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Uniqueness of meromorphic functions sharing two values

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Abstract

In this article, we shall study the uniqueness problems on meromorphic functions sharing nonzero finite value or having fixed points. Our results extend the corresponding results of Fang and Hua, Yang and Hua, and Fang and Qiu. **MSC 2010**: 30D35, 30D30.

Keywords: uniqueness, meromorphic function, sharing value, fixed point

1 Introduction and main results

Let $\mathbb{C}$ denote the complex plane and f be a nonconstant meromorphic function on $\mathbb{C}$. We assume the reader is familiar with the standard notion used in the Nevanlinna value distribution theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$ (see, e.g., [1-4]), and $S(r, f)$ denotes any quantity that satisfies the condition $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure. A meromorphic function a is called a small function with respect to f, provided that $T(r, a) = S(r, f)$.

Let f and g be two nonconstant meromorphic functions. Let a be a small function of f and g. We say that f, g share a counting multiplicities (CM) if $f - a$, $g - a$ have the same zeros with the same multiplicities and we say that f, g share a ignoring multiplicities (IM) if we do not consider the multiplicities. We denote $\overline{N}_0(r, \infty)$ the reduced counting function of the common poles of f and g. If $\overline{N}(r, f) - \overline{N}_0(r, \infty) = S(r, f)$, and $\overline{N}(r, g) - \overline{N}_0(r, \infty) = S(r, g)$, we say that f and g share ∞ "IM". We denote by $N_{k)}(r, \frac{1}{f-a})$ (or $\overline{N}_k(r, \frac{1}{f-a})$) the counting function for zeros of $f - a$ with multiplicity $\leq k$ (IM), and by $N_{(k}(r, \frac{1}{f-a})$ (or $\overline{N}_k(r, \frac{1}{f-a})$) the counting function for zeros of $f - a$ with multiplicity $\leq k$ (IM). Moreover, we set $N_k(r, \frac{1}{f-a}) = \bar{N}(r, \frac{1}{f-a}) + \bar{N}_2(r, \frac{1}{f-a}) + \bar{N}_3(r, \frac{1}{f-a}) + \cdots + \bar{N}_k(r, \frac{1}{f-a})$.

We say that a finite value z_0 is called a fixed point of f if $f(z_0) = z_0$ or z_0 is a zero of $f(z) - z$.

The following theorem in the value distribution theory is well known [5,6].

Theorem A. *Let f be a transcendental meromorphic function, $n \leq 1$ a positive integer. Then $f^n f' = 1$ has infinitely many solutions.*

Fang and Hua [7], Yang and Hua [8] got a unicity theorem, respectively, corresponding to Theorem A.

Theorem B. *Let f and g be two nonconstant entire (meromorphic) functions, $n \le 6(n \le 11)$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

Considering the uniqueness question of entire or meromorphic functions having fixed points, Fang and Qiu [9] obtained the following result.

Theorem C. *Let f and g be two nonconstant meromorphic (entire) functions, $n \le 11(n \le 6)$ a positive integer. If $f^n f'$ and $g^n g'$ share z CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2, and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

For more results in such directions, we refer the readers to [7-24].

We recall the following result by Xu et al. [25] or Zhang and Li [26], respectively.

Theorem D. *Let f be a transcendental meromorphic function, $n(\le 2)$, k be two positive integers. Then $f^n f^{(k)}$ takes every finite nonzero value infinitely many times or has infinitely many fixed points.*

Corresponding to Theorem D, one may ask, what can be said about the relationship between two meromorphic functions f and g, if $f^n f^{(k)}$ and $g^n g^{(k)}$ have the same fixed points or share one nonzero complex number, where n and k are positive numbers? In this direction, we will prove:

Theorem 1.1. *Let f and g be two transcendental meromorphic functions, whose zeros are of multiplicities at least k, where k is a positive integer. Let $n > \max\{2k - 1, k + 4/k + 4\}$ be a positive integer. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share z CM, f and g share ∞ IM, one of the following two conclusions holds:*

(i) $f^n f^{(k)} = g^n g^{(k)}$;

(ii) $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2, and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.

Theorem 1.2. *Let f and g be two nonconstant meromorphic functions, whose zeros are of multiplicities at least k, where k is a positive integer. Let $n > \max\{2k - 1, k + 4/k + 4\}$ be a positive integer. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share 1 CM, f and g share ∞ IM, one of the following two conclusions holds:*

(i) $f^n f^{(k)} = g^n g^{(k)}$;

(ii) $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, where c_3, c_4, and d are constants such that $(-1)^k (c_3c_4)^{n+1} d^{2k} = 1$.

Remark 1.1. *Theorems 1.1 and 1.2 are also true of entire functions when $n > 4/k+2$ be a positive integer.*

Theorem 1.3. *Let f and g be two nonconstant meromorphic functions, whose zeros are of multiplicities at least k + 1, where k is a positive integer with $1 \le k \le 5$. Let $n \le 10$ be a positive integer. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share 1 CM, $f^{(k)}$ and $g^{(k)}$ share 1 CM, f and g share ∞ IM, one of the following two conclusions holds:*

(i) $f \equiv tg$ for a constant t such that $t^{n+1} = 1$;

(ii) $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k (c_3c_4)^{n+1} d^{2k} = 1$.

2 Preliminary lemmas

Lemma 2.1. [3] *Let f be a nonconstant meromorphic function, and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N\left(r,\ \frac{1}{f^{(k)}}\right) \leq N\left(r,\ \frac{1}{f}\right) + k\bar{N}\left(r,\ f\right) + S\left(r,\ f\right).$$

By using the similar method of Yang and Hua [8], we can prove the following lemma.

Lemma 2.2. *Let f and g be two nonconstant meromorphic functions, a be a finite nonzero constant. If f and g share a CM and* ∞ *"IM", one of the following cases holds:*

(*i*) $T(r, f) \leq N_2(r, 1/f) + N_2(r, 1/g) + 3\overline{N}(r, f) + S(r, f) + S(r, g)$, *the same inequality holding for* $T(r, g)$;

(*ii*) $fg \equiv a^2$; (*iii*) $f \equiv g$.

Lemma 2.3. [1, Theorem 3.10] *Suppose that f is a nonconstant meromorphic function,* $k \leq 2$ *is an integer. If*

$$N(r,\ f) + N(r,\ 1/f) + N(r,\ 1/f^{(k)}) = S(r,\ f'/f),$$

Then $f(z) = e^{az+b}$, *where* $a \neq 0$, b *are constants.*

Lemma 2.4. [27] *Let f and g be nonconstant meromorphic functions. Suppose that f and g share the values 0 and* ∞ *CM,* $f^{(k)}$ *and* $g^{(k)}$ *share the value 0 CM for* $k = 1, 2, 3, 4, 5, 6$. *Then f and g satisfy one of the following cases:*

(i) $f = tg$, *where* $t(\neq 0)$ *is a constant.*

(*ii*) $f(z) = e^{az+b}$, $g(z) = e^{cz}+^{d}$, *where a, b, c, and d are constants with* $ac \neq 0$.

(*iii*) $f(z) = \frac{a}{1-be^{\alpha(z)}}$, $g(z) = \frac{a}{e^{-\alpha(z)}-b}$, *where a, b are nonzero consants, and* $\alpha(z)$ *is a nonconstant entire function.*

(*iv*) $f(z) = a(1 - be^{cz})$, $g(z) = d(e^{-cz} - b)$, *where a, b, c, and d are nonzero constants.*

To prove Theorems 1.1 and 1.2, we also need the following results.

Lemma 2.5. *Let f, g be two nonconstant meromorphic functions, whose zeros are of multiplicities at least k, where k is a positive integer. Let* $n > 2k - 1$ *be a positive integer. If* $f^n f^{(k)} g^n g^{(k)} = z^2$, *f and g share* ∞ *IM, then* $f(z) = c_1 e^{cz^2}$ $g(z) = c_2 e^{-cz^2}$, *where* c_1, c_2, *and c are constants such that* $4(c_1c_2)^{n+1}c^2 = -1$.

Lemma 2.6. *Let f, g be two nonconstant meromorphic functions, whose zeros are of multiplicities at least k, where k is a positive integer. Let* $n > 2k - 1$ *be a positive integer. If* $f^n f^{(k)} g^n g^{(k)} = 1$, *f and g share* ∞ *IM, then* $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, *where* c_3, c_4, *and d are constants such that* $(-1)^k(c_3c_4)^{n+1}d^{2k} = 1$.

3 Proofs of Lemmas 2.5 and 2.6

3.1 Proof of Lemma 2.5

Since f and g share ∞ IM, we have from

$$f^n f^{(k)} g^n g^{(k)} = z^2 \tag{3.1}$$

that f and g are entire functions.

Suppose that f has a zero z_0 of multiplicity $p \leq k$, then z_0 is a zero of $f^n f^{(k)}$ with multiplicity $np + p - k \leq 2k^2$. In view of (3.1), we get $k = 1$, $n = 2$, and $z_0 = 0$.

Moreover, g has no zero. Therefore,

$$f(z) = ze^{\alpha_1(z)},\ g(z) = e^{\beta_1(z)}, \tag{3.2}$$

where $\alpha_1(z)$, $\beta_1(z)$ are nonconstant entire functions. So from 3.2, we get

$$z^2(1+z\alpha_1'(z))\beta_1' e^{3(\alpha_1(z)+\beta_1(z))}=z^2, \tag{3.3}$$

namely

$$(1+z\alpha_1'(z))\beta_1' e^{3(\alpha_1(z)+\beta_1(z))}=1, \tag{3.4}$$

which is impossible since α_1 and β_1 are non-constant entire functions $\left(\alpha'_1 \equiv 0,\ \beta'_1 \equiv 0\right)$. Thus f has no zero. similarly, we get that g has no zero. So, we have

$$f(z)=e^{\alpha(z)}, g(z)=e^{\beta(z)}, \tag{3.5}$$

where $\alpha(z)$, $\beta(z)$ are nonconstant entire functions. Then

$$T\left(r, \frac{f'}{f}\right)=T(r,\ \alpha'). \tag{3.6}$$

We claim that $\alpha + \beta \equiv C$, where C is a constant.

Let $F = f^n f^{(k)}$, $G = g^n g^{(k)}$. Then we have

$$\begin{aligned} nT(r,\ f)\ &=\ T(r,\ f^n)\ =\ T\left(r, \frac{F}{f^{(k)}}\right) \\ &\leq\ T(r,\ F)\ +\ T(r,\ f^{(k)})+S(r,\ f)\ \leq\ T(r,\ F)\ +\ T(r,\ f)\ +\ S(r,\ f), \end{aligned}$$

We obtain from (3.7) that

$$T(r,\ f)\ =\ O(T(r,\ F)), \tag{3.8}$$

as $r \in E$ and $r \to \infty$, where $E \subset (0, +\infty)$ is some subset of finite linear measure. Note that

$$\begin{aligned} T\,(r,\,F)\ &=\ T\left(r,\ f^n f^{(k)}\right)\ +\ nT\left(r,\ f\right)\ +T\left(r,\ f^{(k)}\right)\ +S\left(r,\ f\right) \\ &\leq\ (n+\ 1)\,T\left(r,\ f\right)\ +\ S\left(r,\ f\right), \end{aligned} \tag{3.9}$$

We obtain from (3.9) that

$$T(r,F)\ =\ O(T(r,f)), \tag{3.10}$$

as $r \in E$ and $r \to \infty$, where $E \subset (0, +\infty)$ is some subset of finite linear measure.

Thus from (3.8), (3.10) and the standard reasoning of removing exceptional set (see [2, Lemma 1.1.1]) we deduce $\sigma\ (f) = \sigma\ (F)$. Similarly, we have $\sigma(g) = \sigma(G)$. It follows from (3.1) that $\sigma(F\ ) = \sigma(G)$, we get $\sigma\ (f) = \sigma(g)$.

We deduce that either both α and β are transcendental functions or both α and β are polynomials. Moreover, we have

$$N(r,\ 1/f^{(k)})\ \leq\ N(r,\ 1/z^2)\ =\ O(\log r).$$

From this and (3.5) we get

$$N(r,f)+N(r,\ 1/f)\ +\ N(r,\ 1/f^{(k)})\ =\ O(\log r).$$

If $k \leq 2$, then it follows from (3.6) and Lemma 2.3 that α' is a polynomial, and so α is a nonconstant polynomial. Similarly, we can deduce that β is also a nonconstant polynomial.

We deduce from (3.5) that

$$f^{(k)} = [(\alpha')^k + P_{k-1}(\alpha')]e^{\alpha}, \ g^{(k)} = [(\beta')^k + Q_{k-1}(\beta')]e^{\beta},$$

where $P_{k-1}(\alpha')$ and $Q_{k-1}(\beta')$ are differential polynomials in α' and β' of degree at most k - 1, respectively. Thus, we obtain

$$[(\alpha')^k + P_{k-1}(\alpha')]\,[(\beta')^k + Q_{k-1}(\beta')]e^{(n+1)(\alpha+\beta)} = z^2. \tag{3.11}$$

We deduce from (3.11) that $\alpha(z) + \beta(z) \equiv C$ for a constant C.

If $k = 1$, from (3.1) and (3.5) we get

$$\alpha'\beta' e^{(n+1)(\alpha+\beta)} = z^2. \tag{3.12}$$

Next, we let $\alpha + \beta = \gamma$ and suppose that α, β are transcendental entire functions.

If γ is a constant, then $\alpha' + \beta' = 0$, and from (3.12) we have $\alpha'\beta' e^{(n+1)\gamma} = -\alpha'^2 e^{(n+1)\gamma} = z^2$ which implies that α' is a nonconstant polynomial of degree deg $(\alpha') = 1$. This together with $\alpha' + \beta' = 0$ implies that β' is also a nonconstant polynomial of degree $\deg(\beta') = 1$.

If γ is not a constant, then (3.12) implies that

$$\alpha'(\gamma' - \alpha')e^{(n+1)\gamma} = z^2. \tag{3.13}$$

Since $T(r, \gamma') = m(r, \gamma') \leq m\left(r, \frac{(e^{(n+1)\gamma})'}{e^{(n+1)\gamma}}\right) + O(1) = S(r, e^{(n+1)\gamma})$. Thus (3.13) implies that

$$\begin{aligned} T(r, e^{(n+1)\gamma}) \leq\ & T\left(r, \frac{z^2}{\alpha'(\gamma' - \alpha')}\right) + O(1) \\ \leq\ & (2 + o(1))T(r, \alpha') + S(r, e^{n\gamma}), \end{aligned}$$

which implies that

$$T(r, e^{(n+1)\gamma}) = O(T(r, \alpha')),$$

Similarly, we have

$$T(r, \alpha') = O(T(r, e^{(n+1)\gamma})).$$

Thus $T(r, \gamma') = S(r, e^{(n+1)\gamma}) = S(r, \alpha')$.

In view of (3.13) and by the second fundamental theorem for small functions, we get

$$T(r, \alpha') \leq \bar{N}\left(r, \frac{1}{\alpha'}\right) + \bar{N}\left(r, \frac{1}{\alpha' - \gamma'}\right) + S(r, \alpha') \leq O(\log r) + S(r, \alpha').$$

Thus α' is a polynomial, which contradicts that α is a transcendental entire function.

Thus α and β are both polynomials and $\alpha(z) + \beta(z) \equiv C$ for a constant C.

Hence, from (3.11) we get

$$(-1)^k(\alpha')^{2k} = z^2 + \tilde{P}_{2k-1}(\alpha'), \tag{3.14}$$

where $\tilde{P}_{2k-1}$ is a differential polynomial in α' of degree at most $2k$ - 1. From (3.14) we have

$$2kT(r,\ \alpha') = 2\log r + S(r,\ \alpha'). \tag{3.15}$$

From (3.15) we can see that α' is a nonconstant polynomial of degree 1 and that k = 1.

By induction we get

$$\alpha' + \beta' = 0,$$
$$e^{(n+1)C}\alpha'\beta' = z^2.$$

By computation we get

$$\alpha' = l_1 z, \quad \beta' = -l_2 z, \tag{3.16}$$

Hence

$$\alpha = cz^2 + l_3, \quad \beta = -cz^2 + l_4, \tag{3.17}$$

We can rewrite f and g as

$$f = c_1 e^{cz^2}, \quad g = c_2 e^{-cz^2},$$

where c_1, c_2, and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.

This completes the proof of Lemma 2.5.

3.2 Proof of Lemma 2.6

By the same reasons as in Lemma 2.5, we get

$$f(z) = e^{\alpha(z)}, \quad g(z) = e^{\beta(z)}, \tag{3.18}$$

and

$$(\alpha')^{2k} = 1 + \tilde{P}_{2k-1}(\alpha'), \tag{3.19}$$

where $\tilde{P}_{2k-1}$ is a differential polynomial in α' of degree at most $2k$ - 1. From (3.19) we have

$$2kT(r,\ \alpha') \quad \leq \quad (2k-1)T(r,\ \alpha') + S(r,\ \alpha'), \tag{3.20}$$

which implies that α' is a nonzero constant. Thus $\alpha = dz + l_5$, $\beta = -dz + l_6$. By (3.18), rewrite f and g as

$$f = c_3 e^{dz}, \quad g = c_4 e^{-dz}, \tag{3.21}$$

where c_3, c_4 and d are constants such that $(-1)^k(c_3c_4)^{n+1}d^{2k} = 1$.

This completes the proof of Lemma 2.6.

4 Proof of Theorem 1.1

Let $F = f^n f^{(k)}$, $G = g^n g^{(k)}$, $F^*(z) = F(z)/z$, $G^*(z) = G(z)/z$. Note that f and g are transcendental, so z is a small function with respect to both F and G. Then F^* and G^* share 1 CM and ∞ "IM". By Lemma 2.2, we consider three cases.

Case 1. Suppose that

$$T(r,\ F^*) \leq N_2(r,\ 1/F^*) + N_2(r,\ 1/G^*) + 3\bar{N}(r,\ f) + S(r,\ f) + S(r,\ g). \tag{4.1}$$

We deduce from (4.1) that

$$T(r,\ F) \leq N_2(r,\ 1/F) + N_2(r,\ 1/G) + 3\bar{N}(r,\ f) + S(r,\ f) + S(r,\ g).$$

Obviously,

$$N(r,\ F) = (n+1)N(r,\ f) + k\bar{N}(r,\ f) + S(r,\ f). \tag{4.2}$$

Since

$$\begin{aligned} nm(r,f) &= m(r,F/f^{(k)}) \leq m(r,F) + m(r,1/f^{(k)}) + S(r,f) \\ &= m(r,F) + T(r,f^{(k)}) - N(r,1/f^{(k)}) + S(r,f) \\ &\leq m(r,F) + T(r,f) + k\bar{N}(r,f) - N(r,1/f^{(k)}) + S(r,f). \end{aligned} \tag{4.3}$$

It follows from (4.2), (4.3), and Lemma 2.1 that

$$\begin{aligned} (n-1)T(r,f) \leq\ & T(r,F) - N(r,\ f) - N(r,\ 1/f^{(k)}) + S(r,\ f) \\ \leq\ & N_2(r,1/F) + N_2(r,\ 1/G) + 3\bar{N}(r,\ f) \\ & - N(r,\ f) - N(r,\ 1/f^{(k)}) + S(r,\ f) + S(r,\ g) \\ \leq\ & N_2(r,\ 1/f^n) + N_2(r,\ 1/f^{(k)}) + N_2(r,\ 1/g^n) + N_2(r,\ 1/g^{(k)}) \\ & + 3N(r,\ f) - N(r,\ f) - N(r,\ 1/f^{(k)}) + S(r,\ f) + S(r,\ g) \\ \leq\ & 2\bar{N}(r,\ 1/f) + 2\bar{N}(r,\ 1/g) + N(r,\ 1/f^{(k)}) + N(r,\ 1/g^{(k)}) \\ & + 2N(r,\ f) - N(r,\ 1/f^{(k)}) + S(r,\ f) + S(r,\ g) \\ \leq\ & \frac{2}{k}N(r,\ 1/f) + \frac{2}{k}N(r,\ 1/g) + N(r,\ 1/g) + k\bar{N}(r,\ g) \\ & + 2N(r,\ g) + S(r,\ f) + S(r,\ g) \\ \leq\ & \frac{2}{k}(T(r,\ f) + T(r,\ g)) + (k+3)T(r,\ g) + S(r,\ f) + S(r,\ g). \end{aligned}$$

Similarly, we have

$$(n-1)T(r,\ g) \leq \frac{2}{k}(T(r,\ f) + T(r,\ g)) + (k+3)T(r,\ f) + S(r,\ f) + S(r,\ g).$$

Combining the above two inequalities gives

$$(n-1)(T(r,\ f) + T(r,\ g)) \leq \left(\frac{4}{k} + k + 3\right)(T(r,\ f) + T(r,\ g)) + S(r,\ f) + S(r,\ g). \tag{4.4}$$

Note that $n > k + 4/k + 4$, we get a contradiction from (4.4).

Case 2. Suppose that $f^n f^{(k)} g^n g^{(k)} = z^2$. Then, by Lemma 2.5, we get conclusion (ii).

Case 3. Suppose that $f^n f^{(k)} = g^n g^{(k)}$. Then we have the conclusion (i) of Theorem 1.1. This completes the proof of Theorem 1.1.

By Lemma 2.6, using the same argument as in the proof of Theorem 1.1, we can prove Theorem 1.2.

5 Proof of Theorem 1.3

Since $n \le 10$, we have $n > \max\{2k - 1, k + 4/k + 4\}$ for $1 \le k \le 5$. By Theorem 1.2, one of the following two conclusions holds:

(i) $f^n f^{(k)} = g^n g^{(k)}$;

(ii) $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, where c_3, c_4, and d are constants such that $(-1)^k (c_3 c_4)^{n+1} d^{2k} = 1$.

We only need to consider the case (1). Namely

$$f^n f^{(k)} = g^n g^{(k)}. \tag{5.1}$$

Let

$$h = g/f. \tag{5.2}$$

We claim that h is a nonzero constant.

If h is not a constant, we claim that

$$h = e^{\gamma}, \tag{5.3}$$

where γ is a nonconstant entire function. Thus we need to prove that

(a) f and g share ∞ CM;

(b) f and g share 0 CM.

We prove the above two conclusions step-by-step.

Step 1. We prove (a).

Note that f and g share ∞ IM. Suppose that z_0 is a pole of f with multiplicity p, a pole of g with multiplicity q. From (5.1) we have

$$np + p + k = nq + q + k, \tag{5.4}$$

namely

$$(n + 1)(p - q) = 0, \tag{5.5}$$

which implies $p = q$, thus f and g share ∞ CM, we get (a).

Step 2. We prove (b).

From (5.1) and the assumptions that $f^n f^{(k)}$ and $g^n g^{(k)}$ share 1 CM, $f^{(k)}$ and $g^{(k)}$ share 1 CM, f and g share ∞ IM, we can deduce f and g share 0 IM.

Suppose that z_2 is a zero of f with multiplicity m_1, a zero of g with multiplicity m_2. Then we have

$$nm_1 + m_1 - k = nm_2 + m_2 - k, \tag{5.6}$$

which implies $m_1 = m_2$. Thus f and g share 0 CM, and we get (b). Therefore, (5.3) holds. Moreover, from (5.1) we get

$$f^{(k)}/g^{(k)} = e^{n\gamma}. \tag{5.7}$$

Thus $f^{(k)}$ and $g^{(k)}$ share 0 and ∞ CM. By Lemma 2.4 and note that h is not a constant, we consider three cases.

Case 1. $f(z) = e^{az+b}$, $g(z) = e^{cz+d}$, where a, b, c, and d are constants with $ac \neq 0$. We deduce from (5.1) that

$$(a/c)^k e^{(n+1)(b-d)} e^{(n+1)(a-c)z} = 1, \tag{5.8}$$

which implies $a = c$, thus $f = t_1 g$, where t_1 is a nonzero constant, which is a contradiction since h is not a constant. Case 1 has been ruled out.

Case 2. $f(z) = \frac{a}{1-be^{\alpha(z)}}$, $g(z) = \frac{a}{e^{-\alpha(z)}-b}$, where a, b are nonzero constants, and $\alpha(z)$ is a nonconstant entire function. Thus we have $h = g/f = e^{\alpha(z)}$. From (5.1) it is easy to obtain

$$S(r,f) = S(r,g). \tag{5.9}$$

Moreover, we have from the expressions of f and g that

$$T(r,\ f) = T(r,\ e^{\alpha(z)}) + O(1) = T(r,\ h) + S(r,\ h), \quad T(r,\ g) = T(r,\ h) + S(r,\ h). \tag{5.10}$$

Combining (5.9) and (5.10) gives

$$S(r,f) = S(r,g) = S(r,h). \tag{5.11}$$

From (5.1) we have

$$h^{n+1} = \frac{gf^{(k)}}{fg^{(k)}}. \tag{5.12}$$

Note that f and g have no zero. By the first fundamental theorem and the lemma of logarithmic derivative, we deduce from (5.12) that

$$\begin{aligned}(n+1)T(r,\ h) = &\quad T(r,\ h^{n+1}) = T\left(r, \frac{gf^{(k)}}{fg^{(k)}}\right) \le T\left(r, \frac{f^{(k)}}{f}\right) + T\left(r, \frac{g^{(k)}}{g}\right) + O(1) \\ = &\quad m\left(r,\ \frac{f^{(k)}}{f}\right) + N\left(r,\ \frac{f^{(k)}}{f}\right) + m\left(r,\ \frac{g^{(k)}}{g}\right) + N\left(r,\ \frac{g^{(k)}}{g}\right) + O(1) \\ \le &\quad k\bar{N}(r,\ f) + k\bar{N}(r,\ g) + S(r,\ f) + S(r,\ g) \\ \le &\quad kT(r,\ f) + kT(r,\ g) + S(r,\ f) + S(r,\ g) \\ \le &\quad 2kT(r,\ h) + S(r,\ h) \le 10T(r,\ h) + S(r,\ h),\end{aligned} \tag{5.13}$$

which is a contradiction since $n \le 11$. Case 2 has been ruled out.

Case 3. $f(z) = a(1 - be^{cz})$, $g(z) = d(e^{-cz} - b)$, where a, b, c, and d are nonzero constants.

Then all zeros of f and g are simple, which contradicts our assumption. Case 3 has been ruled out.

Therefore, h is a constant. Since g is not a constant, we have $h \ne 0$. Let $t = 1/h$, and we have $f \equiv tg$, we deduce from (5.1) that $t^{n+1} = 1$. This completes the proof of Theorem 1.3.

Conjecture

In this section, we pose the following

Conjecture: Let f and g be two nonconstant meromorphic functions, whose zeros are of multiplicities at least k, where k is a positive integer. Let $n > \max\{2k - 1, k + 4/k + 5\}$ be a positive integer. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share 1 CM, then one of the following two conclusions holds:

(i) $f = tg$ for a constant t such that $t^{n+1} = 1$;

(ii) $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k (c_3 c_4)^{n+1} d^{2k} = 1$.

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Authors' contributions
YH completed the main part of this article, YH and XB corrected the main theorems. All the authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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