Chu *et al. Journal of Inequalities and Applications* 2011, **2011**:94
http://www.journalofinequalitiesandapplications.com/content/2011/1/94

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

A best-possible double inequality between Seiffert and harmonic means

Yu-Ming Chu[1*], Miao-Kun Wang[1] and Zi-Kui Wang[2]

* Correspondence:
chuyuming2005@yahoo.com.cn
[1]Department of Mathematics, Huzhou Teachers College, Huzhou, 313000, China
Full list of author information is available at the end of the article

Abstract

In this paper, we establish a new double inequality between the Seiffert and harmonic means.

The achieved results is inspired by the papers of Sándor (Arch. Math., 76, 34-40, 2001) and Hästö (Math. Inequal. Appl., 7, 47-53, 2004), and the methods from Wang et al. (J. Math. Inequal., 4, 581-586, 2010). The inequalities we obtained improve the existing corresponding results and, in some sense, are optimal.

2010 Mathematics Subject Classification: 26E60.

Keywords: harmonic mean, Seiffert mean, inequality

1 Introduction

For a, b >0 with $a \neq b$, the Seiffert mean $P(a, b)$ was introduced by Seiffert [1] as follows:

$$P(a,b) = \frac{a-b}{4\arctan\sqrt{a/b}-\pi}. \tag{1.1}$$

Recently, the bivariate mean values have been the subject of intensive research. In particular, many remarkable inequalities for the Seiffert mean can be found in the literature [1-9].

Let $H(a, b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a, b) = (a - b)/(\log a - \log b)$, $I(a, b) = 1/e\,(b^b/a^a)^{1/(b-a)}$, $A(a, b) = (a+b)/2$, $C(a, b) = (a^2+b^2)/(a+b)$, and $M_p(a, b) = ((a^p + b^p)/2)^{1/p}$ $(p \neq 0)$ and $M_0(a,b) = \sqrt{ab}$ be the harmonic, geometric, logarithmic, identric, arithmetic, contraharmonic, and p-th power means of two different positive numbers a and b, respectively. Then, it is well known that

$$\begin{aligned}\min\{a,b\} < H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) < L(a,b)\\ < I(a,b) < A(a,b) = M_1(a,b) < C(a,b) < \max\{a,b\}.\end{aligned}$$

For all a, b >0 with $a \neq b$, Seiffert [1] established that $L(a, b) < P(a, b) < I(a, b)$; Jagers [4] proved that $M_{1/2}(a, b) < P(a, b) < M_{2/3}(a, b)$ and $M_{2/3}(a, b)$ is the best-possible upper power mean bound for the Seiffert mean $P(a, b)$; Seiffert [7] established that $P(a, b) > A(a, b)G(a, b)/L(a, b)$ and $P(a, b) > 2A(a, b)/\pi$; Sándor [6] presented that $(A(a,b)+G(a,b))/2 < P(a,b) < \sqrt{A(a,b)(A(a,b)+G(a,b))/2}$ and $\sqrt[3]{A^2(a,b)G(a,b)} < P(a,b) < (G(a,b)+2A(a,b))/3$; Hästö [3] proved that $P(a, b) >$

$M_{\log 2/\log \pi}(a, b)$ and $M_{\log 2/\log \pi}(a, b)$ is the best-possible lower power mean bound for the Seiffert mean $P(a, b)$.

Very recently, Wang and Chu [8] found the greatest value α and the least value β such that the double inequality $A^{\alpha}(a, b)H^{1-\alpha}(a, b) < P(a, b) < A^{\beta}(a, b)H^{1-\beta}(a, b)$ holds for $a, b > 0$ with $a \neq b$; For any $\alpha \in (0, 1)$, Chu et al. [10] presented the best-possible bounds for $P^{\alpha}(a, b)G^{1-\alpha}(a, b)$ in terms of the power mean; In [2], the authors proved that the double inequality $\alpha A(a, b) + (1 - \alpha)H(a, b) < P(a, b) < \beta A(a, b) + (1 - \beta)H(a, b)$ holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 2/\pi$ and $\beta \geq 5/6$; Liu and Meng [5] proved that the inequalities

$$\alpha_1 C(a, b) + (1 - \alpha_1)G(a, b) < P(a, b) < \beta_1 C(a, b) + (1 - \beta_1)G(a, b)$$

and

$$\alpha_2 C(a, b) + (1 - \alpha_2)H(a, b) < P(a, b) < \beta_2 C(a, b) + (1 - \beta_2)H(a, b)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 2/9$, $\beta_1 \geq 1/\pi$, $\alpha_2 \leq 1/\pi$ and $\beta_2 \geq 5/12$.

For fixed $a, b > 0$ with $a \neq b$ and $x \in [0, 1/2]$, let

$$h(x) = H(xa + (1 - x)b, xb + (1 - x)a).$$

Then, it is not difficult to verify that $h(x)$ is continuous and strictly increasing in $[0, 1/2]$. Note that $h(0) = H(a, b) < P(a, b)$ and $h(1/2) = A(a, b) > P(a, b)$. Therefore, it is natural to ask what are the greatest value α and least value β in $(0, 1/2)$ such that the double inequality $H(\alpha a + (1 - \alpha)b, \alpha b + (1 - \alpha)a) < P(a, b) < H(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a)$ holds for all $a, b > 0$ with $a \neq b$. The main purpose of this paper is to answer these questions. Our main result is the following Theorem 1.1.

Theorem 1.1. If $\alpha, \beta \in (0, 1/2)$, then the double inequality

$$H(\alpha a + (1 - \alpha)b, \alpha b + (1 - \alpha)a) < P(a, b) < H(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq (1 - \sqrt{1 - 2/\pi})/2$ and $\beta \geq (6 - \sqrt{6})/12$.

2 Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\lambda = (1 - \sqrt{1 - 2/\pi})/2$ and $\mu = (6 - \sqrt{6})/12$. We first prove that inequalities

$$P(a, b) > H(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a) \tag{2.1}$$

and

$$P(a, b) < H(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a) \tag{2.2}$$

hold for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = \sqrt{a/b} > 1$ and $p \in (0, 1/2)$; then, from (1.1), one has

$$\begin{aligned} &H(pa+(1-p)b, pb+(1-p)a)-P(a,b) \\ &= \frac{2[pt^2+(1-p)][(1-p)t^2+p]}{t^2+1} - \frac{t^2-1}{4\arctan t - \pi} \\ &= \frac{2[pt^2+(1-p)][(1-p)t^2+p]}{(t^2+1)(4\arctan t-\pi)} \\ &\times \left\{ 4\arctan t - \frac{t^4-1}{2[pt^2+(1-p)][(1-p)t^2+p]} - \pi \right\}. \end{aligned} \tag{2.3}$$

Let

$$f(t) = 4\arctan t - \frac{t^4-1}{2[pt^2+(1-p)][(1-p)t^2+p]} - \pi, \tag{2.4}$$

then, simple computations lead to

$$f(1) = 0, \tag{2.5}$$

$$\lim_{t\to+\infty} f(t) = \pi - \frac{1}{2p(1-p)} \tag{2.6}$$

and

$$f'(t) = \frac{f_1(t)}{(t^2+1)[p(1-p)t^4+(2p^2-2p+1)t^2+p(1-p)]^2}, \tag{2.7}$$

where

$$\begin{aligned} f_1(t) = {} & 4p^2(1-p)^2t^8 - (2p^2-2p+1)t^7 + 8p(1-p)(2p^2-2p+1)t^6 \\ & +(2p^2-2p-1)t^5 + 4(6p^4-12p^3+10p^2-4p+1)t^4 \\ & +(2p^2-2p-1)t^3 + 8p(1-p)(2p^2-2p+1)t^2 \\ & -(2p^2-2p+1)t + 4p^2(1-p)^2. \end{aligned} \tag{2.8}$$

Note that

$$f_1(1) = 0, \tag{2.9}$$

$$\lim_{t\to+\infty} f_1(t) = +\infty, \tag{2.10}$$

$$\begin{aligned} f'_1(t) = {} & 32p^2(1-p)^2t^7 - 7(2p^2-2p+1)t^6 + 48p(1-p)(2p^2-2p+1)t^5 \\ & +5(2p^2-2p-1)t^4 + 16(6p^4-12p^3+10p^2-4p+1)t^3 \\ & +3(2p^2-2p-1)t^2 + 16p(1-p)(2p^2-2p+1)t \\ & -(2p^2-2p+1), \end{aligned}$$

$$f_1'(1) = 0, \tag{2.11}$$

$$\lim_{t\to+\infty} f_1'(t) = +\infty. \tag{2.12}$$

Let $f_2(t) = f_1''(t)/2$, $f_3(t) = f_2'(t)/3$, $f_4(t) = f_3'(t)/4$, $f_5(t) = f_4'(t)/5$, $f_6(t) = f_5'(t)/6$ and $f_7(t) = f_6'(t)/7$. Then, simple computations lead to

$$f_2(t) = 112p^2(1-p)^2t^6 - 21(2p^2-2p+1)t^5 + 120p(1-p)(2p^2-2p+1)t^4 \\ +10(2p^2-2p-1)t^3 + 24(6p^4-12p^3+10p^2-4p+1)t^2 \\ +3(2p^2-2p-1)t + 8p(1-p)(2p^2-2p+1),$$

$$f_2(1) = -2(24p^2-24p+5), \tag{2.13}$$

$$\lim_{t\to+\infty} f_2(t) = +\infty, \tag{2.14}$$

$$f_3(t) = 224p^2(1-p)^2t^5 - 35(2p^2-2p+1)t^4 + 160p(1-p)(2p^2-2p+1)t^3 \\ +10(2p^2-2p-1)t^2 + 16(6p^4-12p^3+10p^2-4p+1)t \\ +(2p^2-2p-1),$$

$$f_3(1) = -6(24p^2-24p+5), \tag{2.15}$$

$$\lim_{t\to+\infty} f_3(t) = +\infty, \tag{2.16}$$

$$f_4(t) = 280p^2(1-p)^2t^4 - 35(2p^2-2p+1)t^3 + 120p(1-p)(2p^2-2p+1)t^2 \\ +5(2p^2-2p-1)t + 4(6p^4-12p^3+10p^2-4p+1),$$

$$f_4(1) = 4(16p^4-32p^3-25p^2+41p-9), \tag{2.17}$$

$$\lim_{t\to+\infty} f_4(t) = +\infty, \tag{2.18}$$

$$f_5(t) = 224p^2(1-p)^2t^3 - 21(2p^2-2p+1)t^2 + 48p(1-p)(2p^2-2p+1)t \\ +(2p^2-2p-1),$$

$$f_5(1) = 2(64p^4-128p^3+20p^2+44p-11), \tag{2.19}$$

$$\lim_{t\to+\infty} f_5(t) = +\infty, \tag{2.20}$$

$$f_6(t) = 112p^2(1-p)^2t^2 - 7(2p^2-2p+1)t \\ +8p(1-p)(2p^2-2p+1), \tag{2.21}$$

$$f_6(1) = 96p^4-192p^3+74p^2+22p-7, \tag{2.22}$$

$$\lim_{t\to+\infty} f_6(t) = +\infty, \tag{2.23}$$

$$f_7(t) = 32p^2(1-p)^2t - (2p^2-2p+1) \tag{2.24}$$

and

$$f_7(1) = 32p^4-64p^3+30p^2+2p-1. \tag{2.25}$$

We divide the proof into two cases.

Case 1. $p = \lambda = (1 - \sqrt{1 - 2/\pi})/2$. Then equations (2.6), (2.13), (2.15), (2.17), (2.19), (2.22) and (2.25) become

$$\lim_{t \to +\infty} f(t) = 0, \tag{2.26}$$

$$f_2(1) = -\frac{2(5\pi - 12)}{\pi} < 0, \tag{2.27}$$

$$f_3(1) = -\frac{6(5\pi - 12)}{\pi} < 0, \tag{2.28}$$

$$f_4(1) = -\frac{2(18\pi^2 - 41\pi - 8)}{\pi^2} < 0, \tag{2.29}$$

$$f_5(1) = -\frac{2(11\pi^2 - 22\pi - 16)}{\pi^2} < 0, \tag{2.30}$$

$$f_6(1) = -\frac{7\pi^2 - 11\pi - 24}{\pi^2} < 0 \tag{2.31}$$

and

$$f_7(1) = \frac{\pi + 8 - \pi^2}{\pi^2} > 0. \tag{2.32}$$

From (2.24), we clearly see that $f_7(t)$ is strictly increasing in $[1, +\infty)$, and then (2.32) leads to the conclusion that $f_7(t) > 0$ for $t \in [1, +\infty)$. Thus, $f_6(t)$ is strictly increasing in $[1, +\infty)$.

It follows from (2.23) and (2.31) together with the monotonicity of $f_6(t)$ that there exists $t_1 > 1$ such that $f_6(t) < 0$ for $t \in (1, t_1)$ and $f_6(t) > 0$ for $t \in (t_1, +\infty)$. Thus, $f_5(t)$ is strictly decreasing in $[1, t_1]$ and strictly increasing in $[t_1, +\infty)$.

From (2.20) and (2.30), together with the piecewise monotonicity of $f_5(t)$, we clearly see that there exists $t_2 > t_1 > 1$ such that $f_4(t)$ is strictly decreasing in $[1, t_2]$ and strictly increasing in $[t_2, +\infty)$. Then, equation (2.18) and inequality (2.29) lead to the conclusion that there exists $t_3 > t_2 > 1$ such that $f_3(t)$ is strictly decreasing in $[1, t_3]$ and strictly increasing in $[t_3, +\infty)$.

It follows from (2.16) and (2.28) together with the piecewise monotonicity of $f_3(t)$ we conclude that there exists $t_4 > t_3 > 1$ such that $f_2(t)$ is strictly decreasing in $[1, t_4]$ and strictly increasing in $[t_4, +\infty)$. Then, equation (2.14) and inequality (2.27) lead to the conclusion that there exists $t_5 > t_4 > 1$ such that $f_1'(t)$ is strictly decreasing in $[1, t_5]$ and strictly increasing in $[t_5, +\infty)$.

From equations (2.11) and (2.12), together with the piecewise monotonicity of $f_1'(t)$, we know that there exists $t_6 > t_5 > 1$ such that $f_1(t)$ is strictly decreasing in $[1, t_6]$ and strictly increasing in $[t_6, +\infty)$. Then, equations (2.7)-(2.10) lead to the conclusion that there exists $t_7 > t_6 > 1$ such that $f(t)$ is strictly decreasing in $[1, t_7]$ and strictly increasing in $[t_7, +\infty)$.

Therefore, inequality (2.1) follows from equations (2.3)-(2.5) and (2.26) together with the piecewise monotonicity of $f(t)$.

Case 2. $p = \mu = (6 - \sqrt{6})/12$. Then, equations (2.13), (2.15), (2.17), (2.19) and (2.21) become

$$f_2(1) = 0, \tag{2.33}$$

$$f_3(1) = 0, \tag{2.34}$$

$$f_4(1) = \frac{17}{18} > 0, \tag{2.35}$$

$$f_5(1) = \frac{17}{9} > 0 \tag{2.36}$$

and

$$f_6(t) = \frac{1}{36}(175t^2 - 147t + 35) > 0 \tag{2.37}$$

for $t > 1$.

From inequality (2.37), we know that $f_5(t)$ is strictly increasing in $[1, +\infty)$, and then inequality (2.36) leads to the conclusion that $f_5(t) > 0$ for $t \in [1, +\infty)$. Thus, $f_4(t)$ is strictly increasing in $[1, +\infty)$.

It follows from inequality (2.35) and the monotonicity of $f_4(t)$ that $f_3(t)$ is strictly increasing in $[1, +\infty)$.

Therefore, inequality (2.2) follows easily from equations (2.3)-(2.5), (2.7), (2.9), (2.11), (2.33), and (2.34) together with the monotonicity of $f_3(t)$.

Next, we prove that $\lambda = (1 - \sqrt{1 - 2/\pi})/2$ is the best-possible parameter such that inequality (2.1) holds for all a, $b > 0$ with $a \neq b$. In fact, if $(1 - \sqrt{1 - 2/\pi})/2 = \lambda < p < 1/2$, then equation (2.6) leads to

$$\lim_{t \to +\infty} f(t) = \pi - \frac{1}{2p(1 - p)} > 0. \tag{2.38}$$

Inequality (2.38) implies that there exists $T = T(p) > 1$ such that

$$f(t) > 0 \tag{2.39}$$

for $t \in (T, +\infty)$.

From equations (2.3) and (2.4), together with inequality (2.39), we clearly see that $P(a, b) < H(pa + (1 - p)b, pb + (1 - p)a)$ for $a/b \in (T^2, +\infty)$.

Finally, we prove that $\mu = (6 - \sqrt{6})/12$ is the best-possible parameter such that inequality (2.2) holds for all a, $b > 0$ with $a \neq b$. In fact, if $0 < p < \mu = (6 - \sqrt{6})/12$, then equation (2.13) leads to

$$f_2(1) = -2(24p^2 - 24p + 5) < 0. \tag{2.40}$$

Inequality (2.40) implies that there exists $\delta = \delta(p) > 0$ such that

$$f_2(t) < 0 \tag{2.41}$$

for $t \in (1, 1 + \delta)$.

Therefore, $P(a, b) > H(pa + (1 - p)b, pb + (1 - p)a)$ for $a/b \in (1, (1 + \delta)^2)$ follows from equations (2.3)-(2.5), (2.7), (2.9), and (2.11) together with inequality (2.41).

Acknowledgements
This study is partly supported by the Natural Science Foundation of China (Grant no. 11071069), the Natural Science Foundation of Hunan Province (Grant no. 09JJ6003), and the Innovation Team Foundation of the Department of Education of Zhejiang Province(Grant no. T200924).

Author details
[1]Department of Mathematics, Huzhou Teachers College, Huzhou, 313000, China [2]Department of Mathematics, Hangzhou Normal University, Hangzhou, 310012, China

Authors' contributions
Y-MC provided the main idea in this paper. M-KW carried out the proof of inequality (2.1) in this paper. Z-KW carried out the proof of inequality (2.2) in this paper. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 12 June 2011 Accepted: 26 October 2011 Published: 26 October 2011

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doi:10.1186/1029-242X-2011-94
Cite this article as: Chu *et al.*: A best-possible double inequality between Seiffert and harmonic means. *Journal of Inequalities and Applications* 2011 **2011**:94.