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Some normality criteria of functions related a Hayman conjecture

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Abstract

In the article, we study the normality of families of meromorphic functions concerning shared values. We consider whether a family meromorphic functions $\mathcal{F}$ is normal in D, if for every pair of functions f and g in $\mathcal{F}$, $f^n f'$ and $g^n g'$ share a nonzero value a. Two examples show that the conditions in our results are best possible in a sense.

1 Introduction and main results

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in a domain $D \subseteq \mathbf{C}$, and let a be a finite complex value. We say that f and g share a CM (or IM) in D provided that $f - a$ and $g - a$ have the same zeros counting (or ignoring) multiplicity in D. When $a = \infty$ the zeros of $f - a$ means the poles of f (see [1]). It is assumed that the reader is familiar with the standard notations and the basic results of Nevanlinna's value-distribution theory [1-4].

Bloch's principle [5] states that every condition which reduces a meromorphic function in the plane $\mathbf{C}$ to be a constant forces a family of meromorphic functions in a domain D normal. Although the principle is false in general (see [6]), many authors proved normality criterion for families of meromorphic functions corresponding to Liouville-Picard type theorem (see [4]).

It is also more interesting to find normality criteria from the point of view of shared values. In this area, Schwick [7] first proved an interesting result that a family of meromorphic functions in a domain is normal if in which every function shares three distinct finite complex numbers with its first derivative. And later, more results about normality criteria concerning shared values have emerged, for instance, (see [8-10]). In recent years, this subject has attracted the attention of many researchers worldwide.

We now first introduce a normality criterion related to a Hayman normal conjecture [11].

Theorem 1.1 *Let $\mathcal{F}$ be a meromorphic function family on domain D, $n \in \mathbf{N}$. If each function f(z) of family $\mathcal{F}$ satisfies $f^n(z) f'(z) \neq 1$, then $\mathcal{F}$ is normal in D.*

The proof of Theorem 1.1 is because of Gu [12] for $n \geq 3$, Pang [13] for $n = 2$, Chen and Fang [14] for $n = 1$. In 2004, by the ideas of shared values, Fang and Zalcman [15] obtained:

Theorem 1.2 *Let $\mathcal{F}$ be a family of meromorphic functions in D, n be a positive integer. If for each pair of functions f and g in $\mathcal{F}$, f and g share the value 0 and $f^n f'$ and $g^n g'$ share a nonzero value a in D, then $\mathcal{F}$ is normal in D.*

In 2008, Zhang [10] obtained a criterion for normality of $\mathcal{F}$ in terms of the multiplicities of the zeros and poles of the functions in $\mathcal{F}$ and use it to improve Theorem 1.2 as follows.

Theorem 1.3 *Let $\mathcal{F}$ be a family of meromorphic functions in D satisfying that all zeros and poles of $f \in \mathcal{F}$ have multiplicities at least 3. If for each pair of functions f and g in $\mathcal{F}$, f' and g' share a nonzero value a in D, then $\mathcal{F}$ is normal in D.*

Theorem 1.4 *Let $\mathcal{F}$ be a family of meromorphic functions in D, n be a positive integer. If $n \geq 2$ and for each pair of functions f and g in $\mathcal{F}$, $f^n f'$ and $g^n g'$ share a nonzero value a in D, then $\mathcal{F}$ is normal in D.*

Zhang [10] gave the following example to show that Theorem 1.4 is not true when $n = 1$, and therefore the condition $n \geq 1$ is best possible.

Example 1.1 *The family of holomorphic functions $\mathcal{F} = \{f_j(z) = \sqrt{j}(z + \frac{1}{j}) : j = 1, 2, \ldots,\}$ is not normal in $D = \{z : |z| < 1\}$ This is deduced by $f_j^{\#}(0) = \frac{j\sqrt{j}}{j+1} \to \infty$, as $j \to \infty$ and Marty's criterion* [2], *although for any $f_j(z) \in \mathcal{F}, f_j f_j' = jz + 1$. Hence, for each pair m, j, $f_m f_m'$ and $f_j f_j'$ share the value* 1.

Here $f^{\#}(\xi)$ denotes the spherical derivative

$$f^{\#}(\xi) = \frac{|f'(\xi)|}{1 + |f(\xi)|^2}.$$

In this article, we will improve Theorem 1.3 and use it to consider Theorem 1.4 when $n = 1$. Our main results are as follows:

Theorem 1.5 *Let $\mathcal{F}$ be a family of meromorphic functions in D satisfying that all zeros of $f \in \mathcal{F}$ have multiplicities at least 4 and all poles of $f \in \mathcal{F}$ are multiple. If for each pair of functions f and g in $\mathcal{F}$, f' and g' share a nonzero value a in D, then $\mathcal{F}$ is normal in D.*

Theorem 1.6 *Let $\mathcal{F}$ be a family of meromorphic functions in D satisfying that all zeros of $f \in \mathcal{F}$ are multiple. If for each pair of functions f and g in $\mathcal{F}$, ff' and gg' share a nonzero value a in D, then $\mathcal{F}$ is normal in D.*

Since normality of families of $\mathcal{F}$ and $\mathcal{F}^* = \{\frac{1}{f} | f \in \mathcal{F}\}$ is the same by the famous Marty's criterion, we obtain the following criteria from above results.

Theorem 1.7 *Let $\mathcal{F}$ be a family of meromorphic functions in D, n be a positive integer. If $n \geq 4$ and for each pair of functions f and g in $\mathcal{F}$, $f^{-n} f'$ and $g^{-n} g'$ share a nonzero value a in D, then $\mathcal{F}$ is normal in D.*

Theorem 1.8 *Let $\mathcal{F}$ be a family of meromorphic functions in D satisfying that all poles of $f \in \mathcal{F}$ are multiple. If for each pair of functions f and g in $\mathcal{F}$, $f^{-3} f'$ and $g^{-3} g'$ share a nonzero value a in D, then $\mathcal{F}$ is normal in D.*

Theorem 1.9 *Let $\mathcal{F}$ be a family of meromorphic functions in D satisfying that all zeros and poles of $f \in \mathcal{F}$ have multiplicities at least 3. If for each pair of functions f and g in $\mathcal{F}$, $f^{-2} f'$ and $g^{-2} g'$ share a nonzero value a in D, then $\mathcal{F}$ is normal in D.*

Theorem 1.10 *Let $\mathcal{F}$ be a family of meromorphic functions in D satisfying that all poles of $f \in \mathcal{F}$ have multiplicities at least 4 and all zeros of $f \in \mathcal{F}$ are multiple. If for each pair of functions f and g in $\mathcal{F}$, $f^{-2} f'$ and $g^{-2} g'$ share a nonzero value a in D, then $\mathcal{F}$ is normal in D.*

Example 1.2 *The family of holomorphic functions $\mathcal{F} = \{f_j(z) = je^z - j - 1 : j = 1, 2, \ldots,\}$ is not normal in $D = \{z : |z| < 1\}$ This is deduced by $f_j^{\#}(0) = j \to \infty$, as $j \to \infty$ and Marty's* *criterion* [2], *although for any* $f_j(z) \in \mathcal{F}$, $\frac{f'_j}{f_j} = 1 + \frac{j+1}{je^z - j - 1} \neq 1$. *Hence, for each pair m, j,* $\frac{f'_j}{f_j}$and $\frac{f'_j}{f_j}$ *share the value* 1.

Remark 1.11 Example 1.1 shows that the condition that all zeros of $f \in \mathcal{F}$ are multiple in Theorem 1.6 is best possible. Both Examples 1.1 and 1.2 show that above results are best possible in a sense.

2 Preliminary lemmas

To prove our result, we need the following lemmas. The first is the extended version of Zalcman's [16] concerning normal families.

Lemma 2.1 [17]*Let* $\mathcal{F}$ *be a family of meromorphic functions on the unit disc satisfying all zeros of functions in* $\mathcal{F}$ *have multiplicity* $\geq p$ *and all poles of functions in* $\mathcal{F}$ *have multiplicity* $\geq q$. *Let* α *be a real number satisfying* $-q < \alpha < p$. *Then,* $\mathcal{F}$ *is not normal at* 0 *if and only if there exist*

(a) a number $0 < r < 1$;
(b) points z_n *with* $|z_n| < r$;
(c) functions $f_n \in \mathcal{F}$;
(d) positive numbers $\rho_n \to 0$

such that $g_n(\zeta) := \rho^{-\alpha} f_n(z_n + \rho_n\zeta)$ *converges spherically uniformly on each compact subset of* **C** *to a nonconstant meromorphic function* $g(\zeta)$, *whose all zeros of functions in* $\mathcal{F}$ *have multiplicity* $\geq p$ *and all poles of functions in* $\mathcal{F}$ *have multiplicity* $\geq q$ *and order is at most 2.*

Remark 2.2 If $\mathcal{F}$ is a family of holomorphic functions on the unit disc in Lemma 2.1, then $g(\zeta)$ is a nonconstant entire function whose order is at most 1.

The order of g is defined using Nevanlinna's characteristic function $T(r, g)$:

$$\rho(g) = \lim_{r\to\infty} \sup \frac{\log T(r,g)}{\log r}.$$

Lemma 2.3 [18] or [19] Let $f(z)$ be a meromorphic function and $c \in \mathbf{C}\backslash\{0\}$. If $f(z)$ has neither simple zero nor simple pole, and $f'(z) \neq c$, then $f(z)$ is constant.

Lemma 2.4 [20] Let $f(z)$ be a transcendental meromorphic function of finite order in **C**, and have no simple zero, then $f'(z)$ assumes every nonzero finite value infinitely often.

3 Proof of the results

Proof of Theorem 1.5 Suppose that $\mathcal{F}$ is not normal in D. Then, there exists at least one point z_0 such that $\mathcal{F}$ is not normal at the point z_0. Without loss of generality, we assume that $z_0 = 0$. By Lemma 2.1, there exist points $z_j \to 0$, positive numbers $\rho_j \to 0$ and functions $f_j \in \mathcal{F}$ such that

$$g_j(\xi) = \rho_j^{-1} f_j(z_j + \rho_j\xi) \Rightarrow g(\xi) \tag{3.1}$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function in **C** satisfying all its zeros have multiplicities at least 4 and all its poles are multiple. Moreover, the order of g is ≤ 2.

From (3.1), we know

$$g_j'(\xi) = f_j'(z_j + \rho_j\xi) \Rightarrow g'(\xi)$$

and

$$\begin{aligned} f'_j(z_j + \rho_j\xi) - a &= g'_j(\xi) - a \\ &\Rightarrow g'(\xi) - a \end{aligned} \tag{3.2}$$

also locally uniformly with respect to the spherical metric.

If $g' - a \equiv 0$, then $g \equiv a\xi + c$, where c is a constant. This contradicts with g satisfying all its zeros have multiplicities at least 4. Hence, $g' - a$ ⊠ 0.

If $g' - a \neq 0$, by Lemma 2.3, then g is also a constant which is a contradiction with g being not any constant. Hence, $g' - a$ is a nonconstant meromorphic function and has at least one zero.

Next, we prove that $g' - a$ has just a unique zero. By contraries, let ξ_0 and ξ_0^* be two distinct zeros of $g' - a$, and choose δ (> 0) small enough such that $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \phi$ where $D(\xi_0, \delta) = \{\xi : |\xi - \xi_0| < \delta\}$ and $D(\xi_0^*, \delta) = \{\xi : |\xi - \xi_0^*| < \delta\}$. From (3.2), by *Hurwitz's* theorem, there exist points $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ such that for sufficiently large j

$$f_j'(z_j + \rho_j\xi_j) - a = 0, \quad f_j'(z_j + \rho_j\xi_j^*) - a = 0.$$

By the hypothesis that for each pair of functions f and g in $\mathcal{F}$, $f' - a$ and $g' - a$ share 0 in D, we know that for any positive integer m

$$f_m'(z_j + \rho_j\xi_j) - a = 0, \quad f_m'(z_j + \rho_j\xi_j^*) - a = 0.$$

Fix m, take $j \to \infty$, and note $z_j + \rho_j\xi_j \to 0$, $z_j + \rho_j\xi_j^* \to 0$, then $f_m'(0) - a = 0$. Since the zeros of $f_m' - a$ have no accumulation point, so

$$z_j + \rho_j\xi_j = 0, z_j + \rho_j\xi_j^* = 0.$$

Hence,$\xi_j = -\frac{z_j}{\rho_j}$, $\xi_j^* = -\frac{z_j}{\rho_j}$. This contradicts with $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ and $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \phi$. Hence, $g' - a$ has just a unique zero, which can be denoted by ξ_0. By Lemma 2.4, g is not any transcendental function.

If g is a nonconstant polynomial, then $g' - a = A(\xi - \xi_0)^l$, where A is a nonzero constant, l is a positive integer. Thus, $g' = A(\xi - \xi_0)^l$ and $g'' = Al(\xi - \xi_0)^{l-1}$. Noting that the zeros of g are of multiplicity ≥ 4, and g'' has only one zero ξ_0, we see that g has only the same zero ξ_0 too. Hence, $g'(\xi_0) = 0$ which contradicts with $g'(\xi_0) = a \neq 0$. Therefore, g is a rational function which is not polynomial, and $g' + a$ has just a unique zero ξ_0.

Next, we prove that there exists no rational function such as g. Now, we can set

$$g(\xi) = A\frac{(\xi - \xi_1)^{m_1}(\xi - \xi_2)^{m_2}\cdots(\xi - \xi_s)^{m_s}}{(\xi - \eta_1)^{n_1}(\xi - \eta_2)^{n_2}\cdots(\xi - \eta_t)^{n_t}}, \tag{3.3}$$

where A is a nonzero constant, $s \geq 1$, $t \geq 1$, $m_i \geq 4$ (i = 1, 2, ..., s), $n_j \geq 2$ (j = 1, 2, ..., t). For stating briefly, denote

$$m = m_1 + m_2 + \cdots + m_s \geq 4s, \quad N = n_1 + n_2 + \cdots + n_t \geq 2t. \tag{3.4}$$

From (3.3), then

$$g'(\xi) = \frac{A(\xi-\xi_1)^{m_1-1}(\xi-\xi_2)^{m_2-1}\cdots(\xi-\xi_s)^{m_s-1}h(\xi)}{(\xi-\eta_1)^{n_1+1}(\xi-\eta_2)^{n_2+1}\cdots(\xi-\eta_t)^{n_t+1}} = \frac{p_1(\xi)}{q_1(\xi)}, \tag{3.5}$$

where

$$\begin{aligned} h(\xi) &= (m-N-t)\xi^{s+t-1} + a_{s+t-2}\,\xi^{s+t-2} + \cdots + a_0, \\ p_1(\xi) &= A(\xi-\xi_1)^{m_1-1}(\xi-\xi_2)^{m_2-1}\cdots(\xi-\xi_s)^{m_s-1}h(\xi), \\ q_1(\xi) &= (\xi-\eta_1)^{n_1+1}(\xi-\eta_2)^{n_2+1}\cdots(\xi-\eta_t)^{n_t+1} \end{aligned} \tag{3.6}$$

are polynomials. Since $g'(\xi) + a$ has only a unique zero ξ_0, set

$$g'(\xi) + a = \frac{B(\xi-\xi_0)^l}{(\xi-\eta_1)^{n_1+1}(\xi-\eta_2)^{n_2+1}\cdots(\xi-\eta_t)^{n_t+1}}, \tag{3.7}$$

where B is a nonzero constant, so

$$g''(\xi) = \frac{(\xi-\xi_0)^{l-1}p_2(\xi)}{(\xi-\eta_1)^{n_1+2}(\xi-\eta_2)^{n_2+2}\cdots(\xi-\eta_t)^{n_t+2}}, \tag{3.8}$$

where $p_2(\xi) = B(l - N - 2t)\,\xi^t + b_{t-1}\xi^{t-1} + \ldots + b_0$ is a polynomial. From (3.5), we also have

$$g''(\xi) = \frac{(\xi-\xi_1)^{m_1-2}(\xi-\xi_2)^{m_2-2}\cdots(\xi-\xi_s)^{m_s-2}p_3(\xi)}{(\xi-\eta_1)^{n_1+2}(\xi-\eta_2)^{n_2+2}\cdots(\xi-\eta_t)^{n_t+2}}, \tag{3.9}$$

where $p_3(\xi)$ is also a polynomial.

We use deg(p) to denote the degree of a polynomial $p(\xi)$.

From (3.5), (3.6) then

$$\deg(h) \le s+t-1, \quad \deg(p_1) \le m+t-1, \quad \deg(q_1) = N+t. \tag{3.10}$$

Similarly from (3.8), (3.9) and noting (3.10) then

$$\deg(p_2) \le t, \tag{3.11}$$

$$\deg(p_3) \le \deg(p_1) + t - 1 - (m-2s) \le 2t + 2s - 2. \tag{3.12}$$

Note that $m_i \ge 4$ ($i = 1, 2, \ldots, s$), it follows from (3.5) and (3.7) that $g'(\xi_0) = 0$ ($i = 1, 2, \ldots, s$) and $g'(\xi_0) = a \ne 0$. Thus, $\xi_0 \ne \xi_i$ ($i = 1, 2, \ldots, s$), and then $(\xi - \xi_0)^{l-1}$ is a factor of $p_3(\xi)$. Hence, we get that $l - 1 \le \deg(p_3)$. Combining (3.8) and (3.9), we also have $m - 2s = \deg(p_2) + l - 1 - \deg(p_3) \le \deg(p_2)$. By (3.11), we obtain

$$m - 2s \le \deg(p_2) \le t. \tag{3.13}$$

Since $m \ge 4s$, we know by (3.13) that

$$2s \le t. \tag{3.14}$$

If $l \ge N + t$, by (3.12), then

$$3t - 1 \le N + t - 1 \le l - 1 \le \deg(p_3) \le 2t + 2s - 2.$$

Noting (3.14), we obtain1 ≤ 0, a contradiction.

If $l < N + t$, from (3.5) and (3.7), then $\deg(p_1) = \deg(q_1)$. Noting that $\deg(h) \leq s + t - 1$, $\deg(p_1) \leq m + t - 1$ and $\deg(q_1) = N + t$, hence $m \geq N + 1 \geq 2t + 1$. By (3.13), then $2t + 1 \leq 2s + t$. From (3.14), we obtain $1 \leq 0$, a contradiction.

The proof of Theorem 1.5 is complete.

Proof of Theorem 1.6 Set $\mathcal{F}^* = \{\frac{f^2}{2} | f \in \mathcal{F}\}$.

Noting that all zeros of $g \in \mathcal{F}^*$ have multiplicities at least 4 and all poles of $g \in \mathcal{F}^*$ are multiple, and for each pair of functions f and g in $\mathcal{F}^*$, f' and g' share a nonzero value a in D, we know that $\mathcal{F}^*$ is normal in D by Theorem 1.5. Therefore, $\mathcal{F}$ is normal in D.

The proof of Theorem 1.6 is complete.

Proof of Theorem 1.7 Set $\mathcal{F}^* = \{\frac{1}{f} | f \in \mathcal{F}\}, F := \frac{1}{f}$.

Noting that $f^{-n} f' = - F' F^{n-2}$ and $n \geq 4$ implies $n - 2 \geq 2$, by Theorem 1.4, we know that $\mathcal{F}^*$ is normal in D.

Since normality of families of $\mathcal{F}$ and $\mathcal{F}^* = \{\frac{1}{f} | f \in \mathcal{F}\}$ is the same by the famous Marty's criterion,

Therefore, $\mathcal{F}$ is normal in D.

The proof of Theorem 1.7 is complete.

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Authors' contributions

WY and JL carried out the design of the study and performed the analysis. BZ participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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