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On a more accurate half-discrete Hilbert's inequality

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Abstract

By using the way of weight coefficients and the idea of introducing parameters and by means of Hadamard's inequality, we give a more accurate half-discrete Hilbert's inequality with a best constant factor. We also consider its best extension with parameters, the equivalent forms, the operator expressions as well as some reverses.
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1 Introduction

If $a_n, b_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then we have the following well-known Hilbert's inequality (cf. [1]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{m=1}^{\infty}a_m^2\sum_{n=1}^{\infty}b_n^2\right)^{1/2}, \tag{1}$$

where the constant factor π is the best possible. The integral analogue of inequality (1) is given as follows (cf. [2]): If $0 < \int_0^{\infty} f^2(x)dx < \infty$ and $0 < \int_0^{\infty} g^2(x)dx < \infty$, then

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \pi\left(\int_0^{\infty}f^2(x)dx\int_0^{\infty}g^2(x)dx\right)^{1/2}, \tag{2}$$

where the constant factor π is the best possible. We named inequality (2) as Hilbert's integral inequality. Hardy et al. [3] proved the following more accurate Hilbert's inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n-1} < \pi\left(\sum_{m=1}^{\infty}a_m^2\sum_{n=1}^{\infty}b_n^2\right)^{1/2}, \tag{3}$$

where the constant factor π is still the best possible. Inequalities (1)-(3) are important in analysis and its applications [4]. There are lots of improvements, generalizations, and applications of inequalities (1-3), for more details, refer to literatures [5-18].

We find a few results on the half-discrete Hilbert-type inequalities with the non-homogeneous kernel, which were published early ([[3], Th. 351], [19]). Recently, Yang [20-22] gave some half-discrete Hilbert-type inequalities. A half-discrete Hilbert's inequality with the homogeneous kernel was derived as follows [20]: If $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \sum_{n=1}^\infty a_n^2 < \infty$, then

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} \frac{f(x)}{x+n} dx < \pi \left(\sum_{n=1}^{\infty} a_n^2 \int_0^{\infty} f^2(x) dx \right)^{1/2}, \tag{4}$$

where the constant factor π is the best possible.

In this article, by using the way of weight coefficients and the idea of introducing parameters and by means of Hadamard's inequality, we give a more accurate inequality of (4) with a best constant factor as follows:

$$\sum_{n=1}^{\infty} a_n \int_{-\frac{1}{2}}^{\infty} \frac{f(x)}{x+n} dx < \pi \left(\sum_{n=1}^{\infty} a_n^2 \int_{-\frac{1}{2}}^{\infty} f^2(x) dx \right)^{1/2}. \tag{5}$$

We also consider its best extension with parameters, the equivalent forms, the operator expressions as well as some reverses.

2 Some lemmas

Lemma 1 *Suppose* $0 < \alpha \leq 1$, $0 \leq \beta \leq \frac{1}{2}$, $\gamma \in (-\infty, \infty)$, $\lambda_1 > 0$, $0 < \lambda_2 \alpha \leq 1$, $\lambda = \lambda_1 + \lambda_2$. *Define the beta function (cf.* [18]*) and the weight coefficients as follows:*

$$B(u, v) := \int_0^{\infty} \frac{t^{u-1} dt}{(1+t)^{u+v}} = \int_0^1 \frac{t^{u-1} + t^{v-1}}{(1+t)^{u+v}} dt \ (u, v > 0), \tag{6}$$

$$\omega(n) := (n-\beta)^{\lambda_2 \alpha} \int_{\gamma}^{\infty} \frac{(x-\gamma)^{\lambda_1 \alpha - 1}}{[(x-\gamma)^{\alpha} + (n-\beta)^{\alpha}]^{\lambda}} dx \, (n \in \mathrm{N}), \tag{7}$$

$$\varpi(x) := (x-\gamma)^{\lambda_1 \alpha} \sum_{n=1}^{\infty} \frac{(n-\beta)^{\lambda_2 \alpha - 1}}{[(x-\gamma)^{\alpha} + (n-\beta)^{\alpha}]^{\lambda}} \ (x \in (\gamma, \infty)). \tag{8}$$

Setting $k_{\lambda_1}(\alpha) := \frac{1}{\alpha} B(\lambda_1, \lambda_2)$, *we have the following inequalities:*

$$0 < k_{\lambda_1}(\alpha)(1 - \theta_{\lambda}(x)) < \varpi(x) < \omega(n) = k_{\lambda_1}(\alpha), \tag{9}$$

where, $\theta_{\lambda}(x) := \frac{1}{B(\lambda_1, \lambda_2)} \int_0^{\left(\frac{1-\beta}{x-\gamma}\right)^{\alpha}} \frac{u^{\lambda_2 - 1}}{(1+u)^{\lambda}} du > 0$ *and* $\theta_{\lambda}(x) = O\left(\frac{1}{(x-\gamma)^{\lambda_2 \alpha}}\right) \ (x \in (\gamma, \infty))$.

Proof. *Putting* $u = \left(\frac{x-\gamma}{n-\beta}\right)^{\alpha}$ *in (7), we have*

$$\omega(n) = \frac{1}{\alpha}\int_0^{\infty} \frac{u^{\lambda_1 - 1}}{(1+u)^{\lambda}} du = \frac{1}{\alpha} B(\lambda_1, \lambda_2) = k_{\lambda_1}(\alpha). \tag{10}$$

For fixed $x \in (\gamma, \infty)$, *setting*

$$f(t) := \frac{(x-\gamma)^{\lambda_1 \alpha}(t-\beta)^{\lambda_2 \alpha - 1}}{[(x-\gamma)^{\alpha} + (t-\beta)^{\alpha}]^{\lambda}} (t \in (\beta, \infty)), \tag{11}$$

in view of the conditions, we find $f'(t) < 0$ *and* $f''(t) > 0$. *By the following Hadamard's inequality (cf.* [18]*):*

$$f(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(t)dt \, (n \in \mathrm{N}), \tag{12}$$

and putting $u = \left(\frac{t-\beta}{x-\gamma}\right)^{\alpha}$, *it follows*

$$\begin{aligned} \varpi(x) &= \sum_{n=1}^{\infty} f(n) < \sum_{n=1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(t)dt = \int_{\frac{1}{2}}^{\infty} f(t)dt \\ &< \int_{\beta}^{\infty} f(t)dt = \frac{1}{\alpha}\int_0^{\infty} \frac{u^{\lambda_2 - 1}}{(1+u)^{\lambda}} du = \frac{1}{\alpha} B(\lambda_2, \lambda_1) = k_{\lambda_1}(\alpha), \\ \varpi(x) &= \sum_{n=1}^{\infty} f(n) > \int_1^{\infty} f(t)dt = \int_{\beta}^{\infty} f(t)dt - \int_{\beta}^{1} f(t)dt \\ &= k_{\lambda_1}(\alpha) - \frac{1}{\alpha}\int_0^{\left(\frac{1-\beta}{x-\gamma}\right)^{\alpha}} \frac{u^{\lambda_2 - 1}}{(1+u)^{\lambda}} du = k_{\lambda_1}(\alpha)(1 - \theta_{\lambda}(x)) > 0, \end{aligned}$$

where

$$\begin{aligned} 0 < \theta_{\lambda}(x) &= \frac{1}{B(\lambda_1, \lambda_2)} \int_0^{\left(\frac{1-\beta}{x-\gamma}\right)^{\alpha}} \frac{u^{\lambda_2 - 1}}{(1+u)^{\lambda}} du \\ &< \frac{1}{B(\lambda_1, \lambda_2)} \int_0^{\left(\frac{1-\beta}{x-\gamma}\right)^{\alpha}} u^{\lambda_2 - 1} du = \frac{(1-\beta)^{\lambda_2 \alpha}}{\lambda_2 B(\lambda_1, \lambda_2)} \frac{1}{(x-\gamma)^{\lambda_2 \alpha}}. \end{aligned}$$

Hence, we prove that (9) is valid.

Lemma 2 *Suppose that* $\frac{1}{p} + \frac{1}{q} = 1 (p \neq 0, 1)$, $0 < \alpha \leq 1$, $0 \leq \beta \leq \frac{1}{2}$, $\gamma \in (-\infty + \infty)$, $\lambda_1 > 0$, $0 < \lambda_2 \alpha \leq 1$, $\lambda = \lambda_1 + \lambda_2$, $\alpha_n \geq 0$, $f(x) \geq 0$ *is a real measurable function in* (γ, ∞), *then (i) for* $p > 1$, *we have the following*

$$J := \left\{ \sum_{n=1}^{\infty} (n-\beta)^{p\lambda_2\alpha-1} \left[\int_{\gamma}^{\infty} \frac{f(x)}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}} dx \right]^p \right\}^{1/p} \le (k_{\lambda_1}(\alpha))^{1/q} \left\{ \int_{\gamma}^{\infty} \varpi(x)(x-\gamma)^{p(1-\lambda_1\alpha)-1} f^p(x) dx \right\}^{1/p}, \tag{13}$$

$$L_1 := \left\{ \int_{\gamma}^{\infty} \frac{(x-\gamma)^{q\lambda_1\alpha-1}}{\varpi^{q-1}(x)} \left[\sum_{n=1}^{\infty} \frac{a_n}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}} \right]^q dx \right\}^{1/q} < \left\{ k_{\lambda_1}(\alpha) \sum_{n=1}^{\infty} (n-\beta)^{q(1-\lambda_2\alpha)-1} a_n^q \right\}^{1/q}, \tag{14}$$

where $\varpi(x)$ and $\omega(n)$ are indicated by (7) and (8).

(ii) for $p < 1 (p \neq 0)$, we have the reverses of (13) and (14).

Proof. *(i) By (7)-(9) and Hölder's inequality (cf.* [18]*), we find*

$$\begin{aligned} \left[\int_{\gamma}^{\infty} \frac{f(x)}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}} dx \right]^p &= \left\{ \int_{\gamma}^{\infty} \frac{1}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}} \right. \\ &\quad \times \left. \left[\frac{(x-\gamma)^{(1-\lambda_1\alpha)/q}}{(n-\beta)^{(1-\lambda_2\alpha)/q}} f(x) \right] \left[\frac{(n-\beta)^{(1-\lambda_2\alpha)/q}}{(x-\gamma)^{(1-\lambda_1\alpha)/q}} dx \right] \right\}^p \\ &\le \int_{\gamma}^{\infty} \frac{1}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}} \frac{(x-\gamma)^{(1-\lambda_1\alpha)(p-1)}}{(n-\beta)^{1-\lambda_2\alpha}} f^p(x) dx \\ &\quad \times \left[\int_{\gamma}^{\infty} \frac{1}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}} \frac{(n-\beta)^{(1-\lambda_2\alpha)(q-1)}}{(x-\gamma)^{1-\lambda_1\alpha}} dx \right]^{p-1} \\ &= \int_{\gamma}^{\infty} \frac{f^p(x)(x-\gamma)^{(1-\lambda_1\alpha)(p-1)}}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}} \frac{dx}{(n-\beta)^{1-\lambda_2\alpha}} \left[(n-\beta)^{q(1-\lambda_2\alpha)-1} \omega(n) \right]^{p-1} \\ &= (n-\beta)^{1-p\lambda_2\alpha} k_{\lambda_1}^{p-1}(\alpha) \int_{\gamma}^{\infty} \frac{f^p(x)(x-\gamma)^{(1-\lambda_1\alpha)(p-1)}}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}} \frac{1}{(n-\beta)^{1-\lambda_2\alpha}} dx, \end{aligned} \tag{15}$$

$$\begin{aligned} J^p &\le k_{\lambda_1}^{p-1}(\alpha) \sum_{n=1}^{\infty} \int_{\gamma}^{\infty} \frac{f^p(x)}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}} \frac{(x-\gamma)^{(1-\lambda_1\alpha)(p-1)}}{(n-\beta)^{1-\lambda_2\alpha}} dx \\ &= k_{\lambda_1}^{p-1}(\alpha) \int_{\gamma}^{\infty} \sum_{n=1}^{\infty} \frac{(n-\beta)^{\lambda_2\alpha-1}}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}} (x-\gamma)^{\lambda_1\alpha+p(1-\lambda_1\alpha)-1} f^p(x) dx \\ &= k_{\lambda_1}^{p-1}(\alpha) \int_{\gamma}^{\infty} \varpi(x)(x-\gamma)^{p(1-\lambda_1\alpha)-1} f^p(x) dx. \end{aligned} \tag{16}$$

Hence (13) is valid. Using Hölder's inequality again, we have

$$\begin{aligned}
\left[\sum_{n=1}^{\infty}\frac{a_n}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}}\right]^{q} &= \left\{\sum_{n=1}^{\infty}\frac{1}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}}\right.\\
&\times\left.\left[\frac{(x-\gamma)^{(1-\lambda_1\alpha)/q}}{(n-\beta)^{(1-\lambda_2\alpha)/q}}\right]\left[\frac{(n-\beta)^{(1-\lambda_2\alpha)/q}}{(x-\gamma)^{(1-\lambda_1\alpha)/q}}a_n\right]\right\}^{q}\\
&\leq\left[\varpi(x)(x-\gamma)^{p(1-\lambda_1\alpha)-1}\right]^{q-1}\sum_{n=1}^{\infty}\frac{a_n^q}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}}\frac{(n-\beta)^{(1-\lambda_2\alpha)q/p}}{(x-\gamma)^{1-\lambda_1\alpha}}\\
&=\varpi^{q-1}(x)(x-\gamma)^{1-q\lambda_1\alpha}\sum_{n=1}^{\infty}\frac{(n-\beta)^{(1-\lambda_2\alpha)(q-1)|}}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}}\frac{1}{(x-\gamma)^{1-\lambda_1\alpha}}a_n^q,
\end{aligned}\tag{17}$$

$$\begin{aligned}
L_1^q &\leq \int_{\gamma}^{\infty}\sum_{n=1}^{\infty}\frac{1}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}}\frac{(n-\beta)^{(1-\lambda_2\alpha)(q-1)}}{(x-\gamma)^{1-\lambda_1\alpha}}a_n^q dx\\
&=\sum_{n=1}^{\infty}[(n-\beta)^{\lambda_2\alpha}\int_{\gamma}^{\infty}\frac{(x-\gamma)^{\lambda_1\alpha-1}}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}}dx](n-\beta)^{q(1-\lambda_2\alpha)-1}a_n^q\\
&=\sum_{n=1}^{\infty}\omega(n)(n-\beta)^{q(1-\lambda_2\alpha)-1}a_n^q = k_{\lambda_1}(\alpha)\sum_{n=1}^{\infty}(n-\beta)^{q(1-\lambda 2\alpha)-1}a_n^q.
\end{aligned}\tag{18}$$

Hence (14) is valid.

(ii) For $0 < p < 1 (q < 0)$ *or* $p < 0 (0 < q < 1)$, *using the reverse Hölder's inequality and in the same way, we have the reverses of (13) and (14).*

Lemma 3 *As the assumptions of Lemmas 1 and 2, we set* $\phi(x) := (x-\gamma)^{p(1-\lambda_1\alpha)-1}$, $\psi(n) := (n-\beta)^{q(1-\lambda_2\alpha)-1}$, $\psi(n) := (n-\beta)^{q(1-\lambda_2\alpha)-1}$,

$$L_{p,\phi}(\gamma,\infty) := \left\{f;\|f\|_{p,\phi} = \left\{\int_{\gamma}^{\infty}\phi(x)|f(x)|^p dx\right\}^{1/p} < \infty\right\},$$

$$l_{q,\psi} := \left\{a = \{a_n\};\|a\|_{,\psi} = \left\{\sum_{n=1}^{\infty}\psi(n)|a_n|^q\right\}^{1/q} < \infty\right\}$$

(Note. if $p > 1$, *then* $L_{p,\varphi}(\gamma,\infty)$ *and* $l_{q,\psi}$ *are normal spaces; if* $0 < p < 1$ *or* $p < 0$, *then both* $L_{p,\varphi}(\gamma,\infty)$ *and* $l_{q,\psi}$ *are not normal spaces, but we still use the formal symbols in the following.) For* $0 < \varepsilon < \min\{1, \lambda_1 p\alpha\}$, *setting* $\tilde{a} = \{\tilde{a}_n\}_{n=1}^{\infty}$ *and* $\tilde{f}(x)$ *as follows*

$$\tilde{a}_n = (n-\beta)^{\lambda_2\alpha-\frac{\varepsilon}{q}-1};\ \tilde{f}(x) = \begin{cases}0, & x\in(\gamma,1+\gamma),\\ (x-\gamma)^{\lambda_1\alpha-\frac{\varepsilon}{p}-1}, & x\in[1+\gamma,\infty),\end{cases}\tag{19}$$

(i) if $p > 1$, *there exists a constant* $k > 0$, *such that*

$$\tilde{I} := \sum_{n=1}^{\infty}\tilde{a}_n\int_{\gamma}^{\infty}\frac{\tilde{f}(x)}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}}dx < k\left\|\tilde{f}\right\|_{p,\phi}\|\tilde{a}\|_{q,\psi},\tag{20}$$

then it follows

$$k\left(\frac{\varepsilon+1-\beta}{(1-\beta)^{\varepsilon+1}}\right)^{1/q} \geq \frac{1}{\alpha}B(\lambda_2+\frac{\varepsilon}{p\alpha},\lambda_1+\frac{\varepsilon}{p\alpha})-\varepsilon O(1); \tag{21}$$

(ii) if $0 < p < 1$, *there exists a constant* $k > 0$, *such that*

$$\tilde{I}-\sum_{n=1}^{\infty}\tilde{a}_n\int_{\gamma}^{\infty}\frac{\tilde{f}(x)}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}}dx > k\left\|\tilde{f}\right\|_{p,\tilde{\phi}}\|\tilde{a}\|_{q,\psi}, \tag{22}$$

then it follows

$$k(1-\varepsilon O(1))^{1/p} < \frac{1}{\alpha}\left(\frac{\varepsilon+1-\beta}{(1-\beta)^{\varepsilon+1}}\right)^{1/p}B\left(\lambda_1-\frac{\varepsilon}{p\alpha},\lambda_2+\frac{\varepsilon}{p\alpha}\right). \tag{23}$$

Proof. *we obtain*

$$\begin{aligned}\left\|\tilde{f}\right\|_{p,\phi} &= \left\{\int_{\gamma}^{\infty}(x-\gamma)^{p(1-\lambda_1\alpha)-1}\tilde{f}^p(x)dx\right\}^{1/p}\\ &= \left\{\int_{1+\gamma}^{\infty}(x-\gamma)^{-1-\varepsilon}dx\right\}^{1/p} = \left(\frac{1}{\varepsilon}\right)^{1/p},\end{aligned} \tag{24}$$

$$\begin{aligned}\|\tilde{a}\|_{q,\psi}^{q} &= \sum_{n=1}^{\infty}(n-\beta)^{q(1-\lambda_2\alpha)-1}\tilde{a}_n^q = \sum_{n=1}^{\infty}(n-\beta)^{-1-\varepsilon}\\ &< (1-\beta)^{-1-\varepsilon}+\int_{1}^{\infty}(x-\beta)^{-1-\varepsilon}dx = \frac{\varepsilon+1-\beta}{\varepsilon(1-\beta)^{\varepsilon+1}}.\end{aligned} \tag{25}$$

(i) For $p > 1$, *then* $q > 1$, $\lambda_2\alpha-\frac{\varepsilon}{q}-1<0$, *by (20), (24), and (25), we find*

$$\begin{aligned}\tilde{I} &< k\left(\frac{1}{\varepsilon}\right)^{1/p}\left[\frac{\varepsilon+1-\beta}{\varepsilon(1-\beta)^{\varepsilon+1}}\right]^{1/q} = \frac{k}{\varepsilon}\left[\frac{\varepsilon+1-\beta}{(1-\beta)^{\varepsilon+1}}\right]^{1/q},\\ \tilde{I} &= \int_{1+\gamma}^{\infty}(x-\gamma)^{\lambda_1\alpha-\frac{\varepsilon}{p}-1}\left(\sum_{n=1}^{\infty}\frac{(n-\beta)^{\lambda_2\alpha-\frac{\varepsilon}{q}-1}}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}}\right)dx\\ &\geq \int_{1+\gamma}^{\infty}(x-\gamma)^{\lambda_1\alpha-\frac{\varepsilon}{p}-1}\left(\int_{1}^{\infty}\frac{(y-\beta)^{\lambda_2\alpha-\frac{\varepsilon}{q}-1}}{[(x-\gamma)^{\alpha}+(y-\beta)^{\alpha}]^{\lambda}}dy\right)dx.\end{aligned} \tag{26}$$

Setting $s = x - \gamma$, $t = (\frac{y-\beta}{x-\gamma})^{\alpha}$ *in the above integral, we have*

$$\tilde{I} \geq \frac{1}{\alpha}\int_{1}^{\infty}s^{-1-\varepsilon}\left[\int_{\left(\frac{1-\beta}{s}\right)^{\varepsilon}}^{\infty}t^{\lambda_2-\frac{\varepsilon}{q\alpha}-1}\frac{1}{(1+t)^{\lambda}}dt\right]ds = A+B, \tag{27}$$

where

$$A: = \frac{1}{\alpha}\int_1^\infty s^{-1-\varepsilon}\int_{(\frac{1-\beta}{s})^\alpha}^1 \frac{t^{\lambda_2-\frac{\varepsilon}{q\alpha}-1}}{(1+t)^\lambda}dtds;$$

$$B: = \frac{1}{\alpha}\int_1^\infty s^{-1-\varepsilon}\int_1^\infty \frac{t^{\lambda_2-\frac{\varepsilon}{q\alpha}-1}}{(1+t)^\lambda}dtds = \frac{1}{\alpha\varepsilon}\int_1^\infty \frac{t^{\lambda_2-\frac{\varepsilon}{q\alpha}-1}}{(1+t)^\lambda}dt \tag{28}$$

$$\overset{u=\frac{1}{t}}{=} \frac{1}{\alpha\varepsilon}\int_0^1 \frac{u^{\lambda_1+\frac{\varepsilon}{q\alpha}-1}}{(1+u)^\lambda}du \geq \frac{1}{\alpha\varepsilon}\int_0^1 \frac{u^{\lambda_1+\frac{\varepsilon}{q\alpha}-1}}{(1+u)^{\lambda+\varepsilon/\alpha}}du.$$

Since

$$0 < \frac{1}{\alpha}\int_{1-\beta}^1 s^{-1-\varepsilon}\int_{(\frac{1-\beta}{s})}^1 \frac{t^{\lambda_2-\frac{\varepsilon}{q\alpha}-1}}{(1+t)^\lambda}dtds \leq \frac{1}{\alpha}\int_{1-\beta}^1\int_{(1-\beta)^\alpha}^1 s^{-2}\frac{t^{\lambda_2-\frac{1}{\alpha}-1}}{(1+t)^\lambda}dtds < \infty,$$

then by Fubini's theorem, we have

$$A = \frac{1}{\alpha}\left[\int_{1-\beta}^\infty s^{-1-\varepsilon}\int_{(\frac{1-\beta}{s})^\alpha}^1 \frac{t^{\lambda_2-\frac{\varepsilon}{q\alpha}-1}}{(1+t)^\lambda}dtds - \int_{1-\beta}^1 s^{-1-\varepsilon}\int_{(\frac{1-\beta}{s})^\alpha}^1 \frac{t^{\lambda_2-\frac{\varepsilon}{q\alpha}-1}}{(1+t)^\lambda}dtds\right]$$

$$= \frac{1}{\alpha}\int_0^1 \frac{t^{\lambda_2-\frac{\varepsilon}{q\alpha}-1}}{(1+t)^\lambda}\left[\int_{(1-\beta)/t^{(1/\alpha)}}^\infty s^{-1-\varepsilon}ds\right] dt - O(1) \tag{29}$$

$$= \frac{(1-\beta)^{-\varepsilon}}{\alpha\varepsilon}\int_0^1 \frac{t^{\lambda_2+\frac{\varepsilon}{q\alpha}-1}}{(1+t)^\lambda}dt - O(1) \geq \frac{1}{\alpha\varepsilon}\int_0^1 \frac{t^{\lambda_2+\frac{\varepsilon}{q\alpha}-1}}{(1+t)^{\lambda+\varepsilon/\alpha}}dt - O(1).$$

In view of (28) and (29) and (6), it follows that

$$A + B \geq \frac{1}{\alpha\varepsilon}\int_0^1 \frac{t^{\lambda_1+\frac{\varepsilon}{q\alpha}-1} + t^{\lambda_2+\frac{\varepsilon}{q\alpha}-1}}{(1+t)^{\lambda+\varepsilon/\alpha}}dt - O(1) = \frac{1}{\alpha\varepsilon}B\left(\lambda_2 + \tfrac{\varepsilon}{p\alpha}, \lambda_1 + \tfrac{\varepsilon}{q\alpha}\right) - O(1).$$

Then by (26) and (27), (21) is valid.

(ii) For $0 < p < 1$, *by (22) and (25), we find (notice that* $q < 0$*)*

$$\tilde{I} > k\left\{\int_{1+\gamma}^\infty \left[1 - O\left(\frac{1}{(x-\gamma)^{\lambda_2\alpha}}\right)\right](x-\gamma)^{-1-\varepsilon}dx\right\}^{1/p}\|\tilde{a}\|_{q,\psi}$$

$$= k\left(\frac{1}{\varepsilon} - \int_{1+\gamma}^\infty O\left(\frac{1}{(x-\gamma)^{\lambda_2\alpha+\varepsilon+1}}\right)dx\right)^{1/p}\|\tilde{a}\|_{q,\psi} \tag{30}$$

$$> k\left(\frac{1}{\varepsilon} - O(1)\right)^{1/p}\left(\frac{\varepsilon+1-\beta}{\varepsilon(1-\beta)^{\varepsilon+1}}\right)^{1/q}$$

$$= \frac{k}{\varepsilon}(1-\varepsilon O(1))^{1/p}\left(\frac{\varepsilon+1-\beta}{(1-\beta)^{\varepsilon+1}}\right)^{1/q}.$$

On the other hand, setting $t = (\frac{x-\gamma}{n-\beta})^{\alpha}$ *in* $\tilde{I}$, *we have*

$$\begin{aligned}\tilde{I} &= \sum_{n=1}^{\infty}(n-\beta)^{-1-\varepsilon}\frac{1}{\alpha}\int_{1/(n-\beta)^{\alpha}}^{\infty}\frac{t^{\lambda_1-\frac{\varepsilon}{p\alpha}-1}}{(1+t)^{\lambda}}dt \\ &\leq \sum_{n=1}^{\infty}(n-\beta)^{-1-\varepsilon}\frac{1}{\alpha}\int_{0}^{\infty}\frac{t^{\lambda_1-\frac{\varepsilon}{p\alpha}-1}}{(1+t)^{\lambda}}dt \\ &< \frac{\varepsilon+1-\beta}{\alpha\varepsilon(1-\beta)^{\varepsilon+1}}B\left(\lambda_1-\frac{\varepsilon}{p\alpha},\lambda_2+\frac{\varepsilon}{p\alpha}\right).\end{aligned} \tag{31}$$

In virtue of (30) and (31), (23) is valid.

3 Main results

Theorem 1 *Suppose that* $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, $0 <\alpha \leq 1$, $0 \leq \beta \leq \frac{1}{2}$, $\gamma \in (-\infty, +\infty)$, $\lambda_1 > 0$, $0 < \lambda_2\alpha \leq 1$, $\lambda = \lambda_1+\lambda_2$, $\phi(x) = (x-\gamma)^{p(1-\lambda_1\alpha)-1}$, $\psi(n) = (n-\beta)^{q(1-\lambda_2\alpha)-1}$, $f(x)$, $a_n \geq 0$, *such that* $f \in L_{p,\varphi}(\gamma, \infty)$, $a = \{a_n\}_{n=1}^{\infty} \in l_{q,\psi}$, $||f||_{p,\varphi} > 0$, $||a||_{q,\psi} > 0$, *then we have the following equivalent inequalities:*

$$\begin{aligned}I &:= \sum_{n=1}^{\infty}a_n\int_{\gamma}^{\infty}\frac{f(x)}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}}dx \\ &= \int_{\gamma}^{\infty}f(x)\sum_{n=1}^{\infty}\frac{a_n dx}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}} < k_{\lambda_1}(\alpha)\,\|f\|_{p,\phi}\,\|a\|_{q,\psi},\end{aligned} \tag{32}$$

$$J = \left\{\sum_{n=1}^{\infty}(n-\beta)^{p\lambda_2\alpha-1}\left[\int_{\gamma}^{\infty}\frac{f(x)\,dx}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}}\right]^{p}\right\}^{\frac{1}{p}} < k_{\lambda_1}(\alpha)\,\|f\|_{p,\phi}, \tag{33}$$

$$L = \left\{\int_{\gamma}^{\infty}(x-\gamma)^{q\lambda_1\alpha-1}\left[\sum_{n=1}^{\infty}\frac{a_n}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}}\right]^{q}dx\right\}^{\frac{1}{q}} < k_{\lambda_1}(\alpha)\,\|a\|_{q,\psi}, \tag{34}$$

where the constant factor $k_{\lambda_1}(\alpha) = \frac{1}{\alpha}B(\lambda_1, \lambda_2)$ *is the best possible.*

Proof. By Lebesgue term-by-term integration theorem [23], we find that there are two expressions of I in (32). By (9), (13) and $0 < ||f||_{p,\varphi} <\infty$, we have (33). By Hölder's inequality, we find

$$\begin{aligned}I &= \sum_{n=1}^{\infty}\left[(n-\beta)^{\lambda_2\alpha-\frac{1}{p}}\int_{\gamma}^{\infty}\frac{f(x)\,dx}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}}\right]\left[(n-\beta)^{\frac{1}{p}-\lambda_2\alpha}a_n\right] \\ &\leq J\left\{\sum_{n=1}^{\infty}(n-\beta)^{q(1-\lambda_2\alpha)-1}a_n^{q}\right\}^{1/q} = J\|a\|_{q,\psi}.\end{aligned} \tag{35}$$

Hence (32) is valid by (33). On the other hand, setting

$$a_n := (n-\beta)^{p\lambda_2\alpha-1}\left[\int_{\gamma}^{\infty}\frac{f(x)}{[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}]^{\lambda}}dx\right]^{p-1}\quad(n \in \mathbf{N}), \tag{36}$$

then we have

$$\|a\|_{q,\psi}^{q} = \sum_{n=1}^{\infty} (n - \beta)^{q(1-\lambda_2\alpha)-1} a_n^q = J^p = I. \tag{37}$$

By (9), (13) and $0 < ||f||_{p,\varphi} < \infty$, it follows that $J < \infty$. If $J = 0$, then (33) is trivially valid. If $J > 0$, then $0 < ||a||_{q,\psi} = J^{p-1} < \infty$. Assuming that (32) is valid, we have

$$\|a\|_{q,\psi}^{q} = J^p = I < k_{\lambda_1}(\alpha) \|f\|_{p,\phi} \|a\|_{q,\psi}, \text{ i.e. } J = \|a\|_{q,\psi}^{q-1} < k_{\lambda_1}(\alpha) \|f\|_{p,\phi}. \tag{38}$$

Hence (33) is valid, which is equivalent to (32).

By (14) and (9), we obtain (34). By Hőlder's inequality again, we have

$$\begin{aligned} I &= \int_{\gamma}^{\infty} \left[(x-\gamma)^{\lambda_1\alpha - \frac{1}{q}} \sum_{n=1}^{\infty} \frac{a_n}{\left[(x-\gamma)^\alpha + (n-\beta)^\alpha\right]^\lambda} \right] \left[(x-\gamma)^{\frac{1}{q}-\lambda_1\alpha} f(x) \right] dx \\ &\le L \left\{ \int_{\gamma}^{\infty} (x-\gamma)^{p(1-\lambda_1\alpha)-1} f^p(x)\, dx \right\}^{1/p} = L \|f\|_{p,\phi}. \end{aligned} \tag{39}$$

Hence (32) is valid by using (34). Assuming that (32) is valid, setting

$$f(x) := (x-\gamma)^{q\lambda_1\alpha-1} \left[\sum_{n=1}^{\infty} \frac{a_n}{\left[(x-\gamma)^\alpha + (n-\beta)\right]^\lambda} \right]^{q-1} \quad (x \in (\gamma, \infty)), \tag{40}$$

then we find

$$\|f\|_{p,\phi}^{p} = \int_{\gamma}^{\infty} (x-\gamma)^{p(1-\lambda_1\alpha)-1} f^p(x)\, dx = L^q = I. \tag{41}$$

By (14) and (9), it follows that $L < \infty$. If $L = 0$, then (34) is trivially valid; if $L > 0$, i.e. $0 < ||f||_{p,\varphi} < \infty$, then by (32), we have

$$\|f\|_{p,\phi}^{p} = L^q = I < k_{\lambda_1}(\alpha) \|f\|_{p,\phi} \|a\|_{q,\psi}, \quad \text{i.e.} \quad L = \|f\|_{p,\phi}^{p-1} < k_{\lambda_1}(\alpha) \|a\|_{q,\psi}.$$

Hence (34) is valid, which is equivalent to (32). It follows that (32), (33), and (34) are equivalent.

If there exists a positive number $k \le k_{\lambda_1}(\alpha)$, such that (32) is still valid as we replace $k_{\lambda_1}(\alpha)$, by k, then in particular, (20) is valid ($\tilde{a}_n$, $\tilde{f}(x)$ are taken as (19)). Then we have (21). For $\varepsilon \to 0^+$ in (21), we have $k \ge \frac{1}{\alpha} B(\lambda_2, \lambda_1) = k_{\lambda_1}(\alpha)$. Hence, $k = k_{\lambda_1}(\alpha)$ is the best value of (32). We conform that the constant factor $k_{\lambda_1}(\alpha)$ in (33) [(34)] is the best possible, otherwise we can get a contradiction by (35) [(39)] that the constant factor in (32) is not the best possible.

Remark 1 *(i) Define a half-discrete Hilbert's operator* $T : L_{p,\phi}(\gamma, \infty) \to l_{p,\psi^{1-p}}$ *as follows: For* $f \in L_{p,\varphi}(\gamma, \infty)$, *we define* $Tf \in l_{p,\psi^{1-p}}$, *satisfying*

$$Tf(n) = \int_{\gamma}^{\infty} \frac{f(x)}{\left[(x-\gamma)^\alpha + (n-\beta)^\alpha\right]^\lambda} dx \quad (n \in \mathbf{N}).$$

Then by (33), it follows $\|Tf\|_{p,\psi^{1-p}} \leq k_{\lambda_1}(\alpha)\|f\|_{p,\phi}$, *i.e. T is the bounded operator with* $\|T\| \leq k_{\lambda_1}(\alpha)$. *Since the constant factor* $k_{\lambda_1}(\alpha)$ *in (33) is the best possible, we have* $\|T\| = k_{\lambda_1}(\alpha)$.

(ii) Define a half-discrete Hilbert's operator $\tilde{T}: l_{q,\psi} \to L_{q,\phi^{1-q}}(\gamma, \infty)$ in the following way: For $a \in l_{q,\psi}$, we define $\tilde{T}a \in L_{q,\phi^{1-q}}$, satisfying

$$\tilde{T}a(x) = \sum_{n=1}^{\infty} \frac{a_n}{\left[(x-\gamma)^\alpha + (n-\beta)^\alpha\right]^\lambda} \quad (x \in (\gamma, \infty)).$$

Then by (34), it follows $\left\|\tilde{T}a\right\|_{q,\phi^{1-q}} \leq k_{\lambda_1}(\alpha)\|a\|_{q,\psi}$, i.e. $\tilde{T}$ is the bounded operator with $\left\|\tilde{T}\right\| \leq k_{\lambda_1}(\alpha)$. Since the constant factor $k_{\lambda_1}(\alpha)$ in (34) is the best possible, we have $\left\|\tilde{T}\right\| = k_{\lambda_1}(\alpha)$.

Theorem 2 *Suppose that* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \alpha \leq 1$, $0 \leq \beta \leq \frac{1}{2}$, $\gamma \in (-\infty, +\infty)$, $\lambda_1 > 0$, $0 < \lambda_2\alpha \leq 1$, $\lambda = \lambda_1 + \lambda_2$, $\psi(n) = (n-\beta)^{q(1-\lambda_2\alpha)-1}$, $\tilde{\phi}(x) = (1-\theta_\lambda(x))(x-\gamma)^{p(1-\lambda_1\alpha)-1}\left(\theta_\lambda(x) = \frac{1}{B(\lambda_1,\lambda_2)}\int_0^{\left(\frac{1-\beta}{x-\gamma}\right)^\alpha} \frac{u^{\lambda_2-1}}{(1+u)^\lambda}du \in (0,1)\right)$, $f(x)$, $a_n \geq 0$, *such that* $f \in L_{p,\tilde{\phi}}(\gamma, \infty)$, $a = \{a_n\}_{n=1}^{\infty} \in l_{q,\psi}$, $\|f\|_{p,\tilde{\phi}} > 0$, $||a||_{q,\psi} > 0$, *then we have the following equivalent inequalities*:

$$\begin{aligned} I &= \sum_{n=1}^{\infty} a_n \int_\gamma^\infty \frac{f(x)}{\left[(x-\gamma)^\alpha + (n-\beta)^\alpha\right]^\lambda}dx \\ &= \int_\gamma^\infty f(x) \sum_{n=1}^{\infty} \frac{a_n dx}{\left[(x-\gamma)^\alpha + (n-\beta)^\alpha\right]^\lambda} > k_{\lambda_1}(\alpha)\|f\|_{p,\tilde{\phi}}\|a\|_{q,\psi}, \end{aligned} \tag{42}$$

$$\begin{aligned} J &= \left\{\sum_{n=1}^{\infty}(n-\beta)^{p\lambda_2\alpha-1}\left[\int_\gamma^\infty \frac{f(x)}{\left[(x-\gamma)^\alpha + (n-\beta)^\alpha\right]^\lambda}dx\right]^p\right\}^{1/p} \\ &> k_{\lambda_1}(\alpha)\|f\|_{p,\tilde{\phi}}, \end{aligned} \tag{43}$$

$$\begin{aligned} \tilde{L} &:= \left\{\int_\gamma^\infty \frac{(x-\gamma)^{q\lambda_1\alpha-1}}{[1-\theta_\lambda(x)]^{q-1}}\left[\sum_{n=1}^{\infty} \frac{a_n}{\left[(x-\gamma)^\alpha + (n-\beta)^\alpha\right]^\lambda}\right]^q dx\right\}^{1/q} \\ &> k_{\lambda_1}(\alpha)\|a\|_{q,\psi}, \end{aligned} \tag{44}$$

where the constant factor $k_{\lambda_1}(\alpha) = \frac{1}{\alpha}B(\lambda_1, \lambda_2)$ *is the best possible.*

Proof. By (9) and the reverse of (13) and $0 < \|f\|_{p,\tilde{\phi}} < \infty$, we have (43). Using the reverse Hӧlder's inequality, we obtain the reverse form of (36) as follows

$$I \geq J\|a\|_{q,\psi}. \tag{45}$$

Then by (43), (42) is valid.

On the other hand, if (42) is valid, setting a_n as (36), then (37) still holds with $0 < p < 1$. By (42), it follows that $J > 0$. If $J = \infty$, then (43) is trivially valid; if $J < \infty$, then $0 < \|a\|_{q,\psi} = J^{p-1} < \infty$, and we have

$$\|a\|_{q,\psi}^{q} = J^{p} = I > k_{\lambda_1}(\alpha)\ \|f\|_{p,\tilde{\phi}}\ \|a\|_{q,\psi}, \quad \text{i.e.} \quad J = \|a\|_{,\psi}^{q-1} > k_{\lambda_1}(\alpha)\ \|f\|_{,\tilde{\phi}},$$

Hence (43) is valid, which is equivalent to (42).

By the reverse of (14), in view of $\varpi(x) > k_{\lambda_1}(\alpha)(1-\theta_\lambda(x))$ and $q < 0$, we have

$$\tilde{L} > k_{\lambda_1}^{\frac{q-1}{q}} L_1 \geq k_{\lambda_1}^{\frac{q-1}{q}}(\alpha)\left\{k_{\lambda_1}(\alpha)\sum_{n=1}^{\infty}(n-\beta)^{q(1-\lambda_2\alpha)-1}a_n^{q}\right\}^{1/q} = k_{\lambda_1}(\alpha)\ \|a\|_{q,\psi},$$

then (44) is valid. By the reverse Hӧlder's inequality again, we have

$$\begin{aligned} I = &\int_{\gamma}^{\infty}\left[\frac{(x-\gamma)^{\gamma 1\alpha-\frac{1}{q}}}{(1-\theta_\lambda(x))^{\frac{1}{q}}}\sum_{n=1}^{\infty}\frac{a_n}{\left[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}\right]^{\lambda}}\right] \\ &\times\left[(1-\theta_\lambda(x))^{\frac{1}{p}}(x-\gamma)^{\frac{1}{q}-\lambda_1\alpha}f(x)\right]dx \\ \geq\ &\tilde{L}\ \|f\|_{p,\tilde{\phi}}. \end{aligned} \tag{46}$$

Hence (42) is valid by (44). On the other hand, if (42) is valid, setting

$$f(x) = \frac{(x-\gamma)^{q\lambda_1\alpha-1}}{[1-\theta_\lambda(x)]^{q-1}}\left[\sum_{n=1}^{\infty}\frac{a_n}{\left[(x-\gamma)^{\alpha}+(n-\beta)^{\alpha}\right]^{\lambda}}\right]^{q-1} \quad (x\in(\gamma,\infty)),$$

then $\|f\|_{p,\tilde{\phi}}^{p} = \int_{\gamma}^{\infty}[1-\theta_\lambda(x)](x-\gamma)^{p(1-\lambda_1\alpha)-1}f^{p}(x)\,dx = \tilde{L}^{q} = I$. By the reverse of (14), it follows that $\tilde{L} > 0$. If $\tilde{L} = \infty$, then (44) is trivially valid; if $0 < \tilde{L} < \infty$, then by (42), we have

$$\|f\|_{p,\tilde{\phi}}^{p} = \tilde{L}^{q} = I > k_{\lambda_1}(\alpha)\ \|f\|_{p,\tilde{\phi}}\ \|a\|_{q,\psi}, \qquad \text{i.e.} \quad \tilde{L}, = \|f\|_{p,\tilde{\phi}}^{p-1} > k_{\lambda_1}(\alpha)\ \|a\|_{q,\psi}.$$

Hence (44) is valid, which is equivalent to (42). It follows that (42), (43), and (44) are equivalent.

If there exists a positive number $k \geq k_{\lambda_1}(\alpha)$, such that (42) is still valid as we replace $k_{\lambda_1}(\alpha)$ by k, then in particular, (22) is valid. Hence we have (23). For $\varepsilon \to 0^+$ in (23), we obtain $k \leq \frac{1}{\alpha}B(\lambda_1, \lambda_2) = k_{\lambda_1}(\alpha)$. Hence $k = k_{\lambda_1}(\alpha)$ is the best value of (42). We conform that the constant factor $k_{\lambda_1}(\alpha)$ in (43) [(44)] is the best possible, otherwise we can get a contradiction by (45) [(46)] that the constant factor in (42) is not the best possible.

In the same way, for $p < 0$, we also have the following result:

Theorem 3 *If the assumption of $p > 1$ in Theorem 1 is replaced by $p < 0$, then the reverses of (32), (33), and (34) are valid and equivalent. Moreover, the same constant factor is the best possible.*

Remark 2 *(i) For $\beta = \gamma = 0$, $\lambda_1 = \frac{1}{q\alpha}$, $\lambda_2 = \frac{1}{p\alpha}$in (32), it follows*

$$\sum_{n=1}^{\infty}a_n\int_{0}^{\infty}\frac{f(x)}{(x^{\alpha}+n^{\alpha})^{1/\alpha}}dx < \frac{1}{\alpha}B\left(\frac{1}{q^{\alpha}}, \frac{1}{p^{\alpha}}\right)\left\{\int_{0}^{\infty}f^{p}(x)\,dx\right\}^{1/p}\left\{\sum_{n=1}^{\infty}a_n^{q}\right\}^{1/q}. \tag{47}$$

In particular, for $\alpha = 1$, $p = q = 2$, (47) reduces to (4). (ii) For $\lambda = \alpha = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$ in (32), it follows

$$\sum_{n=1}^{\infty} a_n \int_{\gamma}^{\infty} \frac{f(x)}{x + n - \gamma - \beta} dx < \frac{\pi}{\sin(\pi/p)} \left\{ \int_{\gamma}^{\infty} f^p(x)\, dx \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} a_n^q \right\}^{1/q}. \tag{48}$$

In particular, for $\gamma = -\frac{1}{2}$, $\beta = \frac{1}{2}$, $p = q = 2$ in (48), we obtain (5). Hence, inequality (32) is the best extension of (4) and (5) with parameters.

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Authors' contributions
QH carried out the study, and wrote the manuscript. BY participated in the design of the study, and reformed the manuscript. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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