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New inequalities of hermite-hadamard type for convex functions with applications

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Abstract

In this paper, some new inequalities of the Hermite-Hadamard type for functions whose modulus of the derivatives are convex and applications for special means are given. Finally, some error estimates for the trapezoidal formula are obtained.
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1. Introduction

A function $f : I \to \mathbb{R}$ is said to be convex function on I if the inequality

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y),$$

holds for all $x, y \in I$ and $\alpha \in [0,1]$.

One of the most famous inequality for convex functions is so called Hermite-Hadamard's inequality as follows: Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$, *with* $a < b$. Then:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In [1], the following theorem which was obtained by Dragomir and Agarwal contains the Hermite-Hadamard type integral inequality.

Theorem 1. *Let* $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $a, b \in I^\circ$ *with* $a < b$. *If* $|f'|$ *is convex on* $[a, b]$, *then the following inequality holds*:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{1.2}$$

In [2] Kirmaci, Bakula, Özdemir and Pečarić proved the following theorem.

Theorem 2. *Let* $f : I \to \mathbb{R}$, $I \subset \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a, b]$, *where* $a, b \in I$, $a < b$. *If* $|f'|^q$ *is concave on* $[a, b]$ *for some* $q > 1$, *then*:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \le \left(\frac{b-a}{4}\right)\left[\frac{q-1}{2q-1}\right]^{\frac{q-1}{q}}\left(\left|f'\left(\frac{a+3b}{4}\right)\right|+\left|f'\left(\frac{3a+b}{4}\right)\right|\right). \quad (1.3)$$

In [3], Kirmaci obtained the following theorem and corollary related to this theorem.

Theorem 3. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and let $p > 1$. If the mapping $|f'|^p$ is concave on $[a, b]$, then we have*

$$\left|f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)dx\right| \le (b-a)\left[K\left|f'\left(\frac{aT+b(K-T)}{K}\right)\right|+M\left|f'\left(\frac{aN+b(M-N)}{M}\right)\right|\right]$$

where

$$K=\frac{A^2+(c-A)^2}{2}, T=\frac{A^3+c^3}{3}-\frac{Ac^2}{2}, M=\frac{(B-c)^2+(1-B)^2}{2},$$
$$N=\frac{B^3+c^3+1}{3}-(1+c^2)\frac{B}{2}.$$

Corollary 1. *Under the assumptions of Theorem 3 with $A = B = c = \frac{1}{2}$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)}{8}\left[\left|f'\left(\frac{5a+b}{6}\right)\right|+\left|f'\left(\frac{a+5b}{6}\right)\right|\right]. \quad (1.4)$$

For recent results and generalizations concerning Hermite-Hadamard's inequality see [1]-[5] and the references given therein.

2. The New Hermite-Hadamard Type Inequalities

In order to prove our main theorems, we first prove the following lemma:

Lemma 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds*:

$$\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du = \frac{(x-a)^2}{b-a}\int_0^1 (t-1)f'(tx+(1-t)a)dt+\frac{(b-x)^2}{b-a}\int_0^1 (1-t)f'(tx+(1-t)b)dt.$$

Proof. We note that

$$J=\frac{(x-a)^2}{b-a}\int_0^1 (t-1)f'(tx+(1-t)a)dt + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)f'(tx+(1-t)b)dt.$$

Integrating by parts, we get

$$\begin{aligned}
J &= \frac{(x-a)^2}{b-a}\left[(t-1)\frac{f(tx+(1-t)a)}{x-a}\bigg|_0^1 - \int_0^1 \frac{f(tx+(1-t)a)}{x-a}dt\right] \\
&+ \frac{(b-x)^2}{b-a}\left[(1-t)\frac{f(tx+(1-t)b)}{x-b}\bigg|_0^1 + \int_0^1 \frac{f(tx+(1-t)b)}{x-b}dt\right] \\
&= \frac{(x-a)^2}{b-a}\left[\frac{f(a)}{x-a} - \frac{1}{(x-a)^2}\int_a^x f(u)du\right] \\
&+ \frac{(b-x)^2}{b-a}\left[-\frac{f(b)}{x-b} + \frac{1}{(x-b)^2}\int_b^x f(u)du\right] \\
&= \frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du.
\end{aligned}$$

□

Using the Lemma 1 the following results can be obtained.

Theorem 4. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L\,[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds*:

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leq \frac{(x-a)^2}{b-a}\left[\frac{|f'(x)|+2|f'(a)|}{6}\right] + \frac{(b-x)^2}{b-a}\left[\frac{|f'(x)|+2|f'(b)|}{6}\right]
\end{aligned}$$

for each $x \in [a, b]$.

Proof. Using Lemma 1 and taking the modulus, we have

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leq \frac{(x-a)^2}{b-a}\int_0^1 (1-t)|f'(tx+(1-t)a)|dt \\
&+ \frac{(b-x)^2}{b-a}\int_0^1 (1-t)|f'(tx+(1-t)b)|dt.
\end{aligned}$$

Since $|f'|$ is convex, then we get

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leq \frac{(x-a)^2}{b-a}\int_0^1 (1-t)[t|f'(x)|+(1-t)|f'(a)|]dt \\
&+ \frac{(b-x)^2}{b-a}\int_0^1 (1-t)[t|f'(x)|+(1-t)|f'(b)|]dt \\
&= \frac{(x-a)^2}{b-a}\left[\frac{|f'(x)|+2|f'(a)|}{6}\right] + \frac{(b-x)^2}{b-a}\left[\frac{|f'(x)|+2|f'(b)|}{6}\right]
\end{aligned}$$

which completes the proof. □

Corollary 2. *In Theorem 4, if we choose* $x = \frac{a+b}{2}$ *we obtain*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{b-a}{12}\left(|f'(a)| + \left|f'\left(\frac{a+b}{2}\right)\right| + |f'(b)|\right).$$

Remark 1. *In Corollary 2, using the convexity of* $|f'|$ *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{b-a}{8}(|f'(a)| + |f'(b)|)$$

which is the inequality in (1.2).

Theorem 5. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f' \in L[a, b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|^{\frac{p}{p-1}}$ *is convex on* $[a, b]$ *and for some fixed* $q > 1$, *then the following inequality holds:*

$$\begin{aligned}&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right| \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}} \\ &\times\left[\frac{(x-a)^2\left[|f'(a)|^q + |f'(x)|^q\right]^{\frac{1}{q}} + (b-x)^2\left[|f'(x)|^q + |f'(b)|^q\right]^{\frac{1}{q}}}{b-a}\right]\end{aligned}$$

for each $x \in [a, b]$ *and* $q = \frac{p}{p-1}$.

Proof. From Lemma 1 and using the well-known Hölder integral inequality, we have

$$\begin{aligned}&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right| \\ &\leq \frac{(x-a)^2}{b-a}\int_0^1 (1-t)|f'(tx+(1-t)a)|dt \\ &\quad+\frac{(b-x)^2}{b-a}\int_0^1 (1-t)|f'(tx+(1-t)b)|dt \\ &\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}} \\ &\quad+\frac{(b-x)^2}{b-a}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|f'|^{\frac{p}{p-1}}$ is convex, by the Hermite-Hadamard's inequality, we have

$$\int_0^1 |f'(tx+(1-t)a)|^q dt \leq \frac{|f'(a)|^q + |f'(x)|^q}{2}$$

and

$$\int_0^1 |f'(tx+(1-t)b)|^q dt \leq \frac{|f'(b)|^q + |f'(x)|^q}{2},$$

so

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}$$
$$\times\left[\frac{(x-a)^2\left[|f'(a)|^q+|f'(x)|^q\right]^{\frac{1}{q}}+(b-x)^2\left[|f'(x)|^q+|f'(b)|^q\right]^{\frac{1}{q}}}{b-a}\right]$$

which completes the proof. □

Corollary 3. *In Theorem 5, if we choose* $x=\frac{a+b}{2}$ *we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}$$
$$\times\left[\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}(|f'(a)|+|f'(b)|).$$

The second inequality is obtained using the following fact: $\sum_{k=1}^n (a_k+b_k)^s\leq\sum_{k=1}^n (a_k)^s+\sum_{k=1}^n (b_k)^s$ *for* $(0\leq s<1)$, $a_1, a_2, a_3,\ldots, a_n\geq 0$; $b_1, b_2, b_3,\ldots, b_n\geq 0$ *with* $0\leq\frac{p-1}{p}<1$, *for* $p>1$.

Theorem 6. *Let* $f: I\subseteq\mathbb{R}\to\mathbb{R}$ *be a differentiable mapping on* I° *such that* $f'\in L\ [a, b]$, *where* $a, b\in I$ *with* $a<b$. *If* $|f'|^q$ *is concave on* $[a, b]$, *for some fixed* $q>1$, *then the following inequality holds:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\left[\frac{q-1}{2q-1}\right]^{\frac{q-1}{q}}\left[\frac{(x-a)^2\left|f'\left(\frac{a+x}{2}\right)\right|+(b-x)^2\left|f'\left(\frac{b+x}{2}\right)\right|}{b-a}\right]$$

for each $x\in[a, b]$.

Proof. As in Theorem 5, using Lemma 1 and the well-known Hölder integral inequality for $q>1$ and $p=\frac{q}{q-1}$, we have

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\le \frac{(x-a)^2}{b-a}\int_0^1 (1-t)|f'(tx+(1-t)a)|dt \\
&\quad+\frac{(b-x)^2}{b-a}\int_0^1 (1-t)|f'(tx+(1-t)b)|dt \\
&\le \frac{(x-a)^2}{b-a}\left(\int_0^1 (1-t)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1 |f'(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}} \\
&\quad+\frac{(b-x)^2}{b-a}\left(\int_0^1 (1-t)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1 |f'(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is concave on $[a, b]$, we can use the Jensen's integral inequality to obtain:

$$\begin{aligned}
\int_0^1 |f'(tx+(1-t)a)|^q dt &= \int_0^1 t^0|f'(tx+(1-t)a)|^q dt \\
&\le \left(\int_0^1 t^0 dt\right)\left|f'\left(\frac{1}{\int_0^1 t^0 dt}\int_0^1 (tx+(1-t)a)dt\right)\right|^q \\
&= \left|f'\left(\frac{a+x}{2}\right)\right|^q
\end{aligned}$$

Analogously,

$$\int_0^1 |f'(tx+(1-t)b)|^q dt \le \left|f'\left(\frac{b+x}{2}\right)\right|^q.$$

Combining all the obtained inequalities, we get

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\le \left[\frac{q-1}{2q-1}\right]^{\frac{q-1}{q}}\left[\frac{(x-a)^2|f'\left(\frac{a+x}{2}\right)|+(b-x)^2|f'\left(\frac{b+x}{2}\right)|}{b-a}\right]
\end{aligned}$$

which completes the proof. □

Remark 2. *In Theorem 6, if we choose* $x=\frac{a+b}{2}$*we have*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\le \left[\frac{q-1}{2q-1}\right]^{\frac{q-1}{q}}\left(\frac{b-a}{4}\right)\left(\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right)
\end{aligned}$$

which is the inequality in (1.3).

Theorem 7. *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* $I°$ *such that* $f' \in L[a, b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|^q$ *is convex on* $[a, b]$ *and for some fixed* $q \ge 1$, *then the following inequality holds:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2\left[|f'(x)|^q+2|f'(a)|^q\right]^{\frac{1}{q}}+(b-x)^2\left[|f'(x)|^q+2|f'(b)|^q\right]^{\frac{1}{q}}}{b-a}\right]$$

for each $x \in [a, b]$.

Proof. Suppose that $q \geq 1$. From Lemma 1 and using the well-known power-mean inequality, we have

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(x-a)^2}{b-a}\int_0^1 (1-t)|f'(tx+(1-t)a)|dt$$
$$+\frac{(b-x)^2}{b-a}\int_0^1 (1-t)|f'(tx+(1-t)b)|dt$$
$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)|f'(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)|f'(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is convex, therefore we have

$$\int_0^1 (1-t)|f'(tx+(1-t)a)|^q dt$$
$$\leq \int_0^1 (1-t)\left[t|f'(x)|^q+(1-t)|f'(a)|^q\right]dt$$
$$=\frac{|f'(x)|^q+2|f'(a)|^q}{6}$$

Analogously,

$$\int_0^1 (1-t)|f'(tx+(1-t)b)|^q dt \leq \frac{|f'(x)|^q+2|f'(b)|^q}{6}.$$

Combining all the above inequalities gives the desired result. □

Corollary 4. *In Theorem* 7, *choosing* $x = \frac{a+b}{2}$ *and then using the convexity of* $|f'|^q$ *we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \left(\frac{b-a}{8}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[\left(2|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(2|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$
$$\leq \left(\frac{3^{1-\frac{1}{q}}}{8}\right)(b-a)(|f'(a)|+|f'(b)|).$$

Theorem 8. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I°$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a, b]$, for some fixed $q \geq 1$, then the following inequality holds:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{1}{2}\left[\frac{(x-a)^2|f'\left(\frac{x+2a}{3}\right)|+(b-x)^2|f'\left(\frac{x+2b}{3}\right)|}{b-a}\right].$$

Proof. First, we note that by the concavity of $|f'|^q$ and the power-mean inequality, we have

$$|f'(tx+(1-t)a)|^q \geq t|f'(x)|^q+(1-t)|f'(a)|^q.$$

Hence,

$$|f'(tx+(1-t)a)| \geq t|f'(x)|+(1-t)|f'(a)|,$$

so $|f'|$ is also concave.

Accordingly, using Lemma 1 and the Jensen integral inequality, we have

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leq \frac{(x-a)^2}{b-a}\int_0^1 (1-t)|f'(tx+(1-t)a)|dt \\
&\quad +\frac{(b-x)^2}{b-a}\int_0^1 (1-t)|f'(tx+(1-t)b)|dt \\
&\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 (1-t)dt\right)\left|f'\left(\frac{\int_0^1 (1-t)(tx+(1-t)a)dt}{\int_0^1 (1-t)dt}\right)\right| \\
&\quad +\frac{(b-x)^2}{b-a}\left(\int_0^1 (1-t)dt\right)\left|f'\left(\frac{\int_0^1 (1-t)(tx+(1-t)b)dt}{\int_0^1 (1-t)dt}\right)\right| \\
&\leq \frac{1}{2}\left[\frac{(x-a)^2\left|f'\left(\frac{x+2a}{3}\right)\right|+(b-x)^2\left|f'\left(\frac{x+2b}{3}\right)\right|}{b-a}\right].
\end{aligned}$$

□

Remark 3. *In Theorem 8, if we choose $x = \frac{a+b}{2}$ we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{b-a}{8}\left[\left|f'\left(\frac{5a+b}{6}\right)\right|+\left|f'\left(\frac{a+5b}{6}\right)\right|\right]$$

which is the inequality in (1.4).

3. Applications to Special Means

Recall the following means which could be considered extensions of arithmetic, logarithmic and generalized logarithmic from positive to real numbers.

(1) The arithmetic mean:

$$A = A(a, b) = \frac{a+b}{2}; a, b \in \mathbb{R}$$

(2) The logarithmic mean:

$$L(a, b) = \frac{b-a}{\ln|b| - \ln|a|}; |a| \neq |b|, ab \neq 0, a, b \in \mathbb{R}$$

(3) The generalized logarithmic mean:

$$L_n(a, b) = \left[\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}\right]^{\frac{1}{n}}; n \in \mathbb{Z}\backslash\{-1, 0\}, a, b \in \mathbb{R}, a \neq b$$

Now using the results of Section 2, we give some applications to special means of real numbers.

Proposition 1. *Let* $a, b \in \mathbb{R}$, $a < b$, $0 \notin [a, b]$ *and* $n \in \mathbb{Z}$, $|n| \geq 2$. *Then, for all* $p > 1$

(a)

$$|A(a^n, b^n) - L_n^n(a, b)| \leq |n|(b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}} A\left(|a|^{n-1}, |b|^{n-1}\right) \tag{3.1}$$

and

(b)

$$|A(a^n, b^n) - L_n^n(a, b)| \leq |n|(b-a)\frac{3^{1-\frac{1}{q}}}{4} A\left(|a|^{n-1}, |b|^{n-1}\right). \tag{3.2}$$

Proof. The assertion follows from Corollary 3 and 4 for $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$, $|n| \geq 2$. □

Proposition 2. *Let* $a, b \in \mathbb{R}$, $a < b$, $0 \notin [a, b]$. *Then, for all* $q \geq 1$,

(a)

$$|A(a^{-1}, b^{-1}) - L^{-1}(a, b)| \leq (b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}} A\left(|a|^{-2}, |b|^{-2}\right) \tag{3.3}$$

and

(b)

$$|A(a^{-1}, b^{-1}) - L^{-1}(a, b)| \leq (b-a)\left(\frac{3^{1-\frac{1}{q}}}{4}\right) A\left(|a|^{-2}, |b|^{-2}\right). \tag{3.4}$$

Proof. The assertion follows from Corollary 3 and 4 for $f(x) = \frac{1}{x}$. □

4. The Trapezoidal Formula

Let d be a division $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ of the interval $[a, b]$ and consider the quadrature formula

$$\int_a^b f(x)dx = T(f, d) + E(f, d) \tag{4.1}$$

where

$$T(f, d) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)$$

for the trapezoidal version and $E(f, d)$ denotes the associated approximation error.

Proposition 3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$ and $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$, where $p > 1$. Then in* (4.1), *for every division d of $[a, b]$, the trapezoidal error estimate satisfies*

$$|E(f, d)| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{q}} \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^2}{2}(|f'(x_i)| + |f'(x_{i+1})|).$$

Proof. On applying Corollary 3 on the subinterval $[x_i, x_{i+1}]$ ($i = 0, 1, 2, \ldots, n - 1$) of the division, we have

$$\left|\frac{f(x_i) + f(x_{i+1})}{2} - \frac{1}{x_{i+1} - x_i}\int_{x_i}^{x_{i+1}} f(x)dx\right| \leq \frac{(x_{i+1} - x_i)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{q}}(|f'(x_i)| + |f'(x_{i+1})|).$$

Hence in (4.1) we have

$$\begin{aligned} \left|\int_a^b f(x)dx - T(f, d)\right| &= \left|\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)\right\}\right| \\ &\leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)\right| \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{q}} \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^2}{2}(|f'(x_i)| + |f'(x_{i+1})|) \end{aligned}$$

which completes the proof. □

Proposition 4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a, b]$, for some fixed $q > 1$, Then in* (4.1), *for every division d of $[a, b]$, the trapezoidal error estimate satisfies*

$$|E(f, d)| \leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^2}{4}\left(\left|f'\left(\frac{3x_i + x_{i+1}}{4}\right)\right| + \left|f'\left(\frac{x_i + 3x_{i+1}}{4}\right)\right|\right).$$

Proof. The proof is similar to that of Proposition 3 and using Remark 2. □

Proposition 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a, b]$, for some fixed $q \geq 1$, Then in* (4.1), *for every division d of $[a, b]$, the trapezoidal error estimate satisfies*

$$|E(f, d)| \leq \frac{1}{8}\sum_{i=0}^{n-1} (x_{i+1} - x_i)^2\left(\left|f'\left(\frac{5x_i + x_{i+1}}{6}\right)\right| + \left|f'\left(\frac{x_i + 5x_{i+1}}{6}\right)\right|\right).$$

Proof. The proof is similar to that of Proposition 3 and using Remark 3. □

Authors' contributions
HK and MA carried out the design of the study and performed the analysis.
MEO (adviser) participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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