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Boundary regularity result for quasilinear elliptic systems

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Abstract

We consider boundary regularity for weak solutions of second-order quasilinear elliptic systems under controllable growth condition, and obtain a general criterion for a weak solution to be regular in the neighborhood of a given boundary point. Combined with existing results on interior partial regularity, this result yields an upper bound on the Hausdorff dimension of the singular set at the boundary.

Keywords: quasilinear elliptic systems, controllable growth condition, A-harmonic approximation technique, boundary partial regularity

1 Introduction

In this article we are concerned with boundary regularity for weak solutions of quasilinear elliptic systems of the following type:

$$-D_\alpha \left(A_{ij}^{\alpha\beta}(x, u)D_\beta u^j \right) = B^i(x, u, Du), \tag{1.1}$$

where Ω is a bounded domain in R^n, $n \geq 2$, u and B^i take values in R^n, $N > 1$. Here each $A_{ij}^{\alpha\beta}$ maps $\Omega \times R^N$ into R, and each B^i maps $\Omega \times R^N \times R^{nN}$ into R. In this article, we shall be concerned with weak solutions. In order to define weak solutions to (1.1), one needs to impose certain structural and regularity conditions on $A_{ij}^{\alpha\beta}$ and inhomogeneity B^i as follows:

(H1) There exists $L > 0$ such that

$$A_{ij}^{\alpha\beta}(x, \xi)(v, \tilde{v}) \leq L|v||\tilde{v}| \quad \text{for all } (x, \xi) \in \bar{\Omega} \times R^N, v, \tilde{v} \in R^{nN}.$$

(H2) $A_{ij}^{\alpha\beta}(x, \xi)$ is uniformly strongly elliptic, that is, for some $\lambda > 0$ we have

$$A_{ij}^{\alpha\beta}(x, \xi)(v, v) \geq \lambda|v|^2 \quad \text{for all } (x, \xi) \in \bar{\Omega} \times R^N, v \in R^{nN}.$$

(H3) There exists a monotone nondecreasing concave function $\omega(t, s): [0, \infty) \to [0, \infty)$ with $\omega(t, 0) = 0$, continuous at 0, such that

$$|A_{ij}^{\alpha\beta}(x, u) - A_{ij}^{\alpha\beta}(y, v)| \leq \omega(|x - y|^2 + |u - v|^2),$$

for all $x, y \in \Omega, u, v \in R^N$.

(H4) B^i fulfill the following controllable growth condition:

$$|B^i(x,\ \xi,\ v)| \le C(|v|^{2(1-\frac{1}{r})} + |\xi|^{r-1} + 1),$$

where $r = \frac{2n}{n-2}$ if $n > 2$, or any exponent if $n = 2$; for all $x \in \bar{\Omega}$, $\xi \in R^N$ and $v \in R^{nN}$.

(H5) There exists s with $s > n$ and a function $g \in H^{1,s}(\Omega, R^n)$, such that there holds:

$$u|_{\partial\Omega} = g|_{\partial\Omega}.$$

Note that we trivially have $g \in H^{1,2}(\Omega, R^N)$. Further, by Sobolev embedding theorem we have $g \in C^{0,\kappa}(\Omega, R^n)$ for any $\kappa \in [0, 1 - \frac{n}{s}]$. If $g|_{\partial\Omega} \equiv 0$, we will take $g \equiv 0$ on Ω.

We will be particularly concerned with a model case in which the domain is the upper half unit ball B^+. We reformulate the boundary condition for this case:

(H5)′ There exists s with $s > n$ and a function $g \in H^{1,s}(B^+, R^n)$ such that there holds:

$$u|_D = g|_D.$$

For $x_0 \in R^{n-1} \times \{0\}$ we write $D_\rho(x_0) = \{x \in R^n : x_n = 0, |x - x_0| < \rho\}$, and set $D_\rho = D_\rho(0)$, $D = D_1$.

Definition 1.1. By a weak solution of (1.1) we mean a vector valued function $u \in W^{1,2}(\Omega, R^n)$ such that

$$\int_\Omega A_{ij}^{\alpha\beta}(x,\ u)(D_\beta u^j,\ D_\alpha\varphi^i)dx = \int_\Omega B^i(x,\ u,\ Du)\cdot\varphi^i dx, \tag{1.2}$$

holds for all test-functions $\varphi \in C_0^\infty(\Omega, R^N)$ and, by approximation, for all $\varphi \in W_0^{1,2}(\Omega, R^N)$. Where we have introduced the notation

$$A_{ij}^{\alpha\beta}(x,\ \xi)(v,\ \tilde{v}) = \left(A_{ij}^{\alpha\beta}(x,\ \xi)\ v\right)\cdot\tilde{v}. \tag{1.3}$$

In the current situation the Sobolev embedding theorem yields the existence of a constant C_s depending only on s, n and N such that there holds:

$$\sup_{B_\rho^+(x_0)} |g - g'_{x_0,\rho}| \le C_s\rho^{1-\frac{n}{s}} \|g\|_{H^{1,s}(B_\rho^+(x_0),R^N)}, \tag{1.4}$$

for $x_0 \in D$, $\rho \le 1-|x_0|$. Obviously, the inequality remains true if we replace $\|g\|_{H^{1,s}(B_\rho^+(x_0),R^N)}$ by $\|g\|_{H^{1,s}(B^+,R^N)}$, which we will henceforth abbreviate simply as $\|g\|_{H^{1,s}}$.

We also note here that the Poincare inequality in this setting yields:

$$\int_{B_\rho^+(x_0)} |g - g'_{x_0,\rho}|^2 dx \le C_p\rho^2 \int_{B_\rho^+(x_0)} |Dg|^2 dx, \tag{1.5}$$

for a constant C_p which depending only on n.

Finally, we fix an exponent $\sigma \in (0, 1)$ as follows: if $g \equiv 0$, σ can be chosen arbitrary (but henceforth fixed); otherwise we take σ fixed in $(0, 1 - \frac{n}{s}]$.

Under such assumptions on the structural conditions, full regularity of (1.1) at the boundary cannot, in general, be expected [1], even if the boundary data is smooth. Then, our goal is to establish a partial boundary regularity for weak solutions of systems (1.1).

There are some previous partial regularity results for quasilinear systems. For example, Arkhipova has studied regularity up to the boundary for nonlinear and quasilinear systems [2-4]. And there are more results of particular forms. For systems in diagonal form, boundary regularity was first established by Wiegner [5], and the proof was generalized and extended by Hildebrandt-Widman [6]. The further discussions in this case can be seen in [7,8]. In the case of minima of functionals of the form $\int_\Omega A(x, u)|Du|^2 dx$, Jost-Meier [8] established full regularity in a neighborhood of the boundary. Further more, the dimension of singular set of solutions to non-differentiable elliptic systems are reduced by Kristensen and Mingione [9,10]

The result which is most closely related to that given here was shown that in [11] Grotowski obtained the boundary partial regularity results for more general systems of the form: $-D_\alpha A_i^\alpha(\cdot, u, Du) = f^i(\cdot, u, Du)$, $i = 1,..., N$, under the quadratic growth and natural condition. And in [12], he got the analogous boundary regularity of (1.1) under natural growth condition. However the results in the current article do not follow from those in [11]; in the current situation we need only impose weaker structure conditions, and at the same time can obtain stronger conclusions.

The comparison is made possible by the technique of A-harmonic approximation. This technique is the natural extension of the technique of harmonic approximation. The harmonic approximation technique has its origins in Simon's [13] proof of the regularity theorem of Allard [14]. The technique of A-harmonic approximation then refers to the direct analog of the above situation. The interior version of this technique has previously been applied by Duzaar and Grotowski [15] to obtain a new, elementary proof for interior partial regularity for systems of the form (1.3). Then it has been extended and developed in [16,17].

In this article, we shown optimal boundary regularity of (1.1) under controllable growth condition. As the argument for combining the boundary and interior estimates is relatively standard, we omit it and get the following results:

Theorem 1.1. Let Ω be a bounded domain in R^n, with boundary of class C^1. Let u be a weak solution of (1.1) satisfying the boundary condition (H5), where the structure conditions (H1)-(H3) hold for $A_{ij}^{\alpha\beta}$, and (H4) holds for B^i. Consider a fixed $\gamma \in (0, \sigma]$. Then there exist positive R_1 and ε_0(depending only on n, N, λ, L, $\omega(\cdot)$ and γ) with the property that

$$\fint_{B_R(x_0)\cap\Omega}|u - u'_{x_0,R}|^2 dx + \|g\|^2_{H^{1,s}}R^{2(1-\frac{n}{s})} + R^2 \leq \varepsilon_0^2,$$

for some $R \in (0, R_1]$ for a given $x_0 \in \partial\Omega$ implies $u \in C^{0,\gamma}(\bar{B}_{\frac{R}{2}}(x_0) \cap \bar{\Omega}, R^N)$.

Note in particular that the boundary condition (H5) means that $u'_{x_0,R}$ makes sense: in fact, we have $u'_{x_0,R} = g'_{x_0,R}$.

Combining this result with the analogous interior [18] and a standard covering argument allows us to obtain immediately the following bound on the size of the singular set:

Corollary 1.2. Under the assumptions of Theorem 1.1, the singular set of the weak solution u has $(n - 2)$-dimensional Hausdorff measure zero in $\bar{\Omega}$.

If the domain of the main step in proving Theorem 1.1 is a half ball; the result then follows from a relatively standard transformation argument.

Theorem 1.2. Consider a weak solution of (1.1) on the upper half unit ball B^+ which satisfies the boundary condition $(H5)'$, where the structure conditions (H1)-(H3) hold for $A_{ij}^{\alpha\beta}$, and (H4) holds for B^i. Then there exist positive R_0 and ε_0 (depending only on n, N, λ, L, $\omega(\cdot)$ and γ) with the property that

$$\fint_{B_R^+(x_0)} |u - u'_{x_0,R}|^2 dx + ||g||_{H^{1,s}}^2 R^{2(1-\frac{n}{s})} + R^2 \leq \varepsilon_0^2,$$

for some $R \in (0, R_0]$ for a given $x_0 \in D$ implies $u \in C^{0,\gamma}(\bar{B}_{\frac{R}{2}}(x_0), R^N)$.

Note that analogously to above, the boundary condition $(H5)'$ ensures that $u'_{x_0,R}$ exists, and we have indeed $u'_{x_0,R} = g'_{x_0,R}$.

We close this section by briefly summarizing the notation we use in this article. For a given set X we denote by $L^n(X)$ and $H^k(X)$ its n-dimensional Lebesgue measure and k-dimensional Harsdorff measure, respectively. We write $B_\rho(x_0) = \{x \in R^n : |x - x_0| < \rho\}$, and further $B_\rho = B_\rho(0)$, $B = B_1$. Similarly we denote upper half balls as follows: for $x_0 \in R^{n-1} \times \{0\}$ we write $B_\rho^+(x_0)$ for $\{x \in R^n : x_n > 0, |x - x_0| < \rho\}$, and set $B_\rho^+ = B_\rho^+(0)$, $B^+ = B_1^+$. For bounded $X \subset R^n$ with $L^n(X) > 0$ we denote the average of a given $g \in L^1$ by $\fint_X g dx$, i.e. $\fint_X g dx = \frac{1}{L^n(X)} \int_X g dx$. For $v \in L^1(\partial\Omega)$, $x_0 \in \partial\Omega$ we set $v'_{x_0,R} = \fint_{\partial\Omega \cap \bar{B}_R(x_0)} v dH^{n-1}$. In particular, for $v \in L^1(D_\rho(x_0))$, $x_0 \in D$, we write $v'_{x_0,\rho} = \fint_{D_\rho(x_0)} v dH^{n-1}$. We let α_n denote the volume of the unit-ball in R^n, i.e. $\alpha_n = L^n(B)$. We write $\text{Bil}(R^{nN})$ for the space of bilinear forms on the space R^{nN}.

2 The A-harmonic approximation technique and some preliminary results

In this section we present the A-harmonic approximation lemma [12], and some standard results due to Campanato [19,20].

Lemma 2.1. (A-harmonic approximation lemma) Consider fixed positive λ and L, and n, $N \in N$ with $n \geq 2$. Then for any given $\varepsilon > 0$ there exists $\delta = \delta(n, N, \lambda, L, \varepsilon) \in (0, 1]$ with the following property: for any $A_{ij}^{\alpha\beta} \in \text{Bil}(R^{nN})$ satisfying

$$A_{ij}^{\alpha\beta}(v, v) \geq \lambda |v|^2, \quad \text{for all } v \in R^{nN}, \tag{2.1}$$

and

$$|A_{ij}^{\alpha\beta}(v, \bar{v})| \leq L|v||\bar{v}|, \quad \text{for all } v, \bar{v} \in R^{nN}, \tag{2.2}$$

for any $w \in H^{1,2}(B_\rho^+(x_0), R^N)$ (for some $\rho > 0$, $x_0 \in R^n$) satisfying

$$\rho^{2-n} \int_{B_\rho^+(x_0)} |Dw|^2 dx \leq 1, \tag{2.3}$$

and

$$\left| \rho^{2-n} \int_{B_\rho^+(x_0)} A_{ij}^{\alpha\beta}(Dg, D\varphi) dx \right| \leq \delta\rho \sup_{B_\rho^+(x_0)} |D\varphi|, \tag{2.4}$$

and

$$w|_{D_\rho(x_0)} = 0, \tag{2.5}$$

for all $\varphi \in C_0^1(B_\rho^+(x_0), R^N)$, there exists an A-harmonic function

$$v \in \tilde{H} = \left\{ \tilde{w} \in H^{1,2}(B_\rho^+(x_0), R^N) \,\middle|\, \rho^{2-n} \int_{B_\rho^+(x_0)} |D\tilde{w}|^2 dx \le 1,\ \tilde{w}|_{D_\rho(x_0)} \equiv 0 \right\},$$

with

$$\rho^{-n} \int_{B_\rho^+(x_0)} |v - w|^2 dx \le \varepsilon. \tag{2.6}$$

Next we recall a slight modification of a characterization of Hölder continuous functions [19].

Lemma 2.2. Consider $n \in N$, $n \ge 2$ and $x_0 \in R^{n-1} \times \{0\}$. Suppose that there are positive constants κ and α, with $\alpha \in (0, 1]$ such that, for some $v \in L^2(B_{6R}^+(x_0))$, there hold the following:

$$\inf_{\mu \in R} \left\{ \rho^{-n} \int_{B_\rho^+(y)} |v - \mu|^2 dx \right\} \le \kappa^2 \left(\frac{\rho}{R} \right)^{2\alpha}, \tag{2.7}$$

for all $y \in D_{2R}(x_0)$ and $\rho \le 4R$; and

$$\inf_{\mu \in R} \rho^{-n} \left\{ \int_{B_\rho(y)} |v - \mu|^2 dx \right\} \le \kappa^2 \left(\frac{\rho}{R} \right)^{2\alpha}, \tag{2.8}$$

for all $y \in B_{2R}^+(x_0)$ and $B_\rho(y) \subset B_{2R}^+(x_0)$.

Then there exists a Hölder continuous representatives of the L^2-class of v on $\bar{B}_R^+(x_0)$, and for this representative $\bar{v}$ there holds:

$$|\bar{v}(x) - \bar{v}(z)| \le C_\kappa \left(\frac{|x - z|}{R} \right)^\alpha, \tag{2.9}$$

for all $x, z \in \bar{B}_R^+(x_0)$, for a constant C_κ depending only on n and α.

We close this section by a standard estimate for the solutions to homogeneous second order elliptic systems with constant coefficients, due originally to Campanato [20].

Lemma 2.3. Consider fixed positive λ and L, and $n, N \in N$ with $n \ge 2$. Then there exists C_0 depending only on n, N, λ and L (without loss of generality we take $C_0 \ge 1$) such that for $A_{ij}^{\alpha\beta} \in \mathrm{Bil}(R^{nN})$ satisfying (2.1) and (2.2), any A-harmonic function h on $B_\rho^+(x_0)$ with $h|_{D_\rho(x_0)} \equiv 0$ satisfies

$$\rho^2 \sup_{B_{\frac{\rho}{2}}^+(x_0)} |Dh|^2 \le C_0 \rho^{2-n} \int_{B_\rho^+(x_0)} |Dh|^2 dx.$$

3 Caccioppoli inequality

In this section we prove the following Caccioppoli's inequality.

Theorem 3.1. Let $u \in W^{1,2}(\bar{\Omega}, R^N)$ be a weak solution of systems (1.1) under (H1)-(H5). Then there exists $\rho_0 > 0$, ρ_0 depending only on L, s and $||g||_{H^{1,s}}$, such that for all $B_\rho^+(x_0) \subset B^+$, with $x_0 \in D^+$, $\rho < R < \rho_0$, there holds

$$\int_{B_{\rho/2}^+(x_0)} |Du|^2 dx \leq \frac{C_1}{\rho^2} \int_{B_\rho^+(x_0)} |u(x) - u'_{x_0,\rho}|^2 dx + C_2 \alpha_n \| g \|_{H^{1,s}}^2 \rho^{n(1-\frac{2}{s})}$$

$$+ C_3 \left(\int_{B_\rho^+(x_0)} (|Du|^2 + |u|^r + 1) dx \right)^{2(1-\frac{1}{r})}, \tag{3.1}$$

where C_1, C_2, C_3 depending only on λ, L and $\|g\|_{L^\infty(B,R^N)}$, and C_2 additional on C_p, C_s, and also on s.

Proof. Now we consider a cut-off function $\eta \in C_0^\infty(B_{\rho/2}^+(x_0))$, satisfying $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_{\rho/2}^+(x_0)$ and $|\nabla \eta| < \frac{4}{\rho}$. Then the function $(u - g)\eta^2$ is in $W_0^{1,2}(B_{\rho/2}^+(x_0, R^N))$, and thus can be taken as a test-function.

Using (1.2), (H1), (H4), (H5), for ε positive but arbitrary (to be fixed later), we have:

$$\int_{B_\rho^+(x_0)} A_{ij}^{\alpha\beta}(x, u)(Du, Du)\eta^2 dx$$

$$= -2 \int_{B_\rho^+(x_0)} A_{ij}^{\alpha\beta}(x, u)(Du, D\eta)(u - g)\eta dx$$

$$+ \int_{B_\rho^+(x_0)} A_{ij}^{\alpha\beta}(x, u)(Du, Dg)\eta^2 dx + \int_{B_\rho^+(x_0)} B^i(x, u, Du) \cdot \varphi^i dx$$

$$\leq 2L \int_{B_\rho^+(x_0)} |Du||D\eta||u - g|\eta dx + L \int_{B_\rho^+(x_0)} |Du||Dg|\eta^2 dx$$

$$+ C \int_{B_\rho^+(x_0)} \left(|Du|^{2(1-\frac{1}{r})} + |u|^{r-1} + 1 \right) \cdot \varphi dx$$

$$\leq 2\varepsilon \int_{B_\rho^+(x_0)} |Du|^2 \eta^2 dx + \frac{L^2}{\varepsilon} \int_{B_\rho^+(x_0)} |u - g|^2 |D\eta|^2 dx + \frac{L^2}{\varepsilon} \int_{B_\rho^+(x_0)} |Dg|^2 dx$$

$$+ C \left(\int_{B_\rho^+(x_0)} (|Du|^2 + |u|^r + 1) dx \right)^{(1-\frac{1}{r})} \left(\int_{B_\rho^+(x_0)} |\varphi|^r dx \right)^{\frac{1}{r}} \tag{3.2}$$

$$\leq 2\varepsilon \int_{B_\rho^+(x_0)} |Du|^2 \eta^2 dx + \frac{L^2}{\varepsilon} \int_{B_\rho^+(x_0)} |u - g|^2 |D\eta|^2 dx + \frac{L^2}{\varepsilon} \int_{B_\rho^+(x_0)} |Dg|^2 dx$$

$$+ C \left(\int_{B_\rho^+(x_0)} (|Du|^2 + |u|^r + 1) dx \right)^{(1-\frac{1}{r})} \left(\int_{B_\rho^+(x_0)} |D\varphi|^2 dx \right)^{\frac{1}{2}}$$

$$\leq 3\varepsilon \int_{B_\rho^+(x_0)} |Du|^2 \eta^2 dx + \left(\frac{L^2}{\varepsilon} + C(\varepsilon) \right) \int_{B_\rho^+(x_0)} |u - g|^2 |D\eta|^2 dx + C(\varepsilon) \int_{B_\rho^+(x_0)} |Dg|^2 dx$$

$$+ C(\varepsilon) \left(\int_{B_\rho^+(x_0)} (|Du|^2 + |u|^r + 1) dx \right)^{2(1-\frac{1}{r})}.$$

Using (H2) and the fact that $u'_{x_0,\rho} = g'_{x_0,\rho}$, we thus have:

$$\lambda \int_{B_\rho^+(x_0)} |Du|^2 \eta^2 dx$$

$$\leq 3\varepsilon \int_{B_\rho^+(x_0)} Du|^2 \eta^2 dx + 2\left(\frac{L^2}{\varepsilon} + C(\varepsilon)\right) \int_{B_\rho^+(x_0)} |u - u'_{x_0,\rho}|^2 |D\eta|^2 dx + C(\varepsilon) \int_{B_\rho^+(x_0)} |Dg|^2 dx$$

$$+ 2\left(\frac{L^2}{\varepsilon} + C(\varepsilon)\right) \int_{B_\rho^+(x_0)} |g - g'_{x_0,\rho}|^2 |D\eta|^2 dx + C(\varepsilon) \left(\int_{B_\rho^+(x_0)} (|Du|^2 + |u|^r + 1) dx\right)^{2\left(1-\frac{1}{r}\right)}.$$

Thus, by (1.4), (1.5), we can get:

$$(\lambda - 3\varepsilon) \int_{B_\rho^+(x_0)} |Du|^2 \eta^2 dx \leq 2\left(\frac{L^2}{\varepsilon} + C(\varepsilon)\right) \int_{B_\rho^+(x_0)} |u - u'_{x_0,\rho}|^2 |D\eta|^2 dx$$

$$+ \left[2\left(\frac{L^2}{\varepsilon} + \varepsilon\right) C_p + C(\varepsilon)\right] \alpha_n \|g\|_{H^{1,s}}^2 \rho^{n\left(1-\frac{2}{s}\right)} + C(\varepsilon) \left(\int_{B_\rho^+(x_0)} (|Du|^2 + |u|^r + 1) dx\right)^{2\left(1-\frac{1}{r}\right)}. \quad (3.3)$$

Fix ε small enough such that $\lambda - 3\varepsilon > 0$, thus (3.3) yields the desired inequality, where C_1, C_2, C_3 depending only on $\|g\|_{L^\infty(B,R^N)}$, λ and L, and C_2 additional on C_p, C_s and also on s.

4 The proof of the main theorem

In this section we proceed to the proof of partial regularity result, and hence consider $u \in W^{1,2}(\Omega, R^n)$ to be a weak solution of (1.1). For $R < 1 - |x_0|$, $x_0 \in D$, $y \in D_R(x_0)$, $D_\rho(y) \subset\subset D_R(x_0)$, and $\varphi \in C_0^\infty(B_{\frac{\rho}{2}}^+(y), R^N)$ with $\sup_{B_\rho^+(y)} |D\varphi| \leq 1$, we have

Lemma 4.1.

$$\left|\left(\frac{\rho}{2}\right)^{2-n} \int_{B_{\frac{\rho}{2}}^+(y)} A(y, u_{y,\rho}^+)(Du, D\varphi) dx\right| \leq C_8 \frac{\rho}{2} \sqrt{I}\left(\sqrt{I} + \omega^{\frac{1}{2}}(I)\right) \sup_{B_{\frac{\rho}{2}}^+(x_0)} |D\varphi|, \quad (4.1)$$

here and hereafter, we define

$$I(z, r_0) = f_{B_{r_0}^+(z)} |u - u'_{z,r_0}|^2 dx + \|g\|_{H^{1,s}}^2 r_0^{2(1-n/s)} + r_0^2,$$

for $z \in D$, $r_0 \in (0, 1 - |z|)$.

Proof. Consider $x_0 \in D$ and $y \in D_R(x_0)$, $D_\rho(y) \subset\subset D_R(x_0)$, for $R < 1 - |x_0|$, and $\varphi \in C_0^\infty(B_{\frac{\rho}{2}}^+(y), R^N)$ with $\sup_{B_\rho^+(y)} |D\varphi| \leq 1$. From the definition of weak solution:

$$\int_{B_{\frac{\rho}{2}}^+(y)} A_{ij}^{\alpha\beta}(y, u'_{y,\rho})(Du, D\varphi) dx$$

$$\leq C\left[\int_{B_{\frac{\rho}{2}}^+(y)} \left(|Du|^{2\left(1-\frac{1}{r}\right)} + |u|^{r-1} + 1\right) dx\right] \rho \sup_{B_{\frac{\rho}{2}}^+(y)} |D\varphi| \quad (4.2)$$

$$+ \int_{B_{\frac{\rho}{2}}^+(y)} |A_{ij}^{\alpha\beta}(y, u'_{y,\rho}) - A_{ij}^{\alpha\beta}(x, u)||Du| dx \sup_{B_{\frac{\rho}{2}}^+(y)} |D\varphi|.$$

By Sobolev' s and Hölder's inequalities, and then Young's inequality, we have

$$
C \int_{B^+_{\frac{\rho}{2}}(\gamma)} \left(|Du|^{2\left(1-\frac{1}{r}\right)} + |u|^{r-1} + 1 \right) dx
$$

$$
\leq C \left(\int_{B^+_{\frac{\rho}{2}}(\gamma)} |Du|^2 dx \right)^{1-\frac{1}{r}} \left(\int_{B^+_{\frac{\rho}{2}}(\gamma)} dx \right)^{\frac{1}{r}} + C \left(|u'_{\gamma,\rho/2}|^{r-1} + 1 \right) \alpha_n \rho^n / 2
$$

$$
+ C \left(\int_{B^+_{\frac{\rho}{2}}(\gamma)} |u - u'_{\gamma,\rho/2}|^r dx \right)^{\left(1-\frac{1}{r}\right)} \left(\int_{B^+_{\frac{\rho}{2}}(\gamma)} dx \right)^{\frac{1}{r}}
$$

$$
\leq C \left(\int_{B^+_{\frac{\rho}{2}}(\gamma)} |Du|^2 dx \right)^{1-\frac{1}{r}} \left(\int_{B^+_{\frac{\rho}{2}}(\gamma)} dx \right)^{\frac{1}{r}} + C \left(|u'_{\gamma,\rho/2}|^{r-1} + 1 \right) \alpha_n \rho^n / 2
$$

$$
+ C \left(\int_{B^+_{\frac{\rho}{2}}(\gamma)} |Du|^2 dx \right)^{\frac{r}{2}\left(1-\frac{1}{r}\right)} \left(\int_{B^+_{\frac{\rho}{2}}(\gamma)} dx \right)^{\frac{1}{r}} \tag{4.3}
$$

$$
\leq C \left(\int_{B^+_{\frac{\rho}{2}}(\gamma)} |Du|^2 dx \right)^{1-\frac{1}{r}} \left(\int_{B^+_{\frac{\rho}{2}}(\gamma)} dx \right)^{\frac{1}{r}} + C \left(|u'_{\gamma,\rho/2}|^{r-1} + 1 \right) \alpha_n \rho^n / 2
$$

$$
+ C \left(\int_{B^+_{\frac{\rho}{2}}(\gamma)} |Du|^2 dx \right)^{\left(\frac{r}{2}-1\right)\left(1-\frac{1}{r}\right)} \left(\int_{B^+_{\frac{\rho}{2}}(\gamma)} |Du|^2 dx \right)^{1-\frac{1}{r}} \left(\int_{B^+_{\frac{\rho}{2}}(\gamma)} dx \right)^{\frac{1}{r}}
$$

$$
\leq C(||u||_{W^{1,2}}) \left(\int_{B^+_{\frac{\rho}{2}}(\gamma)} |Du|^2 dx \right)^{1-\frac{1}{r}} \left(\int_{B^+_{\frac{\rho}{2}}(\gamma)} dx \right)^{\frac{1}{r}} + C \left(|u'_{\gamma,\rho/2}|^{r-1} + 1 \right) \frac{\alpha_n \rho^n}{2}
$$

$$
\leq \varepsilon \int_{B^+_{\frac{\rho}{2}}(\gamma)} |Du|^2 dx + C(\varepsilon, ||u||_{W^{1,2}}) \left(|u'_{\gamma,\rho/2}|^{r-1} + 1 \right) \alpha_n \rho^n / 2,
$$

and

$$
C_3 \left(\int_{B^+_\rho(x_0)} \left(|Du|^2 + |u|^r + 1 \right) dx \right)^{2\left(1-\frac{1}{r}\right)}
$$

$$
\leq \frac{c_4}{2} \alpha_n \rho^{n+2} \left(\fint_{B^+_\rho(x_0)} (|Du|^2 + |u - u'_{x_0,\rho}|^r + |u'_{x_0,\rho}| + 1) dx \right)^{2\left(1-\frac{1}{r}\right)}
$$

$$
\leq C_5 \alpha_n \rho^{n+2} \left[\left(\fint_{B^+_\rho(x_0)} |Du|^2 dx \right)^{2\left(1-\frac{1}{r}\right)} + \left(\fint_{B^+_\rho(x_0)} |Du|^2 dx \right)^{r\left(1-\frac{1}{r}\right)} + \left(|u'_{x_0,\rho}| + 1 \right)^{2\left(1-\frac{1}{r}\right)} \right]
$$

$$
\leq C_6(||u||_{W^{1,2}}, |u'_{x_0,\rho}|) \alpha_n \rho^{n+2},
$$

where we have used $r = \frac{2n}{n-2}$ and $u \in W^{1,2}(\bar{\Omega}, R^N)$.

Then Caccioppoli inequality yields

$$\int_{B_{\rho/2}^+(x_0)} |Du|^2 dx \le \frac{C_1}{\rho^2} \int_{B_\rho^+(x_0)} |u(x) - u_{x_0,\rho}'|^2 dx + C_2 \alpha_n \|g\|_{H^{1,s}}^2 \rho^{n(1-\frac{2}{s})} + C_6 \alpha_n \rho^{n+2} \quad (4.4)$$

Henceforth we restrict to ρ sufficiently small. Applying in turn Young's inequality, (H3), (4.4) and Jensen's inequality we calculate from (4.2):

$$\int_{B_{\frac{\rho}{2}}^+(y)} A_{ij}^{\alpha\beta}(y, u_{y,\rho}')(Du, D\varphi)dx$$

$$\le C\rho \left[\int_{B_{\frac{\rho}{2}}^+(y)} |Du|^2 dx + \left(|u_{y,\rho/2}'|^{r-1} + 1\right)\alpha_n \rho^n/2 \right]$$

$$+ \left[\int_{B_{\frac{\rho}{2}}^+(y)} |A_{ij}^{\alpha\beta}(y, u_{y,\rho}') - A_{ij}^{\alpha\beta}(x, u)|^2 dx \right]^{\frac{1}{2}} \cdot \left[\int_{B_{\frac{\rho}{2}}^+(y)} |Du|^2 dx \right]^{\frac{1}{2}} \quad (4.5)$$

$$\le C\rho \left[\int_{B_{\frac{\rho}{2}}^+(y)} |Du|^2 dx + \left(|u_{y,\rho/2}'|^{r-1} + 1\right)\alpha_n \rho^n/2 \right]$$

$$+ \omega^{\frac{1}{2}} \left(\rho^2 + \fint_{B_\rho^+(y)} |u - u_{y,\rho}'|^2 dx \right) (\alpha_n \rho^n/2) \left(\fint_{B_\rho^+(y)} |Du|^2 dx \right)^{\frac{1}{2}}.$$

For $z \in D$, $r_0 \in (0, 1 - |z|)$. We introduce the notation $I(z, r_0) = \fint_{B_{r_0}^+(z)} |u - u_{z,r_0}'|^2 dx + \|g\|_{H^{1,s}}^2 r_0^{2(1-n/s)} + r_0^2$,

and further write I for $I(y, \rho)$. We have from (4.5) and by Poincare's inequality:

$$\int_{B_{\frac{\rho}{2}}^+(y)} A_{ij}^{\alpha\beta}(y, u_{y,\rho}')(Du, D\varphi)dx \le \frac{C_7}{2} \alpha_n \rho^{n-1} \left[I + \omega^{\frac{1}{2}}(I) I^{\frac{1}{2}} \right].$$

For arbitrary $\varphi \in C_0^\infty(\Omega, R^N)$ we thus have, by recalling:

$$\int_{B_{\frac{\rho}{2}}^+(y)} A_{ij}^{\alpha\beta}(y, u_{y,\rho}')(Du, D\varphi)dx \le \frac{C_7}{2} \alpha_n \rho^{n-1} \left[I + \omega^{\frac{1}{2}}(I) I^{\frac{1}{2}} \right] \sup_{B_{\frac{\rho}{2}}^+(x_0)} |D\varphi|.$$

Multiplying through by $(\frac{\rho}{2})^{2-n}$, this yields:

$$\left| \left(\frac{\rho}{2}\right)^{2-n} \int_{B_{\frac{\rho}{2}}^+(y)} A_{ij}^{\alpha\beta}(y, u_{y,\rho}')(Du, D\varphi)dx \right| \le C_8 \frac{\rho}{2} \left[I + \omega^{\frac{1}{2}}(I) I^{\frac{1}{2}} \right] \sup_{B_{\frac{\rho}{2}}^+(x_0)} |D\varphi|, \quad (4.6)$$

for C_8 define by $C_8 = 2^{n-3} C_7 \alpha_n$.

Lemma 4.2. Consider u satisfying the condition of Theorem 1 and σ fixed, then we can find δ and s_0 together with positive constants C_{11} such that the smallness conditions:

$$0 < \omega(|u|, \ s_0) \le \left(\frac{\delta}{2}\right)^2,$$

and

$$I(x_0, \ R) \le C_{13}^{-1} \min\left\{\frac{\delta^2}{4}, \ s_0\right\},$$

together imply the growth condition

$$I(\gamma, \ \theta\rho) \le \theta^{2\sigma} I(\gamma, \ \rho).$$

Proof. Combining Lemma 2.1 and Theorem 3.1, and applying Lemma 2.2, we can conclude the desired Hölder continuity. As the argument is relatively standard and similarly to [12], we omit it.

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Authors' contributions
Shuhong Chen participated in design of the study and drafted the manuscript. Zhong Tan participated in conceived of the study and the amendment of the paper. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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