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The integral estimate with Orlicz norm in $L^{\phi(x)}$-averaging domain

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Abstract

In this article, we obtain some local and global integral inequalities with Orlicz norm for the A-harmonic tensors in $L^{\phi(x)}$-averaging domain, where $\phi(x)$ satisfies the ϕ_p condition. These estimates indicate that many existing inequalities with L^p-norms are special cases of our results.

Keywords: *A*-harmonic equations, differential forms, Luxemburg norms, homotopy operator

1 Introduction

In recent years, there are many remarkable results about the solutions of the nonhomogeneous A-harmonic equation $d^{\star}A(x, d\omega) = B(x, d\omega)$ have been made, see [1-7]. For example, in [2], the following Caccioppoli inequality has been established.

$$\| du\|_{p,Q} \leq C|Q|^{-1/n} \| u\|_{p,\sigma Q}, \quad p > 1, \quad \sigma > 1. \tag{1.1}$$

In [3] we can find the general Poincaré inequality

$$\| u - u_Q\|_{p,Q} \leq C|Q|\mathrm{diam}(Q) \| du\|_{p,\sigma Q}, \quad p > 0, \quad \sigma > 1. \tag{1.2}$$

In [4-6], many inequalities for the classical operators applied to the differential forms have been studied. These integral inequalities play a crucial role in studying PDE and the properties of the solutions of PDE. However, most of these inequalities are developed with the L^p-norms. Meanwhile, we know the Orlicz spaces is the important tool in studying PDE, see [8]. So, in this article, the normalized L^p-norms are replaced by large norms in the scale of Orlicz spaces. We first introduce the ϕ_p condition, which is a particular class of the Young functions, then using the result that the maximal operator M_ϕ is bounded on $L^p(\mathbb{R}^n)$, see [9], we establish some integral estimates with Orlicz norms. In the global case, we also expand the local results to a relative large class of domains, the L^ϕ-averaging domain. Applying our results, we can easily find that many versions of the existing estimates become the special cases of our new results.

Throughout this article, we assume that Ω is a bounded connected open subset of $\mathbb{R}^n$, Q, and σQ are the cubes with the same center and $\mathrm{diam}(\sigma Q) = \sigma\mathrm{diam}(Q)$, $\sigma > 0$. We use $|E|$ to denote the Lebesgue measure of the set $E \subset \mathbb{R}^n$. Let $\Lambda^l = \Lambda^l(\mathbb{R}^n)$ be the set of all l-forms on $\mathbb{R}^n$, $D'(\Omega, \Lambda^l)$ be the space of all differential l-forms on Ω. A differential l-form $\omega(x)$ is generated by $\{dx_{i1} \wedge dx_{i2} \wedge \ldots \wedge dx_{il}\}$, $l = 0, 1, \ldots, n$, that is $\omega(x) = \sum_I \omega_I(x)dx_I = \sum \omega_{i1,i2,\ldots,il}(x)dx_{i1} \wedge dx_{i2} \wedge \cdots \wedge dx_{il}$, where $I = (i_1, i_2, \ldots, i_l)$, $1 \leq i_1$

$<i_2 < \ldots <i_l \le n$. Let $L^p(\Omega, \Lambda^l)$ be the l-forms $\omega(x) = \sum_I \omega_I(x)dx_I$ on Ω satisfying $\int_\Omega |\omega_I|^p < \infty$ for all ordered l-tuples I, $l = 1, 2, \ldots, n$. We write $||\omega||_{p,\Omega} = \left(\int_\Omega |\omega|^p dx\right)^{1/p} = \left(\int_\Omega \left(\sum_I |\omega_I(x)|^2\right)^{p/2} dx\right)^{1/p}$. we denote the exterior derivative by $d : D'(\Omega, \Lambda^l) \to D'(\Omega, \Lambda^{l+1})$ for $l = 0, 1, \ldots, n - 1$. Its formal adjoint operator $d^\star$: $D'(\Omega, \Lambda^{l+1}) \to D'(\Omega, \Lambda^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(\Omega, \Lambda^{l+1})$, $l = 0, 1, 2, \ldots, n - 1$, here $\star$ is the well known Hodge star operator. A differential l-form $u \in D'(\Omega, \Lambda^l)$ is called a closed form if $du = 0$ in Ω. A homotopy operator $T : C^\infty(\Omega, \Lambda^l) \to C^\infty(\Omega, \Lambda^{l-1})$ is defined in [10], and the decomposition

$$u = d(Tu) + T(du) \tag{1.3}$$

holds for any differential form u. We define the l - form $u_Q \in D'(Q, \Lambda^l)$ by

$$u_Q = |Q|^{-1} \int_Q u(y)dy, \quad l = 0 \quad \text{and} \quad u_Q = dT(u), \tag{1.4}$$

for all $u \in L^p(Q, \Lambda^l)$, $1 \le p < \infty$, then $u_Q = u - T(du)$, $l = 1, 2, \ldots, n$.

In this article, we consider solutions to the non-homogeneous A-harmonic equation of the form

$$d^\star A(x, d\omega) = B(x, d\omega), \tag{1.5}$$

where $A : \Omega \times \Lambda^l(\mathbb{R}^n) \to \Lambda^l(\mathbb{R}^n)$ and $B : \Omega \times \Lambda^l(\mathbb{R}^n) \to \Lambda^{l-1}(\mathbb{R}^n)$ satisfy the conditions: $|A(x, \xi)| \le a|\xi|^{p-1}$, $\langle A(x, \xi), \xi\rangle \ge |\xi|^p$ and $|B(x, \xi)| \le b|\xi|^{p-1}$ for almost every $x \in \Omega$ and all $\xi \in \Lambda^l(\mathbb{R}^n)$. Here, a, $b > 0$ are constants and $1 <p < \infty$ is a fixed exponent associated with (1.5). A solution to (1.5) is an element of the Sobolev space $W^{1,p}_{loc}(\Omega, \Lambda^{l-1})$ such that $\int_\Omega \langle A(x, d\omega), d\phi\rangle + \langle B(x, d\omega), \phi\rangle = 0$ for all $\varphi \in W^{1,p}_{loc}(\Omega, \Lambda^{l-1})$ with compact support.

2 Main results

In this section, we first obtain the local strong-type Orlicz norm inequality for the homotopy operator applied to the solutions of Equation 1.5, then, under the similar method, we establish the Caccioppoli and Poincaré inequalities with the Orlicz norms. We also give the generalized weak reverse Hölder-type inequality for the A-harmonic tensors. Finally, we expand these results to the global case. To prove the main results, we first introduce the following definitions and lemmas.

Definition 2.1 Given a Young function $\phi(t)$: $[0, \infty) \to [0, \infty)$, and a cube Q, define the normalized Luxemburg norm on Q by

$$\| u\|_{\varphi(Q)} = inf\{\lambda > 0 : \frac{1}{|Q|}\int_Q \varphi(\frac{|u(x)|}{\lambda})dx \le 1\}. \tag{2.1}$$

If $\phi(t) = t^p$, $1 \le p < \infty$, then $\| u\|_{\varphi(Q)} = (\frac{1}{|Q|}\int_Q |u|^p dx)^{\frac{1}{p}}$ (see [9]) and the Luxemburg norm reduce to the L^p norm.

Given a Young function ϕ, let $\bar{\varphi}$ denote its associate function: the Young function with the property that $t \le \varphi^{-1}(t)\bar{\varphi}^{-1}(t) \le 2t$, $t > 0$. If $\phi(t) = t^p$, then $\bar{\varphi}(t) = t^{p'}$; and if $\phi(t) = t^p log(e+t)^\alpha$, then $\bar{\varphi}(t) \approx t^{p'} log(e + t)^{-\alpha p'/p}$, where p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 2.2 (see [9]) Given p, $1 < p < \infty$, a Young function ϕ satisfies the ϕ_p condition if for some $c > 0$,

$$\int_c^\infty \frac{\varphi(t)}{t^p}\frac{dt}{t} < \infty. \tag{2.2}$$

Given a Young function ϕ, we define the Orlicz maximal operator associated with ϕ by

$$M_\varphi u(x) = \sup_{Q \ni x} \| u\|_{\varphi(Q)}. \tag{2.3}$$

We have the following result taken from [9] that characterizes the boundedness of these maximal functions on $L^p(\mathbb{R}^n)$. This will play an important role in the proofs of our main results.

Lemma 2.3 *Given p,* $1 < p < \infty$, *and a Young function* ϕ, *then for any nonnegative function f, the following result holds.*

$$M_\varphi : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad \text{if and only if} \quad \varphi \in \varphi_p. \tag{2.4}$$

Lemma 2.4 *If A, B, and C are Young functions such that* $A^{-1}(t)B^{-1}(t) \leq C^{-1}(t)$, *then for all functions f and g and any cube Q,*

$$\| fg\|_{C(Q)} \leq 2 \| f\|_{A(Q)} \| g\|_{B(Q)}. \tag{2.5}$$

In particular, given any Young function ϕ,

$$\frac{1}{|Q|}\int_Q |f(x)g(x)|dx \leq 2 \| f\|_{\varphi(Q)} \| g\|_{\bar{\varphi}(Q)}. \tag{2.6}$$

From [10], we know that, for any differential form $u \in L^s_{loc}(Q, \Lambda^l)$, $l = 1, 2, ..., n$, $1 \leq s < \infty$, we have

$$||Tu||_{s,Q} \leq C|Q|\text{diam}(Q)||u||_{s,Q} \tag{2.7}$$

and

$$||\nabla(Tu)||_{s,Q} \leq C|Q|||u||_{s,Q}. \tag{2.8}$$

Theorem 2.5. *Let* $\phi(x)$ *be a Young function satisfying* ϕ_p *condition,* $1 < p < \infty$. *Assume* $\varphi(|u|) \in L^1_{loc}(\Omega)$ *and u is a solution of the nonhomogeneous equation* (1.5) *in* Ω, *T is the homotopy operator,* $\varphi(|Tu|) \in L^1_{loc}(\Omega)$. *Then there exists a constant C, independent of u such that*

$$||Tu||_{\varphi(Q)} \leq C|Q|diam(Q)||u||_{\varphi(\sigma Q)}, \tag{2.9}$$

where Q is any cube with $\sigma Q \subset \Omega$, σ *is constant with* $1 < \sigma < \infty$.

Proof. Using Hölder inequality with $1 = 1/p + (p - 1)/p$ and the definition of the Orlicz maximal operator, we have

$$\begin{aligned} \| Tu\|_{\varphi(Q)} &\leq \frac{1}{|Q|}\int_Q M_\varphi(|Tu|)dx \\ &\leq \frac{1}{|Q|}\left(\int_Q (M_\varphi(|Tu|))^p dx\right)^{\frac{1}{p}}\left(\int_Q 1^{\frac{p}{(p-1)}} dx\right)^{\frac{(p-1)}{p}} \\ &= |Q|^{-1+\frac{(p-1)}{p}}\left(\int_Q (M_\varphi(|Tu|))^p dx\right)^{\frac{1}{p}}. \end{aligned} \tag{2.10}$$

Since $\phi(x)$ satisfies the ϕ_p condition, then using Lemma 2.3, we obtain

$$\| Tu\|_{\varphi(Q)} \leq C_1|Q|^{-\frac{1}{p}} \| Tu\|_{p,Q}. \tag{2.11}$$

Applying (2.7), (2.11) becomes

$$\| Tu\|_{\varphi(Q)} \leq C_2|Q|^{-\frac{1}{p}}|Q|\mathrm{diam}(Q) \| u\|_{p,Q}. \tag{2.12}$$

u is the solution of Equation 1.5 satisfying the weak reverse Hölder inequality $||u||_{s,Q} \leq C||u||_{t,\sigma Q}$, $\sigma > 1$ and $0 < s, t < \infty$ (see [3]), so

$$\begin{aligned} \| Tu\|_{\varphi(Q)} &\leq C_3|Q|^{-\frac{1}{p}}|Q|\mathrm{diam}(Q)|Q|^{\frac{(1-p)}{p}} \| u\|_{1,\sigma Q} \\ &= C_3|Q|\mathrm{diam}(Q)|Q|^{-1} \| u\|_{1,\sigma Q}. \end{aligned} \tag{2.13}$$

Using Lemma 2.4, we can easily have

$$\begin{aligned} \| Tu\|_{\varphi(Q)} &\leq C_4|Q|\mathrm{diam}(Q)|||u|||_{\varphi(\sigma Q)} \| 1\|_{\bar{\varphi}(\sigma Q)} \\ &\leq C_5|Q|\mathrm{diam}(Q) \| u\|_{\varphi(\sigma Q)}. \end{aligned} \tag{2.14}$$

This ends the proof of Theorem 2.5.

Using the similar method and (2.8), under the same condition of Theorem 2.5 we can also prove the following result

$$||\nabla(Tu)||_{\varphi(Q)} \leq C|Q|||u||_{\varphi(\sigma Q)}. \tag{2.15}$$

Remark. Using the similar method, we can expand the result to include a variety of operators-the Green's operator G, the projection operator H and other composite operators such as $T \circ G$, $T \circ H$, $T \circ \Delta \circ G$, and so on.

Using the similar method and the general Caccioppoli inequality (1.1), we can prove the following Caccioppoli inequality with the Luxemburg norm.

Theorem 2.6 *Let $\phi(x)$ be a Young function satisfying ϕ_p condition, $1 < p < \infty$. Assume $\varphi(|u|) \in L^1_{loc}(\Omega)$and u is a solution of the nonhomogeneous equation (1.5) in Ω, $\varphi(|du|) \in L^1_{loc}(\Omega)$. Then, there exists a constant C, independent of u such that*

$$\| du\|_{\varphi(Q)} \leq C|Q|^{-\frac{1}{n}} \| u\|_{\varphi(\sigma Q)}, \tag{2.16}$$

where Q is any cube with $\sigma Q \subset \Omega$, σ is constant with $1 < \sigma < \infty$.

If u is a solution of the equation (1.5), du satisfies the weak reverse Hölder inequality $||du||_{p,Q} \leq C||du||_{q,\sigma Q}$, $\sigma > 1$ and $0 < p, q < \infty$(see [3]). So, applying the general Pioncaré inequality (1.2), under the similar proceeding of Theorem 2.5, we can easily obtain the following Pioncaré-type inequality.

Theorem 2.7 *Let $\phi(x)$ be a Young function satisfying ϕ_p condition, $1 < p < \infty$. Assume $\varphi(|u|) \in L^1_{loc}(\Omega)$and u is a solution of the nonhomogeneous equation (1.5) in Ω, $\varphi(|du|) \in L^1_{loc}(\Omega)$. Then, there exists a constant C, independent of u such that*

$$\| u - u_Q\|_{\varphi(Q)} \leq C|Q|\mathrm{diam}(Q) \| du\|_{\varphi(\sigma Q)}, \tag{2.17}$$

where Q is any cube with $\sigma Q \subset \Omega$, σ is constant with $1 < \sigma < \infty$.

We can also generalize the weak reverse Hölder-type inequality for the A-harmonic tensors.

Theorem 2.8 *Let $\phi_1(x)$ and $\phi_2(x)$ be the Young functions with $\phi_1(x)$ satisfying ϕ_p condition,* $1 < p < \infty$. *Assume that u is a solution of the nonhomogeneous equation* (1.5) *in* Ω $\varphi_1(|u|) \in L^1_{loc}(\Omega)$*and* $\varphi_2(|u|) \in L^1_{loc}(\Omega)$. *Then, there exists a constant C, independent of u such that*

$$\| u\|_{\varphi_1(Q)} \leq C \| u\|_{\varphi_2(\sigma Q)}, \tag{2.18}$$

where Q is any cube with $\sigma Q \subset \Omega$, σ is constant with $1 < \sigma < \infty$.

Proof. Using Hölder inequality with $1 = 1/p + (p - 1)/p$, we have

$$\begin{aligned} \| u\|_{\varphi_1(Q)} &\leq \frac{1}{|Q|}\int_Q M_{\varphi_1}(|u|)dx \\ &\leq \frac{1}{|Q|}\left(\int_Q (M_{\varphi_1}(|u|))^p dx\right)^{\frac{1}{p}}\left(\int_Q 1^{\frac{p}{(p-1)}} dx\right)^{\frac{(p-1)}{p}} \\ &= |Q|^{-1+\frac{(p-1)}{p}}\left(\int_Q (M_{\varphi_1}(|u|))^p dx\right)^{\frac{1}{p}}. \end{aligned} \tag{2.19}$$

Since $\phi_1(x)$ satisfies the ϕ_p condition, then using Lemma 2.3, we obtain

$$\| u\|_{\varphi_1(Q)} \leq C_1 |Q|^{-\frac{1}{p}} \| u\|_{p,Q}. \tag{2.20}$$

Using the weak reverse Hölder inequality of u, (2.20) becomes

$$\| u\|_{\varphi_1(Q)} \leq C_2 |Q|^{-\frac{1}{p}} |Q|^{\frac{(1-p)}{p}} \| u\|_{1,\sigma Q}. \tag{2.21}$$

Using Lemma 2.4, we can easily have

$$\begin{aligned} \| u\|_{\varphi_1(Q)} &\leq C_2 |Q|^{-1} ||u||_{1,\sigma Q} \\ &\leq C_3 |||u|||_{\varphi_2(\sigma Q)} ||1||_{\tilde{\varphi}_2(\sigma Q)} \\ &\leq C_4 ||u||_{\varphi_2(\sigma Q)}. \end{aligned} \tag{2.22}$$

This ends the proof of Theorem 2.8.

In the following $L^{\phi(x)}$-averaging domains, we will extend the local estimates into the global case.

Definition 2.9 (see [11]). Let $\phi(x)$ be an increasing convex function on $[0, \infty)$ with $\phi(0) = 0$. we call a proper subdomain $\Omega \subset \mathbb{R}^n$ an $L^{\phi(x)}$-averaging domain, if $|\Omega| < \infty$ and there exists a constant C such that

$$\int_\Omega \varphi(\tau |u - u_{Q_0}|)dx \leq C \sup_{Q\subset\Omega} \int_\Omega \varphi(\sigma |u - u_Q|)dx \tag{2.23}$$

for some cube $Q_0 \subset \Omega$ and all u such that $\varphi(|u|) \in L^1_{loc}(\Omega)$, where τ, σ are constants with $0 < \tau < \infty$, $0 < \sigma < \infty$. More properties and applications of the $L^{\phi(x)}$-averaging domain can be founded in [11,12].

Theorem 2.10 *Let $\phi(x)$ be a Young function satisfying ϕ_p condition,* $1 < p < \infty$, *and let Ω be any bounded $L^{\phi(x)}$-averaging domain. Assume that* $\varphi(|du|) \in L^1_{loc}(\Omega)$*and u is a solution of the nonhomogeneous equation* (1.5) *in* Ω, *T is the homotopy operator,* $\varphi(|Tu|) \in L^1_{loc}(\Omega)$. *Then there exists a constant C, independent of u such that*

$$||Tu - (Tu)_{Q_0}||_{\varphi(\Omega)} \leq C|\Omega|\text{diam}(\Omega)||u||_{\varphi(\Omega)}, \tag{2.24}$$

where $Q_0 \subset \Omega$ is some fixed cube.

Proof. In $L^{\phi(x)}$-averaging domain, since $\varphi(|Tu|) \in L^1_{loc}(\Omega)$, Tu satisfies (2.23), so

$$||Tu - (Tu)_{Q_0}||_{\varphi(\Omega)} \leq C_1 \sup_{Q\subset\Omega} ||Tu - (Tu)_Q||_{\varphi(Q)}. \tag{2.25}$$

For any differential form u, we know that

$$\| u_Q\|_{s,Q} \leq C_2 \| u\|_{s,Q}. \tag{2.26}$$

Using (2.7) and (2.26), we have

$$\begin{aligned} \| Tu - (Tu)_Q\|_{s,Q} &= \| Td(Tu)\|_{s,Q} \\ &\leq C_2 |Q|\mathrm{diam}(Q) \| dTu\|_{s,Q} \\ &= C_2 |Q|\mathrm{diam}(Q) \| u_Q\|_{s,Q} \\ &\leq C_3 |Q|\mathrm{diam}(Q) \| u\|_{s,Q}. \end{aligned} \tag{2.27}$$

Using the similar method of Theorem 2.5, for $\sigma > 1$, we can prove

$$||Tu - (Tu)_Q||_{\varphi(Q)} \leq C_4 |Q|\mathrm{diam}(Q) \| u\|_{\varphi(\sigma Q)}. \tag{2.28}$$

Substituting (2.28) in (2.25), we obtain

$$\begin{aligned} \| Tu - (Tu)_{Q_0}\|_{\varphi(\Omega)} &\leq C_5 \sup_{Q\subset\Omega} |Q|\mathrm{diam}(Q) \| u\|_{\varphi(\sigma Q)} \\ &\leq C_6 \sup_{Q\subset\Omega} |\Omega|\mathrm{diam}(\Omega) \| u\|_{\varphi(\Omega)} \\ &\leq C_7 |\Omega|\mathrm{diam}(\Omega) \| u\|_{\varphi(\Omega)}. \end{aligned} \tag{2.29}$$

This ends the proof of Theorem 2.10.

Similarly, we can extend Theorem 2.7 into the global case. Under the conditions of Theorem 2.7, we have

$$\| u - u_{Q_0}\|_{\varphi(\Omega)} \leq C|\Omega|\mathrm{diam}(\Omega) \| du\|_{\varphi(\Omega)}. \tag{2.30}$$

Remark. If $\phi(t) = t^p$, then $\| u\|_{\varphi(\Omega)} = (\frac{1}{|\Omega|}\int_\Omega |u|^p)^{\frac{1}{p}}$ and the Luxemburg norm reduce to the L^p norm. Note that a typical Young function that belongs to the class ϕ_p is $\phi(t) = t^s$ with $1 \leq s < p$. We can easily increase $s \to \infty$ as $p \to \infty$, then for $p \geq 1$, our results can be held with L^p -norms. So some existing inequalities in [2-5] become the special cases of our results.

3 Examples

Example 1 We consider the Young function $\phi(t)$ given by

$$\varphi_1(t) = \frac{t^p}{\log^{1+\delta}(e+t)} \tag{3.1}$$

with $\delta > 0$, which satisfies the ϕ_p condition. We defines the Luxemburg norm $\| u\|_{\varphi_1(\Omega)} = \inf\{\lambda > 0 : \frac{1}{|\Omega|}\int_\Omega \varphi_1(\frac{|u(x)|}{\lambda})dx \leq 1\}$ in the Orlicz space $L^{\varphi_1}(\Omega)$. There is an advantage in using the following integral expression instead of $||u||_{\varphi_1}(\Omega)$

$$[u]_\Omega = \left(\frac{1}{|\Omega|} \int_\Omega \frac{|u(x)|^p}{\log^{1+\delta}\left(e + \frac{|u(x)|}{\|u\|_{p,\Omega}}\right)} dx \right)^{\frac{1}{p}}. \tag{3.2}$$

This is not a norm, but compares well with the Luxemburg norm. Using the elementary inequality

$$\min\{1, \lambda\} \le \frac{\log(e + \lambda t)}{\log(e + t)} \le \max\{1, \lambda\}, \tag{3.3}$$

we prove

$$C_1 \parallel u\|_{\varphi_1(\Omega)} \le [u]_\Omega \le C_2 \parallel u\|_{\varphi_1(\Omega)}. \tag{3.4}$$

Under the same conditions of Theorem 2.10, we have

$$[Tu - (Tu)_{Q_0}]_\Omega \le C|\Omega|\mathrm{diam}(\Omega)[u]_\Omega. \tag{3.5}$$

Example 2

we can consider another particular example given by

$$\varphi_2(t) = \frac{t^p}{e^{(1+t)^{(1-\delta)}}} \tag{3.6}$$

with $1 < \delta < 2p$, which is continuous, convex and increasing satisfying $\phi_2(0) = 0$ and $\phi_2(t) \to \infty$ as $t \to \infty$, so it is a Young function. It also satisfies the ϕ_p condition.

Competing interests

The author declares that they have no competing interests.

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