivalent constraints which are the extended results beyond [13], as well as the sufficient global optimality conditions for standard SDP problem (SDP), where the objective function is linear, the variables satisfy LMIs constraints and box constraints.

2 Preliminaries and notations

Firstly we present some notations that will be used throughout this article. The real line is denoted by R and the n-dimensional Euclidean space is denoted by R^n. For vectors $x, y \in R^n$, $x \geq y$ means that $x_k \geq y_k$, for $k = 1,\ldots, n$. The notation $A \succcurlyeq B$ means $A - B$ is a positive semidefinite and $A \preccurlyeq O$ means $- A \succcurlyeq O$. A diagonal matrix with diagonal elements $\alpha_1,\ldots,\alpha_n$ is denoted by $\mathrm{diag}(\alpha_1,\ldots,\alpha_n)$. We let $U = \{x = (x_1,\ldots, x_n)^T \mid x_i \in [u_i, v_i], i \in M, x_j \in \{p_j, p_j + 1,\ldots,q_j\}, j \in N\}$; The feasible set Ω of (SDP_f) is given by $\Omega = U \cap F^{-1}(S)$, where $S = \{M \in S_m | M \succcurlyeq O\}$ is the closed convex cone of positive semidefinite $(m \times m)$ matrices, $F^{-1}(S) := \{x \in R^n | F(x) \succcurlyeq O\}$ and $F(x) := F_0 + \sum_{k=1}^{n} F_k x_k$. The inner product in S_m is defined by $(N1, N2) = \mathrm{Tr}[N1\ N2]$, where $\mathrm{Tr}[\cdot]$ is the trace operation. The dual cone of S is denoted by $S^+ := \{\theta \in S_m | (\theta, Z) \geq 0, \forall Z \in S\}$, then $S^+ = S$. Let $\hat{F}(x) = \sum_{k=1}^{n} x_k F_k$, $x = (x_1,\ldots,x_n) \in R^n$, then $\hat{F}$ is a linear operator from R^n to S_m and its dual is defined by $\hat{F}^*(Z) = \left(\mathrm{Tr}[F_1 Z], \ldots, \mathrm{Tr}[F_n Z]^T\right)$ for any $Z \in S$. The Lagrangian function of (SDP_f) is defined as

$$H_Z(x) := f(x) - \hat{F} * (Z)^T x - \mathrm{Tr}[ZF_0],$$

where $Z \in S_m$. For $\bar{x} \in \Omega$ and any $i \in M$, $j \in N$, we let

$$\tilde{\bar{x}}_i := \begin{cases} -1, & \text{if } \bar{x}_i = u_i \\ 1, & \text{if } \bar{x}_i = v_i \\ \mathrm{sign}(\nabla f(\bar{x}) - \hat{F} * (Z))_i, & \text{if } u_i < \bar{x}_i < v_i \end{cases},$$

$$\tilde{\bar{x}}_j := \begin{cases} -1, & \text{if } \bar{x}_j = p_j \\ 1, & \text{if } \bar{x}_j = q_j \\ \mathrm{sign}(\nabla f(\bar{x}) - \hat{F} * (Z))_j, & \text{if } p_j < \bar{x}_j < q_j \end{cases},$$

$$\tilde{\bar{X}} = \mathrm{diag}\left(\tilde{\bar{x}}_1, \ldots, \tilde{\bar{x}}_n\right),$$

$$b_{\bar{x}_i} := \frac{\tilde{\bar{x}}_i\left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_i}{v_i - u_i}$$

$$b_{\bar{x}_j} := \max\left\{\frac{\tilde{\bar{x}}_j\left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_j}{1}, \frac{\tilde{\bar{x}}_j\left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_j}{q_j - p_j}\right\},$$

$$b_{\bar{x}} := \left(b_{\bar{x}_1}, \ldots, b_{\bar{x}_n}\right)^T,$$

where $\mathrm{sign}\left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_k = \begin{cases} -1, & \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_k < 0 \\ 0, & \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_k = 0 \\ 1, & \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_k > 0, \end{cases}$, $k = 1, 2, \ldots, n$. Let G = diag(α_1, α_2,...,α_n)bea diagonal matrix in S_n. Let $\tilde{G} = \mathrm{diag}\,(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$, where $\tilde{\alpha}_i = \min\{0, \alpha_i\}$ for $i \in M$; $\tilde{\alpha}_j = \alpha_j$ for $j \in N$ and let

$$\bar{U} = \left\{x \in R^n | x_i \in [u_i, v_i], i \in M; x_j \in [p_j, q_j], j \in N\right\}.$$

3 Sufficient global optimality conditions for (SDP_f)

In this section, we will derive the sufficient global optimality conditions for problem (SDP*f*).

Theorem 3.1 (Sufficient global optimality conditions for (SDP_f)) *For the problem* (SDP_f), *let* $\bar{x} \in \Omega$. *If there exist* $Z \succcurlyeq O$ *such that* $\mathrm{Tr}\left[ZF(\bar{x})\right] = 0$ *and a diagonal matrix* $G = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) \in S_n$ *such that* $\nabla^2 f(x) - G \succcurlyeq O$ *for each* $x \in \bar{U}$ *and condition* $\mathrm{diag}\,(b_{\bar{x}}) \preccurlyeq \frac{1}{2}\tilde{G}$ *hold, then* $\bar{x}$ *is a global minimizer of* (SDP_f).

Proof. Let $l(x) = \frac{1}{2}x^T G x + \left(\nabla f(\bar{x}) - \hat{F} * (Z) - G\bar{x}\right)^T x$, $x \in R^n$, and $\varphi(x) = H_Z(x) - l(x)$, $x \in \bar{U}$. Then we have that $\nabla^2\varphi(x) = \nabla^2 f(x) - \nabla^2 l(x) = \nabla^2 f(x) - G \succcurlyeq O$, $\forall x \in \bar{U}$. Thus $\varphi(x)$ is convex on $\bar{U}$, and $\nabla\phi(\bar{x}) = \nabla H_Z(\bar{x}) - \nabla l(\bar{x}) = 0$. So we get that $\phi(x) \geq \phi(\bar{x}), \forall x \in \bar{U}$ and $H_Z(x) - H_Z(\bar{x}) \geq l(x) - l(\bar{x})$ holds. As $\mathrm{Tr}[ZF(x)] \geq 0, \forall x \in F^{-1}(S)$, $\mathrm{Tr}\left[ZF(\bar{x})\right] = 0$, we have

$$\begin{aligned} f(x) - f(\bar{x}) &\geq f(x) - \mathrm{Tr}\left[ZF(x)\right] - \left(f(\bar{x}) + \mathrm{Tr}\left[ZF(\bar{x})\right]\right) \\ &= H_Z(x) - H_Z(\bar{x}), \\ &\geq l(x) - l(\bar{x}), \quad \forall x \in F^{-1}(S). \end{aligned}$$

So we have

$$f(x) - f(\bar{x}) \geq l(x) - l(\bar{x}), \quad \forall x \in \Omega,$$

where

$$l(x) - l(\bar{x}) = \sum_{k=1}^{n} \left[\frac{1}{2}\alpha_k (x_k - \bar{x}_k)^2 + \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_k (x_k - \bar{x}_k) \right]. \quad (1)$$

If $l(x) - l(\bar{x}) \geq 0$, $\forall x \in \Omega$, then $\bar{x}$ is a global minimizer of (SDP$_f$). In the following, we prove if condition $\operatorname{diag}(b_{\bar{x}}) \preccurlyeq \frac{1}{2}\tilde{G}$ hold, then for any $k = 1,\ldots, n$,

$$\frac{1}{2}\alpha_k (x_k - \bar{x}_k)^2 + \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_k (x_k - \bar{x}_k) \geq 0, \quad \text{for any } x \in \Omega. \quad (2)$$

hold, thus we have

$$\sum_{k=1}^{n} \left[\frac{1}{2}\alpha_k (x_k - \bar{x}_k)^2 + \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_k (x_k - \bar{x}_k) \right] \geq 0, \quad \text{for any } x \in \Omega. \quad (3)$$

i.e. $\bar{x}$ is a global minimizer of (SDP$_f$). We consider the following cases:
1°. If $\bar{x}_i = u_i$, then (2) is equivalent to

$$\begin{aligned} &\frac{1}{2}\alpha_i (x_i - \bar{x}_i) + \left(\nabla f(\bar{x}) - \hat{F}^* (Z)\right)_i \geq 0, \quad \text{for any } x_i \in (u_i, v_i] \\ \Leftrightarrow &\begin{cases} \left(\nabla f(\bar{x}) - \hat{F}^*(Z)\right)_i \geq 0, & \text{if } \alpha_i \geq 0 \\ \left(\nabla f(\bar{x}) - \hat{F}^*(Z)\right)_i \geq -\dfrac{(v_i - u_i)\alpha_i}{2}, & \text{if } \alpha_i < 0 \end{cases} \\ \Leftrightarrow &\tilde{\tilde{x}}_i \left(\nabla f(\bar{x}) - \hat{F}^* (Z)\right)_i \leq \min\left\{0, \frac{(v_i - u_i)\alpha_i}{2}\right\}. \end{aligned}$$

2°. If $\bar{x}_i = v_i$, then (2) is equivalent to

$$\begin{aligned} &\frac{1}{2}\alpha_i (x_i - \bar{x}_i) + \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_i \leq 0, \quad \text{for any } x_i \in [u_i, v_i) \\ \Leftrightarrow &\begin{cases} \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_i \leq 0, & \text{if } \alpha_i \geq 0 \\ \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_i \leq -\dfrac{(v_i - u_i)\alpha_i}{2}, & \text{if } \alpha_i < 0 \end{cases} \\ \Leftrightarrow &\tilde{\tilde{x}}_i \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_i \leq \min\left\{0, \frac{(v_i - u_i)\alpha_i}{2}\right\}. \end{aligned}$$

3°. If $u_i < \bar{x}_i < v_i$, then (2) is equivalent to

$$\begin{aligned} &\begin{cases} \frac{1}{2}\alpha_i (x_i - \bar{x}_i) + \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_i \geq 0, \text{ for any } x_i \in (\bar{x}_i, v_i] \\ \frac{1}{2}\alpha_i (x_i - \bar{x}_i) + \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_i \leq 0, \text{ for any } x_i \in [u_i, \bar{x}_i) \end{cases} \\ \Leftrightarrow &\left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_i = 0, \ \alpha_i \geq 0 \\ \Leftrightarrow &\tilde{\tilde{x}}_i \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_i \leq \min\left\{0, \frac{(v_i - u_i)\,\alpha_i}{2}\right\}. \end{aligned}$$

4°. If $\bar{x}_j = p_j$, then (2) is equivalent to

$$\frac{1}{2}\alpha_j\left(x_j-\bar{x}_j\right)+\left(\nabla f(\bar{x})-\hat{F}*(Z)\right)_j \geq 0, \quad \text{for any } x_j \in \left\{p_j+1, p_j+2, \ldots, q_j\right\}$$

$$\Leftrightarrow \begin{cases} \left(\nabla f(\bar{x})-\hat{F}*(Z)\right)_j \geq -\frac{\alpha_j}{2}, & \text{if } \alpha_j \geq 0 \\ \left(\nabla f(\bar{x})-\hat{F}*(Z)\right)_j \geq -\dfrac{(q_j-p_j)\alpha_j}{2}, & \text{if } \alpha_j < 0 \end{cases}$$

$$\Leftrightarrow \tilde{\bar{x}}_j\left(\nabla f(\bar{x})-\hat{F}*(Z)\right)_j \leq \min\left\{\frac{\alpha_j}{2}, \frac{(q_j-p_j)\alpha_j}{2}\right\}.$$

5°. If $\bar{x}_j = q_j$, then (2) is equivalent to

$$\frac{1}{2}\alpha_j\left(x_j-\bar{x}_j\right)+\left(\nabla f(\bar{x})-\hat{F}*(Z)\right)_j \leq 0, \quad \text{for any } x_j \in \left\{p_j p_j+1, \ldots, q_j-1\right\}$$

$$\Leftrightarrow \begin{cases} \left(\nabla f(\bar{x})-\hat{F}*(Z)\right)_j \leq \frac{\alpha_j}{2}, & \text{if } \alpha_j \geq 0 \\ \left(\nabla f(\bar{x})-\hat{F}*(Z)\right)_j \leq -\dfrac{(q_j-p_j)\alpha_j}{2}, & \text{if } \alpha_j < 0 \end{cases}$$

$$\Leftrightarrow \tilde{\bar{x}}_j\left(\nabla f(\bar{x})-\hat{F}*(Z)\right)_j \leq \min\left\{\frac{\alpha_j}{2}, \frac{(q_j-p_j)\alpha_j}{2}\right\}.$$

6°. If $\bar{x}_j \in \left\{p_j+1, \ldots, q_j-1\right\}$, then (2) is equivalent to

$$\begin{cases} \frac{1}{2}\alpha_j(x_j-\bar{x}_j)+\left(\nabla f(\bar{x})-\hat{F}*(Z)\right)_j \leq 0, \text{ for any } x_j \in \left\{p_j, \ldots, \bar{x}_j-1\right\} \\ \frac{1}{2}\alpha_j(x_j-\bar{x}_j)+\left(\nabla f(\bar{x})-\hat{F}*(Z)\right)_j \geq 0, \text{ for any } x_j \in \left\{\bar{x}_j+1, \ldots, q_j\right\} \end{cases}$$

$$\Leftrightarrow -\frac{\alpha_j}{2} \leq \left(\nabla f(\bar{x})-\hat{F}*(Z)\right)_j \leq \frac{\alpha_j}{2}, \alpha_j \geq 0$$

$$\Leftrightarrow \tilde{\bar{x}}_j\left(\nabla f(\bar{x})-\hat{F}*(Z)\right)_j \leq \min\left\{\frac{\alpha_j}{2}, \frac{(q_j-p_j)\,\alpha_j}{2}\right\}.$$

By the above discussion, we know that if condition $\operatorname{diag}(b_{\bar{x}}) \preccurlyeq \frac{1}{2}\tilde{G}$ hold, then for any $k = 1,\ldots,n$, $\frac{1}{2}\alpha_k(x_k-\bar{x}_k)^2+\left(\nabla f(\bar{x})-\hat{F}*(Z)\right)_k(x_k-\bar{x}_k) \geq 0$, for any $x \in \Omega$., i.e. $\bar{x}$ is a global minimizer of (SDP_f).

If $f(x)$ is convex, then we have the following results.

Corollary 3.1 *For the problem* (SDP_f), *let* $\bar{x} \in \Omega$. *If f is convex on* $\bar{U}$ *and there exist Z* $\succcurlyeq O$ *such that* $\mathrm{Tr}\left[ZF(\bar{x})\right] = 0$ *and condition* $\operatorname{diag}(b_{\bar{x}}) \preccurlyeq O$ *hold, then* $\bar{x}$ *is a global minimizer of* (SDP_f).

Proof. We can get the results from the proof of Theorem 3.1 by taken $G = O$.

Then we consider the following special cases. At first consider the following minimization problem with LMI and box constraints:

$$(\mathrm{SDP}'_f) \qquad \min_{x\in R^n} f(x)$$
$$s.t.\ F_0+\sum_{k=1}^{n} F_k x_k \succcurlyeq O,$$
$$x \in \prod_{k=1}^{n}[u_k, v_k].$$

Theorem 3.2 *For the problem* (SDP'_f), *let* $\bar{x} \in F^{-1}(S) \bigcap \prod_{k=1}^{n} [u_k, v_k]$. *If there exist* $Z \succcurlyeq O$ *such that* $\mathrm{Tr}\left[ZF(\bar{x})\right] = 0$ *and a diagonal matrix* $G = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) \in S_n$ *such that* $\nabla^2 f(x) - G \succcurlyeq O$ *for each* $x \in \prod_{k=1}^{n} [u_k, v_k]$ *and condition* $\frac{1}{2}\tilde{\alpha}_k (v_k - u_k) - \tilde{\tilde{x}}_k \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_k \geq 0$ *hold, then* $\bar{x}$ *is a global minimizer of* (SDP'_f).

Remark 3.1 *This is just the result of* [13, Theorem 3.1] *when* $f(x) = \frac{1}{2}x^T A x + a^T x$.

In the second we consider the following minimization problem with LMI and bivalent constraints:

$$
\begin{aligned}
(\mathrm{SDP}''_f) \qquad & \min_{x \in R^n} f(x) \\
& s.t.\ F_0 + \sum_{k=1}^{n} F_k x_k \succcurlyeq O, \\
& x \in \prod_{k=1}^{n} \{-1, 1\}.
\end{aligned}
$$

Theorem 3.3 *For the problem* (SDP''_f), *let* $\bar{x} \in F^{-1}(S) \bigcap \prod_{i=1}^{n} \{-1, 1\}$. *If there exist* $Z \succcurlyeq O$ *such that* $\mathrm{Tr}\left[ZF(\bar{x})\right] = 0$ *and* $\nabla^2 f(x) - \mathrm{diag}\left(\tilde{\tilde{X}}\left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)\right) \succcurlyeq O$ *for each* $x \in \prod_{k=1}^{n} [-1, 1]$ *hold, then* $\bar{x}$ *is a global minimizer of* (SDP''_f).

Proof. From the proof of Theorem3.1, we know if there exists a diagonal matrix $G = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) \in S_n$ such that $\nabla^2 f(x) - G \succcurlyeq O$ on $\prod_{k=1}^{n} [-1, 1]$ and $l(x) - l(\bar{x}) \geq 0$ hold for each $x \in \prod_{k=1}^{n} \{-1, 1\}$, where $l(x) = \frac{1}{2}x^T G x + \left(\nabla f(\bar{x}) - \hat{F} * (Z) - G\bar{x}\right)^T x$, then $\bar{x}$ is a global minimizer of (SDP''_f).

Suppose condition $\nabla^2 f(x) - \mathrm{diag}\left(\tilde{\tilde{X}}\left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)\right) \succcurlyeq O$ hold on $\prod_{k=1}^{n} [-1, 1]$, we let $\alpha_k = \tilde{\tilde{x}}_k \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_k$, then we have $\nabla^2 f(x) - G \succcurlyeq O$. For each $k = 1, 2, \ldots, n$ and each $x \in \prod_{k=1}^{n} \{-1, 1\}$, we only have $x_k = \bar{x}_k$ or $x_k = -\bar{x}_k$. Obviously If $x_k = \bar{x}_k$, then we have

$$
l(x) - l(\bar{x}) = \sum_{k=1}^{n} \left[\frac{1}{2}\alpha_k (x_k - \bar{x}_k)^2 + \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_k (x_k - \bar{x}_k)\right] = 0.
$$

if $x_k = -\bar{x}_k$, then we have

$$
\begin{aligned}
& \frac{1}{2}\alpha_k (x_k - \bar{x}_k)^2 + \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_k (x_k - \bar{x}_k) \\
&= 2\alpha_k (\bar{x}_k)^2 - 2\left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_k \bar{x}_k \\
&= 2\left(\alpha_k - \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_k \tilde{\tilde{x}}_k\right) \\
&= 0,
\end{aligned}
$$

so we get

$$
l(x) - l(\bar{x}) = \sum_{k=1}^{n} \left[\frac{1}{2}\alpha_k (x_k - \bar{x}_k)^2 + \left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)_k (x_k - \bar{x}_k)\right] = 0.
$$

That is for each $x \in \prod_{k=1}^{n} \{-1, 1\}, l(x) - l(\bar{x}) = 0$.

Remark 3.2 *This is just the result of* [13, Theorem 4.1] *when* $f(x) = \frac{1}{2}x^T Ax + a^T x$.

Example 3.1 *Consider the following programming problem with LMI and bivalent constraints:*

$$(EXP1) \quad \min f(x) := \frac{2}{3}x_1^3 - x_1^2 + 2x_2^2 + x_1x_2 - x_2$$
$$s.t. \ F_0 + \sum_{k=1}^{2} x_k F_k \succcurlyeq O,$$
$$x \in \prod_{k=1}^{2} \{-1, 1\},$$

Where $F_0 = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$..

We can check the point $\bar{x} = (-1, 1)^T$ *satisfies the sufficient global optimization conditions of* (SDP''_f). *Since* $F(x) = \begin{pmatrix} 3 + x_2 & 2 + x_1 & 0 \\ 2 + x_1 & 1 + x_2 & 0 \\ 0 & 0 & 1 + x_1 + x_2 \end{pmatrix}, F(\bar{x}) = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *Let* $Z = \begin{pmatrix} 1 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$, *and* $\text{Tr}\left[ZF(\bar{x})\right] = 0$. *We can get*

$\nabla^2 f(x) - \text{diag}\left(\tilde{\tilde{X}}\left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)\right) = \begin{pmatrix} 4x_1 - 2 + 13 & 1 \\ 1 & 4 - 1 \end{pmatrix} \succcurlyeq O$ *so*

$\nabla^2 f(x) - \text{diag}\left(\tilde{\tilde{X}}\left(\nabla f(\bar{x}) - \hat{F} * (Z)\right)\right) = \begin{pmatrix} 4x_1 - 2 + 13 & 1 \\ 1 & 4 - 1 \end{pmatrix} \succcurlyeq O$ *for any* $x_1 \in$ [-1,1]. *So* $(-1,1)^T$ *is the global minimizer of* $(EXP1)$.

In fact from $\Omega = \left\{x \in \prod_{k=1}^{2} \{-1, 1\} | F_0 + \sum_{k=1}^{2} x_k F_k \succcurlyeq O\right\} = \left\{(-1, -1), (1, 1), (-1, 1)\right\}$, *we can easily check that* $(-1,1)^T$ *is the global minimizer of* $(EXP1)$.

4 Sufficient global optimality conditions for (SDP)

Consider the following standard SDP problem, where the objective function is linear, the variables satisfy linear inequalities constraints and box constraints:

$$(\text{SDP}) \qquad \min_{x \in R^n} f(x) = \sum_{k=1}^{n} c_k x_k$$
$$s.t. \ F_0 + \sum_{i=1}^{n} F_k x_k \succcurlyeq O,$$
$$x \in \prod_{k=1}^{n} [u_k, v_k].$$

Theorem 4.1 *For the problem* (SDP), *let* $\bar{x} \in F^{-1}(S) \bigcap \prod_{k=1}^{n} [u_k, v_k]$. *If there exist* $Z \succcurlyeq O$ *such that* $\text{Tr}\left[ZF(\bar{x})\right] = 0$ *and a diagonal matrix* $G = \text{diag}(\alpha_1, \alpha_2, ..., \alpha_n) \in S_n$ *such that*

-G $\succcurlyeq$ *O for each* $x \in \prod_{k=1}^{n}[u_k, v_k]$*and condition* $\frac{1}{2}\tilde{\alpha}_k(v_k - u_k) - \tilde{\bar{x}}_k\left[c_k - \left(\hat{F}^*(Z)\right)_k\right] \geq 0$, $k = 1, 2, \ldots, n$ *hold, then* $\bar{x}$ *is a global minimizer of* (SDP).

Proof. Obviously we have $\nabla^2 f(x) = O$ for all $x \in \prod_{k=1}^{n}[u_k, v_k]$ by the reason of $f(x) = \sum_{k=1}^{n} c_k x_k$, so we can get this result from Theorem 3.2.

Theorem 4.2 *For the problem* (SDP), *let* $\bar{x} \in F^{-1}(S) \bigcap \prod_{k=1}^{n}[u_k, v_k]$. *If there exist* $Z \succcurlyeq O$ *such that* $\mathrm{Tr}\left[ZF(\bar{x})\right] = 0$*and condition* $-\tilde{\bar{x}}_k\left[c_k - \left(\hat{F} * (Z)\right)_k\right] \geq 0$, $k = 1,2,\ldots,n$ *hold, then* $\bar{x}$ *is a global minimizer of* (SDP).

Proof. Let $G = O$ for all $x \in \prod_{k=1}^{n}[u_k, v_k]$, then the condition $-G \succcurlyeq O$ for each $x \in \prod_{k=1}^{n}[u_k, v_k]$ in Theorem 4.1 is met, so we can get this result from Theorem 4.1.

Example 4.1 *Consider the following programming problem with LMI and box constraints:*

$$(EXP2) \quad \min f(x) := 3x_1 - 2x_2$$
$$s.t. \ \ F_0 + \sum_{k=1}^{2} x_k F_k \succcurlyeq O,$$
$$x \in \prod_{k=1}^{2}[-1, 1].$$

Where $F_0 = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, c = (3, -2)^T$..

We can check the point $\bar{x} = (-1, 1)^T$ *satisfies the sufficient global optimization conditions of* (SDP). *Since* $F(x) = \begin{pmatrix} 3 + x_2 & 2 + x_1 & 0 \\ 2 + x_1 & 1 + x_2 & 0 \\ 0 & 0 & 1 + x_1 + x_2 \end{pmatrix}$, $F(\bar{x}) = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. *Let* $Z = \begin{pmatrix} 1 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$, *and* $\mathrm{Tr}\left[ZF(\bar{x})\right] = 0$ $\mathrm{Tr}[ZF_1] = -8$, $\mathrm{Tr}[ZF_2] = 1$. *We can get* $\tilde{\bar{x}}_1(c - \hat{F} * (Z))_1 = (-1) \times (3-(-4)) = -7 < 0; \tilde{\bar{x}}_2(c - \hat{F} * (Z))_2 = (1) \times ((-2)-5) = -7 < 0$. *So* (-1,, 1)^T *is the global minimizer of* (*EXP*2).

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Authors' contributions
All authors participated in this article's design and coordination, they also read and approved the final manuscript.

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