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Some strong limit theorems for arrays of rowwise negatively orthant-dependent random variables

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Abstract

In this article, the strong limit theorems for arrays of rowwise negatively orthant-dependent random variables are studied. Some sufficient conditions for strong law of large numbers for an array of rowwise negatively orthant-dependent random variables without assumptions of identical distribution and stochastic domination are presented. As an application, the Chung-type strong law of large numbers for arrays of rowwise negatively orthant-dependent random variables is obtained.

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1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$ with value in a real space $\mathbb{R}$. We say that the sequence $\{X_n, n \geq 1\}$ satisfies the strong law of large numbers if there exist some increasing sequence $\{a_n, n \geq 1\}$ and some sequence $\{c_n, n \geq 1\}$ such that

$$\frac{1}{a_n}\sum_{i=1}^{n}(X_i - c_i) \to 0 \quad \text{a.s. as } n \to \infty.$$

Many authors have extended the strong law of large numbers for sequences of random variables to the case of triangular array of random variables and arrays of rowwise random variables. For more details about the strong law of large numbers for triangular array of random variables and arrays of rowwise random variables, one can refer to Gut [1], and so forth. In the case of independence, Hu and Taylor [2] proved the following strong law of large numbers.

Theorem 1.1. *Let $\{X_{ni} : 1 \leq i \leq n, n \geq 1\}$ be a triangular array of rowwise independent random variables. Let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $g(t)$ be a positive, even function such that $g(|t|)/|t|^p$ is an increasing function of $|t|$ and $g(|t|)/|t|^{p+1}$ is a decreasing function of $|t|$, respectively, that is,*

$$\frac{g(|t|)}{|t|^p}\uparrow, \quad \frac{g(|t|)}{|t|^{p+1}}\downarrow \quad as\ |t| \ \uparrow$$

for some nonnegative integer p. If $p \geq 2$ and

$$EX_{ni} = 0,$$
$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} E \frac{g(|X_{ni}|)}{g(a_n)} < \infty,$$
$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} E \left(\frac{X_{ni}}{a_n} \right)^2 \right)^{2k} < \infty,$$

where k is a positive integer, then

$$\frac{1}{a_n} \sum_{i=1}^{n} X_{ni} \to 0 \quad a.s.$$

In this article, we will consider the strong law of large numbers for arrays of rowwise negatively associated (NA) random variables. A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be negatively orthant dependent (NOD) if

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$. An infinite sequence $\{X_n, n \geq 1\}$ is said to be NOD if every finite subcollection is NOD.

An array of random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ is called rowwise NOD random variables if for every $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is a sequence of NOD random variables.

The concept of NOD sequence was introduced by Joag-Dev and Proschan [3]. Obviously, independent random variables are NOD. Joag-Dev and Proschan [3] pointed out that NA (one can refer to Joag-Dev and Proschan [3]) random variables are NOD. They also presented an example in which $X = (X_1, X_2, X_3, X_4)$ possesses NOD, but does not possess NA. So we can see that NOD is weaker than NA. A number of limit theorems for NOD random variables have been established by many authors. We refer to Volodin [4] for the Kolmogorov exponential inequality, Asadian et al. [5] for the Rosental's-type inequality, Amini et al. [6,7], Klesov et al. [8], and Li et al. [9] for almost sure convergence, Amini and Bozorgnia [10,11], Kuczmaszewska [12], Taylor et al. [13], Zarei and Jabbari [14] and Wu [15] for complete convergence, and so on.

The main purpose of this article is to study the strong limit theorems for arrays of rowwise NOD random variables. As an application, the Chung-type strong law of large numbers for arrays of rowwise NOD random variables is obtained. We will give some sufficient conditions for strong law of large numbers for an array of rowwise NOD random variables without assumptions of identical distribution and stochastic domination. The results presented in this article are obtained using the truncated method and the Rosental's-type inequality of NOD random variables.

The main results of this article are depending on the following lemmas:

Lemma 1.1 *(cf. Bozorgnia et al.* [16]). *Let random variables $X_1, X_2, \ldots, X_n$ be NOD, $f_1, f_2, \ldots, f_n$ be all nondecreasing (or all nonincreasing) functions, then random variables $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are NOD.*

Lemma 1.2 *(cf. Asadian et al.* [5]*). Let $p \geq 2$ and $\{X_n, n \geq 1\}$ be a sequence of NOD random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for every $n \geq 1$. Then, there exists a positive constant $C = C(p)$ depending only on p such that for every $n \geq 1$*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leq C\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}.$$

Throughout the article, let $I(A)$ be the indicator function of the set A. C denotes a positive constant which may be different in various places.

2 Main results

In this section, we will give some sufficient conditions for strong law of large numbers for an array of rowwise NOD random variables without assumptions of identical distribution and stochastic domination. Our main results are as follows.

Theorem 2.1. *Let $\{X_{ni} : i \geq 1, n \geq 1\}$ be an array of rowwise NOD random variables and $\{a_n, n \geq 1\}$ be a sequence of positive real numbers. Let $\{g_n(t), n \geq 1\}$ be a sequence of positive, even functions such that $g_n(|t|)$ is an increasing function of $|t|$ and $g_n(|t|)/|t|$ is a decreasing function of $|t|$ for every $n \geq 1$, respectively, that is*

$$g_n(|t|) \uparrow, \quad \frac{g_n(|t|)}{|t|} \downarrow \quad \text{as } |t| \uparrow.$$

If

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{Eg_n(|X_{ni}|)}{g_n(a_n)} < \infty, \tag{2.1}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\left|\frac{1}{a_n}\sum_{i=1}^{n} X_{ni}\right| > \varepsilon\right) < \infty. \tag{2.2}$$

Proof. For fixed $n \geq 1$, define

$$X_i^{(n)} = -a_n I(X_{ni} < -a_n) + X_{ni} I(|X_{ni}| \leq a_n) + a_n I(X_{ni} > a_n), \quad i \geq 1,$$
$$T_j^{(n)} = \frac{1}{a_n}\sum_{i=1}^{j}\left(X_i^{(n)} - EX_i^{(n)}\right), \quad j = 1, 2, \ldots, n.$$

By Lemma 1.1, we can see that for fixed $n \geq 1$, $\{X_i^{(n)}, i \geq 1\}$is still a sequence of NOD random variables. It is easy to check that for any $\varepsilon > 0$,

$$\left(\left|\frac{1}{a_n}\sum_{i=1}^{n} X_{ni}\right| > \varepsilon\right) \subset \left(\max_{1\leq i\leq n}|X_{ni}| > a_n\right) \cup \left(\left|\frac{1}{a_n}\sum_{i=1}^{n} X_i^{(n)}\right| > \varepsilon\right),$$

which implies that

$$\begin{aligned} P\left(\left|\frac{1}{a_n}\sum_{i=1}^{n} X_{ni}\right| > \varepsilon\right) &\leq P\left(\max_{1\leq i\leq n}|X_{ni}| > a_n\right) + P\left(\left|\frac{1}{a_n}\sum_{i=1}^{n} X_i^{(n)}\right| > \varepsilon\right) \\ &\leq \sum_{i=1}^{n} P(|X_{ni}| > a_n) + P\left(\left|T_n^{(n)}\right| > \varepsilon - \left|\frac{1}{a_n}\sum_{i=1}^{n} EX_i^{(n)}\right|\right). \end{aligned} \tag{2.3}$$

First, we will show that

$$\left|\frac{1}{a_n}\sum_{i=1}^{n}EX_i^{(n)}\right| \to 0 \quad \text{as } n\to\infty. \tag{2.4}$$

Actually, by conditions $g_n(|t|)\uparrow$, $g_n(|t|)/|t|\downarrow$ as $|t|\uparrow$ and (2.1), we have that

$$\begin{aligned}\left|\frac{1}{a_n}\sum_{i=1}^{n}EX_i^{(n)}\right| &\le \sum_{i=1}^{n}P\left(|X_{ni}|>a_n\right)+\frac{1}{a_n}\sum_{i=1}^{n}E|X_{ni}|I(|X_{ni}|\le a_n)\\ &\le \sum_{i=1}^{n}\frac{Eg_n(|X_{ni}|)}{g_n(a_n)}+\sum_{i=1}^{n}\frac{Eg_n(|X_{ni}|)I(|X_{ni}|\le a_n)}{g_n(a_n)}\\ &\le 2\sum_{i=1}^{n}\frac{Eg_n(|X_{ni}|)}{g_n(a_n)}\to 0 \quad \text{as } n\to\infty,\end{aligned}$$

which implies (2.4). It follows from (2.3) and (2.4) that for n large enough,

$$P\left(\left|\frac{1}{a_n}\sum_{i=1}^{n}X_{ni}\right|>\varepsilon\right)\le\sum_{i=1}^{n}P\left(|X_{ni}|>a_n\right)+P\left(\left|T_n^{(n)}\right|>\frac{\varepsilon}{2}\right).$$

Hence, to prove (2.2), we only need to show that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}P\left(|X_{ni}|>a_n\right)<\infty \tag{2.5}$$

and

$$\sum_{n=1}^{\infty}P\left(\left|T_n^{(n)}\right|>\frac{\varepsilon}{2}\right)<\infty. \tag{2.6}$$

The conditions $g_n(|t|)\uparrow$ as $|t|\uparrow$ and (2.1) yield that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}P\left(|X_{ni}|>a_n\right)\le\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{Eg_n(|X_{ni}|)}{g_n(a_n)}<\infty,$$

which implies (2.5).

By Markov's inequality, Lemma 1.2 (for $p = 2$), $g_n(|t|)\uparrow$, $g_n(|t|)/|t|\downarrow$ as $|t|\uparrow$ and (2.1), we can get that

$$\begin{aligned}\sum_{n=1}^{\infty}P\left(\left|T_n^{(n)}\right|>\frac{\varepsilon}{2}\right)&\le C\sum_{n=1}^{\infty}E\left|T_n^{(n)}\right|^2\le C\sum_{n=1}^{\infty}\frac{1}{a_n^2}\sum_{i=1}^{n}E\left|X_i^{(n)}\right|^2\\ &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}P\left(|X_{ni}|>a_n\right)+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^2I(|X_{ni}|\le a_n)}{a_n^2}\\ &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{|Eg_n(|X_{ni}|)}{g_n(a_n)}+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^2I(|X_{ni}|\le a_n)}{a_n^2}\\ &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{Eg_n(|X_{ni}|)}{g_n(a_n)}+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{Eg_n(|X_{ni}|)I(|X_{ni}|\le a_n)}{g_n(a_n)}\\ &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{Eg_n(|X_{ni}|)}{g_n(a_n)}<\infty,\end{aligned}$$

which implies (2.6). This completes the proof of the theorem. □

Corollary 2.1. *Under the conditions of Theorem 2.1,*

$$\frac{1}{a_n}\sum_{i=1}^{n} X_{ni} \to 0 \quad a.s.$$

Theorem 2.2. *Let* $\{X_{ni} : i \geq 1, n \geq 1\}$ *be an array of rowwise NOD random variables and* $\{a_n, n \geq 1\}$ *be a sequence of positive real numbers. Let* $\{g_n(t), n \geq 1\}$ *be a sequence of nonnegative, even functions such that* $g_n(|t|)$ *is an increasing function of* $|t|$ *for every* $n \geq 1$. *Assume that there exists a constant* $\delta > 0$ *such that* $g_n(t) \geq \delta t$ *for* $0 < t \leq 1$. *If*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} Eg_n\left(\frac{X_{ni}}{a_n}\right) < \infty, \tag{2.7}$$

then for any $\varepsilon > 0$, *(2.2) holds true.*

Proof. We use the same notations as that in Theorem 2.1. The proof is similar to that of Theorem 2.1.

First, we will show that (2.4) holds true. In fact, by the conditions $g_n(t) \geq \delta t$ for $0 < t \leq 1$ and (2.7), we have that

$$\begin{aligned}\left|\frac{1}{a_n}\sum_{i=1}^{n} EX_i^{(n)}\right| &\leq \sum_{i=1}^{n} P\left(|X_{ni}| > a_n\right) + \sum_{i=1}^{n} E\left(\frac{|X_{ni}|}{a_n} I(|X_{ni}| \leq a_n)\right)\\ &\leq \frac{1}{\delta}\sum_{i=1}^{n} Eg_n\left(\frac{X_{ni}}{a_n}\right) + \frac{1}{\delta}\sum_{i=1}^{n} Eg_n\left(\frac{X_{ni}}{a_n}\right) I(|X_{ni}| \leq a_n)\\ &\leq \frac{2}{\delta}\sum_{i=1}^{n} Eg_n\left(\frac{X_{ni}}{a_n}\right) \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

which implies (2.4).

According to the proof of Theorem 2.1, we only need to prove that (2.5) and (2.6) hold true.

When $|X_{ni}| > a_n > 0$, we have $g_n\left(\frac{X_{ni}}{a_n}\right) \geq g_n(1) \geq \delta$, which yields that

$$P(|X_{ni}| > a_n) = EI(|X_{ni}| > a_n) \leq \frac{1}{\delta} Eg_n\left(\frac{X_{ni}}{a_n}\right).$$

Hence,

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left(|X_{ni}| > a_n\right) \leq \frac{1}{\delta}\sum_{n=1}^{\infty}\sum_{i=1}^{n} Eg_n\left(\frac{X_{ni}}{a_n}\right) < \infty,$$

which implies (2.5).

By Markov's inequality, Lemma 1.2 (for $p = 2$), $g_n(t) \geq \delta t$ for $0 < t \leq 1$ and (2.7), we can get that

$$\begin{aligned}\sum_{n=1}^{\infty} P\left(\left|T_n^{(n)}\right| > \frac{\varepsilon}{2}\right) &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left(|X_{ni}| > a_n\right) + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E|X_{ni}|^2 I(|X_{ni}| \le a_n)}{a_n^2} \\ &\le C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E|X_{ni}| I(|X_{ni}| \le a_n)}{a_n} \\ &\le C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} Eg_n\left(\frac{X_{ni}}{a_n}\right) I(|X_{ni}| \le a_n) \\ &\le C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} Eg_n\left(\frac{X_{ni}}{a_n}\right) < \infty,\end{aligned}$$

which implies (2.6). This completes the proof of the theorem. □

Corollary 2.2. *Let* $\{X_{ni}, i \ge 1, n \ge 1\}$ *be an array of rowwise NOD random variables and* $\{a_n, n \ge 1\}$ *be a sequence of positive real numbers. If there exists a constant* $\beta \in (0, 1]$ *such that*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\left(\frac{|X_{ni}|^{\beta}}{|a_n|^{\beta} + |X_{ni}|^{\beta}}\right) < \infty,$$

then (2.2) holds true.

Proof. In Theorem 2.2, we take

$$g_n(t) \equiv \frac{|t|^{\beta}}{1 + |t|^{\beta}}, \quad 0 < \beta \le 1,\ n \ge 1.$$

It is easy to check that $\{g_n(t), n \ge 1\}$ is a sequence of nonnegative, even functions such that $g_n(|t|)$ is an increasing function of $|t|$ for every $n \ge 1$. And

$$g_n(t) \ge \frac{1}{2}t^{\beta} \ge \frac{1}{2}t, \quad 0 < t \le 1,\ 0 < \beta \le 1.$$

Therefore, by Theorem 2.2, we can easily get (2.2). □

Corollary 2.3. *Under the conditions of Theorem 2.2 or Corollary 2.2,*

$$\frac{1}{a_n}\sum_{i=1}^{n} X_{ni} \to 0 \quad a.s.$$

Theorem 2.3. *Let* $\{X_{ni} : i \ge 1, n \ge 1\}$ *be an array of rowwise NOD random variables and* $\{a_n, n \ge 1\}$ *be a sequence of positive real numbers.* $EX_{ni} = 0$, $i \ge 1$, $n \ge 1$. *Let* $\{g_n(x), n \ge 1\}$ *be a sequence of nonnegative, even functions. Assume that there exist* $\beta \in (1, 2]$ *and* $\delta > 0$ *such that* $g_n(x) \ge \delta x^{\beta}$ *for* $0 < x \le 1$ *and there exists a* $\delta > 0$ *such that* $g_n(x) \ge \delta x$ *for* $x > 1$. *If (2.7) satisfies, then for any* $\varepsilon > 0$, *(2.2) holds true.*

Proof. We use the same notations as that in Theorem 2.1. The proof is similar to that of Theorem 2.1.

First, we will show that (2.4) holds true. Actually, by the conditions $EX_{ni} = 0$, $g_n(x) \ge \delta x$ for $x > 1$ and (2.7), we have that

$$\begin{aligned}\left|\frac{1}{a_n}\sum_{i=1}^{n}EX_i^{(n)}\right| &\le \sum_{i=1}^{n}P\left(|X_{ni}|>a_n\right)+\left|\frac{1}{a_n}\sum_{i=1}^{n}EX_{ni}I\big(|X_{ni}|>a_n\big)\right|\\ &\le 2\sum_{i=1}^{n}E\left(\frac{|X_{ni}|}{a_n}I\big(|X_{ni}|>a_n\big)\right)\\ &\le \frac{2}{\delta}\sum_{i=1}^{n}Eg_n\left(\frac{X_{ni}}{a_n}\right)I\big(|X_{ni}|>a_n\big)\\ &\le \frac{2}{\delta}\sum_{i=1}^{n}Eg_n\left(\frac{X_{ni}}{a_n}\right)\to 0 \quad \text{as } n\to\infty,\end{aligned}$$

which implies (2.4). Hence, to prove (2.2), we only need to show that (2.5) and (2.6) hold true.

The conditions $g_n(x) \ge \delta x$ for $x > 1$ and (2.1) yield that

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{i=1}^{n}P\left(|X_{ni}|>a_n\right) &= \sum_{n=1}^{\infty}\sum_{i=1}^{n}EI\left(|X_{ni}|>a_n\right)\\ &\le \sum_{n=1}^{\infty}\sum_{i=1}^{n}E\left(\frac{|X_{ni}|}{a_n}I\left(|X_{ni}|>a_n\right)\right)\\ &\le \frac{1}{\delta}\sum_{n=1}^{\infty}\sum_{i=1}^{n}Eg_n\left(\frac{X_{ni}}{a_n}\right)I\left(|X_{ni}|>a_n\right)\\ &\le \frac{1}{\delta}\sum_{n=1}^{\infty}\sum_{i=1}^{n}Eg_n\left(\frac{X_{ni}}{a_n}\right)<\infty,\end{aligned}$$

which implies (2.5).

By Markov's inequality, Lemma 1.2 (for $p = 2$), $g_n(x) \ge \delta x^\beta$ for $1 < \beta \le 2$, $0 < x \le 1$ and (2.7), we can get that

$$\begin{aligned}\sum_{n=1}^{\infty}P\left(\left|T_n^{(n)}\right|>\frac{\varepsilon}{2}\right) &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}P\left(|X_{ni}|>a_n\right)+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^2I\big(|X_{ni}|\le a_n\big)}{a_n^2}\\ &\le C+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^\beta I\big(|X_{ni}|\le a_n\big)}{a_n^\beta}\\ &\le C+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}Eg_n\left(\frac{X_{ni}}{a_n}\right)I\big(|X_{ni}|\le a_n\big)\\ &\le C+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}Eg_n\left(\frac{X_{ni}}{a_n}\right)<\infty,\end{aligned}$$

which implies (2.6). This completes the proof of the theorem. □

Corollary 2.4. *Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of rowwise NOD random variables and $\{a_n, n \ge 1\}$ be a sequence of positive real numbers. $EX_{ni} = 0$, $i \ge 1$, $n \ge 1$. If there exists a constant $\beta \in (1, 2]$ such that*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}E\left(\frac{|X_{ni}|^\beta}{a_n|X_{ni}|^{\beta-1}+a_n^\beta}\right)<\infty,$$

then (2.2) holds true.

Proof. In Theorem 2.3, we take

$$g_n(x) \equiv \frac{|x|^\beta}{1+|x|^{\beta-1}}, \quad 1 < \beta \le 2,\ n \ge 1.$$

It is easy to check that $\{g_n(x),\ n \ge 1\}$ is a sequence of nonnegative, even functions satisfying

$$g_n(x) \ge \frac{1}{2}x^\beta, \quad 0 < x \le 1,\ 1 < \beta \le 2 \quad \text{and} \quad g_n(x) \ge \frac{1}{2}x, \quad x > 1.$$

Therefore, by Theorem 2.3, we can easily get (2.2). □

Furthermore, by Corollaries 2.2 and 2.4, we can get the following important Chung-type strong law of large numbers for arrays of rowwise NOD random variables.

Corollary 2.5. *Let* $\{X_{ni},\ i \ge 1,\ n \ge 1\}$ *be an array of rowwise NOD random variables and* $\{a_n,\ n \ge 1\}$ *be a sequence of positive real numbers. If there exists some* $\beta \in (0, 2]$ *such that*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^\beta}{a_n^\beta} < \infty, \tag{2.8}$$

and $EX_{ni} = 0,\ i \ge 1,\ n \ge 1$ *if* $\beta \in (1, 2]$, *then (2.2) holds true and* $\frac{1}{a_n}\sum_{i=1}^{n} X_{ni} \to 0$ *a.s.*.

For $\beta \ge 2$, we have the following result.

Theorem 2.4. *Let* $\{X_{ni} : i \ge 1,\ n \ge 1\}$ *be an array of rowwise NOD random variables and* $\{a_n,\ n \ge 1\}$ *be a sequence of positive real numbers. Let* $\{g_n(x),\ n \ge 1\}$ *be a sequence of nonnegative, even functions. Assume that there exists some* $\beta \ge 2$ *such that* $g_n(x) \ge \delta x^\beta$ *for* $x > 0$. *If*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left[Eg_n\left(\frac{X_{ni}}{a_n}\right)\right]^{1/\beta} < \infty, \tag{2.8a}$$

then for any $\varepsilon > 0$, *(2.2) holds true.*

Proof. We use the same notations as that in Theorem 2.1. The proof is similar to that of Theorem 2.1. It is easily seen that (2.8) implies that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}Eg_n\left(\frac{X_{ni}}{Ma_n}\right) < \infty \tag{2.9}$$

and

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left[Eg_n\left(\frac{X_{ni}}{Ma_n}\right)\right]^{2/\beta} < \infty. \tag{2.10}$$

First, we will show that (2.4) holds true. In fact, by Hölder's inequality, $g_n(x) \ge \delta x^\beta$ for $x > 0$, (2.8) and (2.9), we have that

$$\begin{aligned}
\left|\frac{1}{a_n}\sum_{i=1}^{n} EX_i^{(n)}\right| &\le \sum_{i=1}^{n} P\left(|X_{ni}| > a_n\right) + \sum_{i=1}^{n} E\left(\frac{|X_{ni}|}{a_n} I\left(|X_{ni}| \le a_n\right)\right) \\
&\le \sum_{i=1}^{n} E\left(\frac{|X_{ni}|^\beta}{a_n^\beta} I\left(|X_{ni}| > a_n\right)\right) + \sum_{i=1}^{n}\left[E\left(\frac{|X_{ni}|^\beta}{a_n^\beta} I\left(|X_{ni}| \le a_n\right)\right)\right]^{1/\beta} \\
&\le C\sum_{i=1}^{n} Eg_n\left(\frac{X_{ni}}{a_n}\right) + C\sum_{i=1}^{n}\left[Eg_n\left(\frac{X_{ni}}{a_n}\right) I\left(|X_{ni}| \le a_n\right)\right]^{1/\beta} \\
&\le C\sum_{i=1}^{n} Eg_n\left(\frac{X_{ni}}{a_n}\right) + C\sum_{i=1}^{n}\left[Eg_n\left(\frac{X_{ni}}{a_n}\right)\right]^{1/\beta} \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which implies (2.4). To prove (2.2), we only need to show that (2.5) and (2.6) hold true.

By the condition $g_n(x) \ge \delta x^\beta$ for $x > 0$ again and (2.9), we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left(|X_{ni}| > a_n\right) &= \sum_{n=1}^{\infty}\sum_{i=1}^{n} EI\left(|X_{ni}| > a_n\right) \\
&\le \sum_{n=1}^{\infty}\sum_{i=1}^{n} E\left(\frac{|X_{ni}|^\beta}{a_n^\beta} I\left(|X_{ni}| > a_n\right)\right) \\
&\le \frac{1}{\delta}\sum_{n=1}^{\infty}\sum_{i=1}^{n} Eg_n\left(\frac{X_{ni}}{a_n}\right) < \infty,
\end{aligned}$$

which implies (2.5).

By Markov's inequality, Lemma 1.2 (for $p = 2$), $g_n(x) \ge \delta x^\beta$ for $x > 0$ and (2.10), we can get that

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(\left|T_n^{(n)}\right| > \frac{\varepsilon}{2}\right) &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left(|X_{ni}| > a_n\right) + C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^2 I\left(|X_{ni}| \le a_n\right)}{a_n^2} \\
&\le C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left[E\left(\frac{|X_{ni}|^\beta}{a_n^\beta} I\left(|X_{ni}| \le a_n\right)\right)\right]^{2/\beta} \\
&\le C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left[Eg_n\left(\frac{X_{ni}}{a_n}\right) I\left(|X_{ni}| \le a_n\right)\right]^{2/\beta} \\
&\le C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left[Eg_n\left(\frac{X_{ni}}{a_n}\right)\right]^{2/\beta} < \infty,
\end{aligned}$$

which implies (2.6). This completes the proof of the theorem. □

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Competing interests
The authors declare that they have no competing interests.

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