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The existence of solutions to the nonhomogeneous $\mathcal{A}$-harmonic equation

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Abstract

In this paper, we introduce the obstacle problem about the nonhomogeneous $\mathcal{A}$-harmonic equation. Then, we prove the existence and uniqueness of solutions to the nonhomogeneous $\mathcal{A}$-harmonic equation and the obstacle problem.

Keywords: the obstacle problem, the nonhomogeneous A-harmonic equation, existence and uniqueness of solutions

1 Introduction

In this paper, we study the nonhomogeneous $\mathcal{A}$-harmonic equation

$$-\mathtt{div}\mathcal{A}(x, \nabla u(x)) = f(x),$$

where $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is an operator and f is a function satisfying some assumptions given in the next section. We give the definition of solutions to the nonhomogeneous $\mathcal{A}$-harmonic equation and the obstacle problem. In the mean time, we show some properties of their solutions. Then, we prove the existence and uniqueness of solutions to the Dirichlet problem for the nonhomogeneous $\mathcal{A}$-harmonic equation with Sobolev boundary values.

Let $\mathbb{R}^n$ be the real Euclidean space with the dimension n. Throughout this paper, all the topological notions are taken with respect to $\mathbb{R}^n$. $E \Subset F$ means that $\bar{E}$ is a compact subset of F. $C(\Omega)$ is the set of all continuous functions $u : \Omega \to \mathbb{R}$. $\mathtt{spt}u$ is the smallest closed set such that u vanishes outside $\mathtt{spt}u$.

$$C^k(\Omega) = \{\varphi : \Omega \to \mathbb{R} : \mathtt{the}\ k\mathtt{th\ -\ derivative\ of}\ \varphi\ \mathtt{is\ continuous}\},$$
$$C_0^k(\Omega) = \{\varphi \in C^k(\Omega) : \mathtt{spt}\varphi \Subset \Omega\},$$
$$C^\infty(\Omega) = \bigcap_{k=1}^{\infty} C^k(\Omega)$$

and

$$C_0^\infty(\Omega) = \{\varphi \in C^\infty(\Omega) : \mathtt{spt}\varphi \Subset \Omega\}.$$

Let $L^p(\Omega) = \{\phi: \Omega \to \mathbb{R}: \int_\Omega |\phi|^p \, dx < \infty\}$ and $L^p(\Omega; \mathbb{R}^n) = \{\phi: \Omega \to \mathbb{R}^n : \int_\Omega |\phi|^p \, dx < \infty\}$, $1 < p < \infty$. Denote the norm of $L^p(\Omega)$ and $L^p(\Omega; \mathbb{R}^n)$ by $|| \cdot ||_p$,

$$||\phi||_p = \left(\int_\Omega |\phi|^p \mathrm{d}x\right)^{1/p},$$

where $\varphi \in L^p(\Omega)$(or $L^p(\Omega; \mathbb{R}^n)$).

For $\phi \in C^\infty(\Omega)$, let

$$||\varphi||_{1,p} = \left(\int_\Omega |\varphi|^p \mathrm{d}x\right)^{1/p} + \left(\int_\Omega |\nabla\varphi|^p \mathrm{d}x\right)^{1/p},$$

where $\nabla\phi = (\partial_1\phi, ..., \partial_n\phi)$ is the gradient of ϕ. The Sobolev space $H^{1,p}(\Omega)$ is defined to be the completion of the set $\{\phi \in C^\infty(\Omega): ||\phi||_{1,p} < \infty\}$ with respect to the norm $||\cdot||_{1,p}$. In other words, $u \in H^{1,p}(\Omega)$ if and only if $u \in L^p(\Omega)$ and there is a function $v \in L^p(\Omega; \mathbb{R}^n)$ and a sequence $\phi_i \in C^\infty(\Omega)$, such that

$$\int_\Omega |\varphi_i - u|^p \mathrm{d}x \to 0 \text{ and} \int_\Omega |\nabla\varphi_i - v|^p \mathrm{d}x \to 0, i \to \infty.$$

We call v the gradient of u in $H^{1,p}(\Omega)$ and write $v = \nabla u$.

The space $H_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^{1,p}(\Omega)$. Obviously, $H^{1,p}(\Omega)$ and $H_0^{1,p}(\Omega)$ are Banach space with respect to the norm $||\cdot||_{1,p}$. Moreover, $||\cdot||_{1,p}$ is uniformly convex and the Sobolev space $H^{1,p}(\Omega)$ and $H_0^{1,p}(\Omega)$ are reflexive; see [1] for details.

$$u \in H_{loc}^{1,p}(\Omega) \text{ if and only if } u \in H^{1,p}(\Omega') \text{ for each open set } \Omega' \Subset \Omega.$$

The Dirichlet space $L^{1,p}(\Omega)$ and $L_0^{1,p}(\Omega)$ are defined as follows: $u \in L^{1,p}(\Omega)$ if and only if $u \in H_{loc}^{1,p}(\Omega)$ and $\nabla u \in L^p(\Omega)$; $L_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the semi-norm $p(u) = (\int_\Omega|\nabla u|^p)^{1/p}$. In other words, $L_0^{1,p}(\Omega)$ is the set of all functions $u \in L^{1,p}(\Omega)$, for which there is a sequence $\varphi_j \in C_0^\infty(\Omega)$ such that $\nabla\phi_j \to \nabla u$ in $L^p(\Omega; \mathbb{R}^n)$.

Lemma 1.1 [2]*Let* $1 < p < \infty$ *and* f_i *be a bounded sequence in* $L^p(\Omega)$, *i.e.* $f_i \in L^p(\Omega)$ *and* $\sup_i ||f_i||_p < \infty$. *If* $f_i \to f$ *a.e. in* Ω, *then* f_i *converges to* f *weakly in* $L^p(\Omega)$.

Lemma 1.2 [3]*(1) If* $u \in H_0^{1,p}(\Omega)$ *with* $\nabla u = 0$, *then* $u = 0$.

(2) If $u, v \in H^{1,p}(\Omega)$, *then* $\min\{u, v\}$ *and* $\max\{u, v\}$ *are in* $H^{1,p}(\Omega)$ *with*

$$\nabla \max\{u, v\} = \begin{cases} \nabla u, & u \geq v \\ \nabla v, & u \leq v \end{cases} \quad \text{and} \nabla \min\{u, v\} = \begin{cases} \nabla v, & u \geq v \\ \nabla u, & u \leq v \end{cases}.$$

(3) If $u, v \in H_0^{1,p}(\Omega)$, *then* $\min\{u, v\}$ *and* $\max\{u, v\}$ *are in* $H_0^{1,p}(\Omega)$. *Moreover, if* $u \in H_0^{1,p}(\Omega)$ *is nonnegative, then there is a sequence of nonnegative functions* $\varphi_i \in C_0^\infty(\Omega)$ *converging to* u *in* $H^{1,p}(\Omega)$.

2 The nonhomogeneous $\mathcal{A}$-harmonic equation

The following nonlinear elliptic equation

$$-\mathtt{div}\mathcal{A}(x, \nabla u) = f(x) \tag{2.1}$$

is called the nonhomogeneous $\mathcal{A}$-harmonic equation, where $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is an operator satisfying the following assumptions for some constants $0 < \alpha \leq \beta < \infty$:

(I) the mapping $x \mapsto \mathcal{A}(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^n$ and
the mapping $\xi \mapsto \mathcal{A}(x, \xi)$ is continuous for a.e. $x \in \mathbb{R}^n$;

for all $\xi \in \mathbb{R}^n$ and almost all $x \in \mathbb{R}^n$,

(II) $\mathcal{A}(x, \xi) \cdot \xi \geq \alpha |\xi|^p$,

(III) $|\mathcal{A}(x, \xi)| \leq \beta |\xi|^{p-1}$,

(IV) $(\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0$,

whenever $\xi_1, \xi_2 \in \mathbb{R}^n$, $\xi_1 \neq \xi_2$; and

(V) $\mathcal{A}(x, \lambda\xi) = \lambda|\lambda|^{p-2}\mathcal{A}(x, \xi)$

whenever $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and f is a function satisfying $f \in L^{p/(p-1)}(\Omega)$.

If $f = 0$, the equation (2.1) degenerates into the homogeneous $\mathcal{A}$-harmonic equation

$$-\mathtt{div}\mathcal{A}(x, \nabla u(x)) = 0. \tag{2.2}$$

A continuous solution to (2.2) in Ω is called $\mathcal{A}$-harmonic function. Many well-known results have been developed about (2.2), especially as (2.2) is the corresponding $\mathcal{A}$-harmonic equation of differential forms; see [4-10].

Definition 2.1 *A function $u \in H^{1,p}_{loc}(\Omega)$ is a (weak) solution to the equation (2.1) in Ω, if $-\mathtt{div}\mathcal{A}(x, \nabla u) = f$ weakly in Ω, i.e.*

$$\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla\varphi - f\varphi)dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$.

A function $u \in H^{1,p}_{loc}(\Omega)$ is a supersolution to (2.1) in Ω, if $-\mathtt{div}\mathcal{A}(x, \nabla u) \geq f$ in weakly Ω, i.e.

$$\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla\varphi - f\varphi)dx \geq 0$$

whenever $\varphi \in C_0^\infty(\Omega)$ is nonnegative.

A function $u \in H^{1,p}_{loc}(\Omega)$ is a subsolution to (2.1) in Ω, if $-\mathtt{div}\mathcal{A}(x, \nabla u) \leq f$ weakly in Ω, i.e.

$$\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla\varphi - f\varphi)dx \leq 0$$

whenever $\varphi \in C_0^\infty(\Omega)$ is nonnegative.

Remark: If u is a solution (a supersolution or a subsolution), then $u+\tau$ is also a solution (a supersolution or a subsolution), but $\lambda u + \tau$, λ, $\tau \in \mathbb{R}$ may not.

Proposition 2.1 *A function u is a solution (a supersolution or a subsolution) to (2.1) in Ω if and only if Ω can be covered by open sets where u is a solution (a supersolution or a subsolution).*

Proof. We just give the proof in the case that u is a solution and the others are similar.

(i) Since Ω is covered by itself, it is easy to know that Ω can be covered by open sets where u is a solution.

(ii) Let $\Omega = \bigcup_{\lambda \in I} \Omega_\lambda$ and u be the solution to (2.1) in Ω_λ for each $\lambda \in I$, where I is an index set. For each $\varphi \in C_0^\infty(\Omega)$, there is a subset $\{\Omega_1, \ldots, \Omega_m\}$ of $\{\Omega_\lambda\}_{\lambda \in I}$ such that

$\mathtt{spt}\varphi \subset \bigcup_{i=1}^{m} \Omega_i = D$. Choose a partition of unity of D, $\{g_1, ..., g_m\}$, subordinate to the covering Ω_i, such that $g_i \in C_0^\infty(\Omega_i)$, $0 \le g_i \le 1$ and $\sum_{i=1}^{m} g_i \equiv 1$ in D; see Lemma 2.3.1 in [11]. Thus,

$$\begin{aligned}\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla\varphi - f\varphi)\mathrm{d}x &= \int_D (\mathcal{A}(x, \nabla u) \cdot \nabla\varphi - f\varphi)\mathrm{d}x \\ &= \int_D (\mathcal{A}(x, \nabla u) \cdot \nabla(\sum_{i=1}^{m} g_i\varphi) - f(\sum_{i=1}^{m} g_i\varphi))\mathrm{d}x \\ &= \sum_{i=1}^{m} \int_D (\mathcal{A}(x, \nabla u) \cdot \nabla(g_i\varphi) - g_i\varphi f)\mathrm{d}x.\end{aligned}$$

Note that $g_i \in C_0^\infty(\Omega_i)$ and $\varphi \in C_0^\infty(\Omega)$, it is easy to see that $g_i\varphi \in C_0^\infty(\Omega_i)$. Since u is solution in Ω_i, we have

$$\int_D (\mathcal{A}(x, \nabla u) \cdot \nabla(g_i\varphi) - g_i\varphi f)\mathrm{d}x = \int_{\Omega_i} (\mathcal{A}(x, \nabla u) \cdot \nabla(g_i\varphi) - g_i\varphi f)\mathrm{d}x = 0.$$

Therefore,

$$\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla\varphi - f\varphi)\mathrm{d}x = 0.$$

It means that u is a solution in Ω.

Lemma 2.1 *If* $u \in L^{1,p}(\Omega)$ *is a solution (respectively, a supersolution or a subsolution) to (2.1), then*

$$\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla\varphi - f\varphi)dx = 0 \ (\mathtt{respectively}, \quad \geq 0 \ \mathtt{or} \leq 0)$$

for all $\varphi \in H_0^{1,p}(\Omega)$*(respectively, for all nonnegative* $\varphi \in H_0^{1,p}(\Omega)$*or for all nonnegative* $\varphi \in H_0^{1,p}(\Omega)$*).*

Proof. For all $\varphi \in H_0^{1,p}(\Omega)$, there is a sequence $\varphi_i \in C_0^\infty(\Omega)$, such that $\phi_i \to \phi$ in $H^{1,p}(\Omega)$.

Since $\mathcal{A}$ satisfies the assumption (III), $f \in L^{p/(p-1)}(\Omega)$ and $u \in \Omega\ L^{1,p}(\Omega)$, it follows that

$$\begin{aligned}&\left|\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla\varphi - f\varphi)\mathrm{d}x - \int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla\varphi_i - f\varphi_i)\mathrm{d}x\right| \\ &= \left|\int_\Omega (\mathcal{A}(x, \nabla u) \cdot (\nabla\varphi - \nabla\varphi_i) - f(\varphi - \varphi_i))\mathrm{d}x\right| \\ &\le \int_\Omega |\mathcal{A}(x, \nabla u)||\nabla\varphi - \nabla\varphi_i|\mathrm{d}x + \int_\Omega |f||\varphi - \varphi_i|\mathrm{d}x \\ &\le \beta \int_\Omega |\nabla u|^{p-1}|\nabla\varphi - \nabla\varphi_i|\mathrm{d}x + \int_\Omega |f||\varphi - \varphi_i|\mathrm{d}x \\ &\le \beta(\int_\Omega |\nabla u|^p\mathrm{d}x)^{1-\frac{1}{p}}(\int_\Omega |\nabla\varphi - \nabla\varphi_i|^p\mathrm{d}x)^{\frac{1}{p}} + (\int_\Omega |f|^{p/(p-1)}\mathrm{d}x)^{1-\frac{1}{p}}(\int_\Omega |\varphi - \varphi_i|^p\mathrm{d}x)^{\frac{1}{p}} \\ &\le M\{(\int_\Omega |\nabla\varphi - \nabla\varphi_i|^p\mathrm{d}x)^{\frac{1}{p}} + (\int_\Omega |\varphi - \varphi_i|^p\mathrm{d}x)^{\frac{1}{p}}\} \\ &= M||\varphi - \varphi_i||_{1,p} \to 0,\end{aligned}$$

where $M = \max\{\beta(\int_\Omega |\nabla u|^p \mathrm{d}x)^{1-\frac{1}{p}}, (\int_\Omega |f|^{p/(p-1)} \mathrm{d}x)^{1-\frac{1}{p}}\} < \infty$.

Since u is a solution,

$$\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla \varphi - f\varphi)\mathrm{d}x = \lim_{i\to\infty} \int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla \varphi_i - f\varphi_i)\mathrm{d}x = 0.$$

If $u \in L^{1,p}(\Omega)$ is a supersolution or a subsolution, by Lemma 1.2, there is a sequence of nonnegative functions $\varphi_i \in C_0^\infty(\Omega)$ converging to the nonnegative function ϕ in $H^{1,p}(\Omega)$. By the same discussion, the lemma follows.

Remark: Using the similar method as above, it is easy to prove that, if u is a solution (a supersolution or a subsolution),

$$\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla \varphi - f\varphi)\mathrm{d}x = 0 \ \ (\geq 0 \text{ or } \leq 0)$$

for all (nonegative) $\varphi \in H_0^{1,p}(\Omega)$ with compact support.

Proposition 2.2 *A function u is a solution to (2.1) if and only if u is a supersolution and a subsolution.*

Proof. Obviously, u is both a supersolution and a subsolution if u is a solution.

To establish the converse, for each $\varphi \in C_0^\infty(\Omega)$, let ϕ^+ be the positive part and ϕ^- be the negative part of ϕ. Then, both ϕ^+ and ϕ^- are in $H_0^{1,p}(\Omega)$ and have compact support. Since u is both a supersolution and a subsolution and $\phi^+ \geq 0$, $-\phi^- \geq 0$, the following inequalities hold,

$$\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla \varphi^+ - f\varphi^+)\mathrm{d}x \geq 0,$$

$$\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla (-\varphi^-) - f(-\varphi^-))\, \mathrm{d}x \geq 0,$$

$$\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla \varphi^+ - f\varphi^+)\mathrm{d}x \leq 0$$

and

$$\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla (-\varphi^-) - f(-\varphi^-))\mathrm{d}x \leq 0.$$

By the above inequalities,

$$\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla \varphi^+ - f\varphi^+)\mathrm{d}x = 0 \text{ and } \int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla \varphi^- - f\varphi^-)\mathrm{d}x = 0.$$

Then,

$$\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla \varphi - f\varphi)\mathrm{d}x = \int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla \varphi^+ - f\varphi^+)\mathrm{d}x + \int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla \varphi^- - f\varphi^-)\mathrm{d}x = 0$$

This proves that u is a solution to (2.1).

Lemma 2.2 ***(Comparison Lemma)*** *Let $u \in H^{1,p}(\Omega)$ be a supersolution and $v \in H^{1,p}(\Omega)$ be a subsolution to (2.1). If $\eta = \min\{u - v, 0\} \in H_0^{1,p}(\Omega)$, then $u \geq v$ a.e. in Ω.*

Proof. By $\eta = \min\{u - v, 0\}$ and Lemma 1.2, $\eta \le 0$ and $\nabla\eta = \begin{cases} \nabla u - \nabla v, & u < v \\ 0, & u \ge v \end{cases}$.

Since $u \in H^{1,p}(\Omega)$ is a supersolution and $v \in H^{1,p}(\Omega)$ is a subsolution, the following inequalities hold,

$$-\int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla\eta - f\eta)\mathrm{d}x = \int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla(-\eta) - f(-\eta))\mathrm{d}x \ge 0,$$

and

$$\int_\Omega (\mathcal{A}(x, \nabla v) \cdot \nabla\eta - f\eta)\mathrm{d}x \ge 0.$$

Then, by the assumption (IV),

$$\begin{aligned} 0 &\le \int_\Omega (\mathcal{A}(x, \nabla v) \cdot \nabla\eta - f\eta)\mathrm{d}x - \int_\Omega (\mathcal{A}(x, \nabla u) \cdot \nabla\eta - f\eta)\mathrm{d}x \\ &= \int_\Omega (\mathcal{A}(x, \nabla v) - (\mathcal{A}(x, \nabla u)) \cdot \nabla\eta \,\mathrm{d}x \\ &= \int_{\{u<v\}} (\mathcal{A}(x, \nabla v) - (\mathcal{A}(x, \nabla u)) \cdot \nabla(u - v)\mathrm{d}x \\ &= -\int_{\{u<v\}} (\mathcal{A}(x, \nabla v) - (\mathcal{A}(x, \nabla u)) \cdot \nabla(v - u)\mathrm{d}x \\ &= -\int_{\{u<v\}\cap\{\nabla u \ne \nabla v\}} (\mathcal{A}(x, \nabla v) - (\mathcal{A}(x, \nabla u)) \cdot \nabla(v - u)\mathrm{d}x \le 0 \end{aligned}$$

Therefore, the Lebesgue measure of the set $\{u < v\} \cap \{\nabla u \ne \nabla v\}$ is zero. That is $\nabla\eta = 0$ a.e. in Ω By $\eta \in H_0^{1,p}(\Omega)$ and Lemma 1.2, $\eta = 0$ a.e. in Ω. Thus, $u \ge v$ a.e. in Ω.

3 The obstacle problem

Suppose that Ω is bounded in $\mathbb{R}^n$, $\psi : \Omega \to [-\infty,\infty]$ is a function and $\vartheta \in H^{1,p}(\Omega)$). Let

$$\mathcal{K}_{\psi,\vartheta} = \mathcal{K}_{\psi,\vartheta}(\Omega) = \{v \in H^{1,p}(\Omega) : v \ge \psi \text{ a.e. in } \Omega \text{ and } v - \vartheta \in H_0^{1,p}(\Omega)\}.$$

If $\psi = \vartheta$, write $\mathcal{K}_{\psi,\psi}(\Omega) = \mathcal{K}_\psi(\Omega)$.

The problem is to find a function u in $K_{\psi,\vartheta}$ such that

$$\int_\Omega (\mathcal{A}(x, \nabla u) \cdot (\nabla v - \nabla u) - f(v - u))\mathrm{d}x \ge 0 \tag{3.1}$$

whenever $v \in \mathcal{K}_{\psi,\vartheta}$. We call the function ψ an obstacle.

Definition 3.1 *If a function $u \in \mathcal{K}_{\psi,\vartheta}(\Omega)$ satisfies (3.1) for all $v \in \mathcal{K}_{\psi,\vartheta}(\Omega)$, we say that u is a solution to the obstacle problem with obstacle ψ and boundary vales ϑ or a solution to the obstacle problem in $\mathcal{K}_{\psi,\vartheta}(\Omega)$.*

If u is a solution to the obstacle problem in $\mathcal{K}_{\psi,u}(\Omega)$, we say that u is a solution to the obstacle problem with obstacle ψ.

Proposition 3.1 *(1) A solution u to the obstacle problem is always a supersolution to (2.1) in Ω.*

(2) If u is a supersolution to (2.1) in Ω, u is a solution to the obstacle problem in $\mathcal{K}_{u,u}(D)$*for each open sets D ⊠ Ω. Moreover, if Ω is bounded, u is a solution to the obstacle problem in* $\mathcal{K}_{u,u}(\Omega)$.

(3) A solution u to the obstacle problem in $\mathcal{K}_{-\infty,u}(\Omega)$*is a solution to (2.1) in Ω.*

(4) If u is a solution to (2.1) in Ω, u is a solution to the obstacle problem in $\mathcal{K}_{-\infty,u}(D)$*for each open set D ⊠ Ω. Moreover, if Ω is bounded, u is a solution to the obstacle problem in* $\mathcal{K}_{-\infty,u}(\Omega)$.

Theorem 3.1 *Suppose u is a solution to the obstacle problem in* $\mathcal{K}_{\psi,\vartheta}(\Omega)$. *If* $v \in H^{1,p}(\Omega)$ *is a supersolution to (2.1) in Ω, such that* $\min\{u, v\} \in \mathcal{K}_{\psi,\vartheta}(\Omega)$, *then* $v \geq u$ *a.e. in Ω.*

The proof is similar to Lemma 2.2.

4 The existence of solutions

In this section, we introduce the main work of this paper, to prove the existence and the uniqueness of solutions to the nonhomogeneous $\mathcal{A}$- harmonic equation. We can see this work for the $\mathcal{A}$-harmonic equation (2.2) in [3, Chapter 3 and Appendix I] for details. We use the similar method to prove our results.

First, we introduce the following proposition as the theoretical basis for our work, which is a general result in the theory of monotone operators; see [12]. Let X be a reflexive Banach space and denote its dual by X'. Let $||\cdot||$ be the norm of X and $\langle\cdot,\cdot\rangle$ be a pairing between X' and X. K is a closed convex subset of X.

Definition 4.1 *A mapping* $\mathcal{L} : K \to X'$ *is called monotone, if*

$$\langle \mathcal{L}u - \mathcal{L}v, u - v\rangle \geq 0 \tag{4.1}$$

for all u, v in K.

$\mathcal{L}$*is called coercive on K, if there exists* $\phi \in K$ *such that*

$$\frac{\langle \mathcal{L}u_j - \mathcal{L}\varphi, u_j - \varphi\rangle}{||u_j - \varphi||} \to \infty \tag{4.2}$$

for each sequence u_j *in K with* $||u_j|| \to \infty$.

$\mathcal{L}$*is called weakly continuous on K, if* $\mathcal{L}u_j$ *converges to* $\mathcal{L}u$ *weakly in X', i.e.*

$$\langle \mathcal{L}u_j, v\rangle \to \langle \mathcal{L}u, v\rangle \text{ for each } v \in X, \tag{4.3}$$

whenever $u_j \in K$ *converges to* $u \in K$ *in X.*

Proposition 4.1 *Let K be a nonempty closed convex subset of X and let* $\mathcal{L}: K \to X'$ *be monotone, coercive and weakly continuous on K. Then there exists an element u in K such that*

$$\langle \mathcal{L}u, v - u\rangle \geq 0 \tag{4.4}$$

whenever $v \in K$.

Lemma 4.1 *Let* x_i *be a sequence of X. For any subsequence* x_{i_j} *of* x_i, *there is a subsequence* $x_{i_{j_k}}$ *of,* x_{i_j}*such that* $x_{i_{j_k}}$*converges to* x_0 *weakly in X and the weak limit* x_0 *is independent of the choice of the subsequence of* x_i. *Then* x_i *converges to* x_0 *weakly in X.*

Proof. Suppose that x_i does not converge to x_0 weakly in X. Then, there exist $\varepsilon_0 > 0$, $y_0 \in X'$ and a subsequence x_{i_j} of x_i, such that

$$\langle y_0, x_{i_j} - x_0 \rangle \geq \varepsilon_0$$

for each $j \in \mathbb{N}$.

Obviously, for any subsequence $x_{i_{j_k}}$ of x_{i_j}, $x_{i_{j_k}}$ cannot converge to x_0 weakly in X. This contradicts the condition of the lemma.

Therefore, x_i converges to x_0 weakly in X.

Now let $X = L^p(\Omega) \times L^p(\Omega; \mathbb{R}^n)$. Then, X is a reflexive Banach space and its dual $X' = L^{p/(p-1)}(\Omega) \times L^{p/(p-1)}(\Omega; \mathbb{R}^n)$. The norm of X is

$$||g|| = ||g_1||_p + ||g_2||_p$$

for all $g = (g_1, g_2) \in X$. $\langle \cdot, \cdot \rangle$ is the usual pairing between X' and X,

$$\langle h, g \rangle = \int_\Omega (h_1 g_1 + h_2 \cdot g_2) \mathrm{d}x,$$

where $g = (g_1, g_2)$ is in X and $h = (h_1, h_2)$ in X'.

Let Ω be a bounded open set in $\mathbb{R}^n$, $\vartheta \in H^{1,p}(\Omega)$ and $\psi: \Omega \to [-\infty, \infty]$ be any function. Construct the obstacle set

$$\mathcal{K}_{\psi,\vartheta} = \mathcal{K}_{\psi,\vartheta}(\Omega) = \{v \in H^{1,p}(\Omega) : v \geq \psi \texttt{ a.e. } \texttt{in } \Omega \texttt{ and } v - \vartheta \in H_0^{1,p}(\Omega)\}$$

and suppose that $\mathcal{K}_{\psi,\vartheta}$ is not empty.

Let $K = \{(v, \nabla v) : v \in \mathcal{K}_{\psi,\vartheta}\}$. Then, K is also not empty.

Lemma 4.2 *K is a nonempty closed convex subset of X.*

Proof. (i) Suppose that $(v, \nabla v) \in K$. Because $v \in \mathcal{K}_{\psi,\vartheta}$, v is in $H^{1,p}(\Omega)$. Then, $v \in L^p(\Omega)$ and $\nabla v \in L^p(\Omega)$. That means $(v, \nabla v) \in X$. Therefore, $K \subset X$.

(ii) If $(v_i, \nabla v_i) \in K$ is a sequence which converges to (v, ϕ) in X, where $\phi = (\phi_1, \ldots, \phi_n) \in L^p(\Omega; \mathbb{R}^n)$, it follows that

$$\int_\Omega |(v_i - \vartheta) - (v - \vartheta)|^p \mathrm{d}x = \int_\Omega |v_i - v|^p \mathrm{d}x \to 0,$$

and

$$\int_\Omega |(\nabla v_i - \nabla \vartheta) - (\varphi - \nabla \vartheta)|^p \mathrm{d}x = \int_\Omega |\nabla v_i - \varphi|^p \mathrm{d}x \to 0.$$

Since $v_i - \vartheta \in H_0^{1,p}(\Omega)$,$v - \vartheta \in H_0^{1,p}(\Omega)$ and $\nabla v = \phi$.

Since $v_i \to v$ in $L^p(\Omega)$, there exists a subsequence v_{i_j}, such that $v_{i_j} \to v$ a.e. in Ω. By $v_i \geq \psi$ a.e. in Ω, $v \geq \psi$ a.e. in Ω.

By the argumentation above, we have $v \in \mathcal{K}_{\psi,\vartheta}$ and $\nabla v = \phi$. So $(v, \phi) = (v, \nabla v) \in K$. This means K is closed in X.

(iii) Let $(u, \nabla u) \in K$, $(v, \nabla v) \in K$ and $\lambda \in [0, 1]$.

$$\lambda u + (1 - \lambda) v \geq \lambda \psi + (1 - \lambda) \psi = \psi \texttt{ a.e. in } \Omega,$$
$$\lambda u + (1 - \lambda) v - \vartheta = \lambda (u - \vartheta) + (1 - \lambda)(v - \vartheta) \in H_0^{1,p}(\Omega).$$

It means that $\lambda u + (1 - \lambda) v \in K_{\psi,\vartheta}$. Then,

$$\lambda(u, \nabla u) + (1 - \lambda)(v, \nabla v) = (\lambda u + (1 - \lambda) v, \nabla(\lambda u + (1 - \lambda) v)) \in K.$$

Therefore, K is convex in X.

Define a mapping $\mathscr{L}: K \to X'$ by $\mathscr{L}(v, \nabla v) = (-f, \mathcal{A}(x, \nabla v))$ for each $(v, \nabla v) \in K$. For convenience, we denote $\mathscr{L}(v, \nabla v)$ simply by $\mathscr{L}v$. For any element $h = (h_1, h_2) \in X$,

$$\langle \mathscr{L}v, h\rangle = \int_\Omega ((-f)h_1 + \mathcal{A}(x, \nabla v) \cdot h_2)\mathrm{d}x = \int_\Omega (\mathcal{A}(x, \nabla v) \cdot h_2 - fh_1)\mathrm{d}x.$$

Since $f \in L^{p/(p-1)}(\Omega)$, by the assumption (III) and the Hölder inequality, we have

$$\begin{aligned}
&|\int_\Omega (\mathcal{A}(x, \nabla v) \cdot h_2 - fh_1)\mathrm{d}x| \\
&\leq \int_\Omega |\mathcal{A}(x, \nabla v)||h_2|\mathrm{d}x + \int_\Omega |f||h_1|\mathrm{d}x \\
&\leq \beta \int_\Omega |\nabla v|^{p-1}|h_2|\mathrm{d}x + \int_\Omega |f||h_1|\mathrm{d}x \\
&\leq \beta(\int_\Omega |\nabla v|^p \mathrm{d}x)^{1-\frac{1}{p}} (\int_\Omega |h_2|^{\frac{1}{p}}\mathrm{d}x)^{\frac{1}{p}} + (\int_\Omega |f|^{\frac{p}{p-1}}\mathrm{d}x)^{1-\frac{1}{p}} (\int_\Omega |h_1|^{\frac{1}{p}}\mathrm{d}x)^{\frac{1}{p}} \\
&\leq M[(\int_\Omega |h_2|^{\frac{1}{p}}\mathrm{d}x)^{\frac{1}{p}} + (\int_\Omega |h_1|^{\frac{1}{p}}\mathrm{d}x)^{\frac{1}{p}}] \\
&= M||h||,
\end{aligned} \tag{4.5}$$

where $M = \max\{\beta(\int_\Omega |\nabla v|^p \mathrm{d}x)^{1-\frac{1}{p}}, (\int_\Omega |f|^{\frac{p}{p-1}}\mathrm{d}x)^{1-\frac{1}{p}}\} < \infty$.

By the inequality (4.5), $\mathscr{L}v \in X'$ for each $(v, \nabla v) \in K$. The mapping $\mathscr{L}$ is well defined.

The following three lemmas show that $\mathscr{L}$ is monotone, coercive and weakly continuous on K.

Lemma 4.3 *$\mathscr{L}$ is monotone on K, i.e. $\langle \mathscr{L}u - \mathscr{L}v, u - v\rangle \geq 0$ for all $(u, \nabla u)$, $(v, \nabla v)$ in K.*

Proof. For all $(u, \nabla u)$, $(v, \nabla v)$ in K, $\mathscr{L}u = (-f, \mathcal{A}(x, \nabla u))$ and $\mathscr{L}v = (-f, \mathcal{A}(x, \nabla v))$.

Then, $\mathscr{L}u - \mathscr{L}v = (-f, \mathcal{A}(x, \nabla u)) - (-f, \mathcal{A}(x, \nabla v)) = (0, \mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla v))$.

Since $(u - v, \nabla u - \nabla v) \in X$, by the assumption (IV), we have

$$\langle \mathscr{L}u - \mathscr{L}v, u - v\rangle = \int_\Omega (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla v)) \cdot (\nabla u - \nabla v)\mathrm{d}x \geq 0.$$

This proves the lemma.

Lemma 4.4 *$\mathscr{L}$ is coercive on K, i.e. there exists $\phi \in K$ such that*

$$\frac{\langle \mathscr{L}u_j - \mathscr{L}\varphi, u_j - \varphi\rangle}{||u_j - \varphi||} \to \infty$$

for each sequence u_j in K with $||u_j|| \to \infty$.

Proof. Fix $(\phi, \nabla\phi) \in K$. For each $(u, \nabla u) \in K$, by assumptions (II), (III) and the Hölder inequality,

$$\begin{aligned}\langle \mathscr{L}u - \mathscr{L}\varphi, u-\varphi\rangle &= \int_\Omega (\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla\varphi))\cdot(\nabla u - \nabla\varphi)\mathrm{d}x \\ &\geq \alpha(||\nabla u||_p^p + ||\nabla\varphi||_p^p) - \beta(||\nabla u||_p^{p-1}||\nabla\varphi||_p + ||\nabla u||_p||\nabla\varphi||_p^{p-1}).\end{aligned} \tag{4.6}$$

Using the inequality $(a+b)^r \leq 2^r(a^r + b^r)$ for all $a \geq 0$, $b \geq 0$ and $r > 0$, the following inequalities hold.

$$\begin{aligned}&||\nabla u + \nabla\varphi||_p^p \leq (||\nabla u||_p + ||\nabla\varphi||_p)^p \leq 2^p(||\nabla u||_p^p + ||\nabla\varphi||_p^p),\\ &||\nabla u||_p^{p-1} \leq (||\nabla u||_p + ||\nabla\varphi - \nabla u||_p)^{p-1} \leq 2^{p-1}(||\nabla u||_p^{p-1} + ||\nabla u - \nabla\varphi||_p^{p-1})\end{aligned}$$

and

$$||\nabla u||_p \leq \ ||\nabla u||_p + ||\nabla\varphi - \nabla u||_p.$$

Putting the above inequalities into (4.6), we get

$$\begin{aligned}\langle \mathscr{L}u - \mathscr{L}\varphi, u-\varphi\rangle n &\geq \alpha 2^{-p}||\nabla u - \nabla\varphi||_p^p - \beta 2^{p-1}||\nabla\varphi||_p(||\nabla\varphi||_p^{p-1} + ||\nabla u - \nabla\varphi||_p^{p-1}) \\ &\quad - \beta||\nabla\varphi||_p^{p-1}(||\nabla\varphi||_p + ||\nabla u - \nabla\varphi||_p) \\ &= \alpha 2^{-p}||\nabla u - \nabla\varphi||_p^p - \beta 2^{p-1}||\nabla\varphi||_p||\nabla u - \nabla\varphi||_p^{p-1} \\ &\quad - \beta||\nabla\varphi||_p^{p-1}||\nabla u - \nabla\varphi||_p - \beta(2^{p-1}+1)||\nabla\varphi||_p^p.\end{aligned}$$

Then, we have

$$\begin{aligned}\frac{\langle \mathscr{L}u - \mathscr{L}\varphi, u-\varphi\rangle}{||\nabla u - \nabla\varphi||_p} &\geq \alpha 2^{-p}||\nabla u - \nabla\varphi||_p^{p-1} - \beta 2^{p-1}||\nabla\varphi||_p||\nabla u - \nabla\varphi||_p^{p-2} \\ &\quad - \beta||\nabla\varphi||_p^{p-1} - \beta(2^{p-1}+1)||\nabla\varphi||_p^p\frac{1}{||\nabla u - \nabla\varphi||_p}.\end{aligned} \tag{4.7}$$

Since $(u, \nabla u)$, $(\phi, \nabla\phi) \in K$, both u and ϕ are in $\mathcal{K}_{\psi,\vartheta}$. Thus, $u - \varphi = u - \vartheta - (\varphi - \vartheta) \in H_0^{1,p}(\Omega)$.

By the Poincaré inequality,

$$||u - \varphi||_p \leq C\,\mathtt{diam}\Omega||\nabla u - \nabla\varphi||_p, \tag{4.8}$$

where C is a constant independent of u and ϕ.

By the definition of the norm of X and the inequality (4.8), we obtain

$$||\nabla u - \nabla\varphi||_p \leq \ ||u-\varphi||_p + ||\nabla u - \nabla\varphi||_p = \ ||u - \varphi|| \ \leq (C\,\mathtt{diam}\Omega + 1))||\nabla u - \nabla\varphi||_p. \tag{4.9}$$

Combining the inequality $||u_j - \phi|| \geq ||u_j|| - ||\phi||$ and (4.9), we have

$$(C\,\mathtt{diam}\Omega + 1)||\nabla u_j - \nabla\varphi||_p \geq \ ||u_j - \varphi|| \ \geq \ ||u_j|| - ||\varphi||.$$

For each sequence $(u_j, \nabla u_j) \in K$ with $||u_j|| \to \infty$, $||\nabla u_j - \nabla\phi||_p \to \infty$.

Thus,

$$\begin{aligned}&\alpha 2^{-p}||\nabla u_j - \nabla\varphi||_p^{p-1} - \beta 2^{p-1}||\nabla\varphi||_p||\nabla u_j - \nabla\varphi||_p^{p-2} \\ &= \ ||\nabla u_j - \nabla\varphi||_p^{p-1}(\alpha 2^{-p} - \beta 2^{p-1}||\nabla\varphi||_p\frac{1}{||\nabla u_j - \nabla\varphi||_p}) \to \infty, \\ &\beta(2^{p-1}+1)||\nabla\varphi||_p^p\frac{1}{||\nabla u - \nabla\varphi||_p} \to 0.\end{aligned} \tag{4.10}$$

Combining (4.10) with (4.7), we obtain

$$\frac{\langle \mathscr{L}u - \mathscr{L}\varphi, u - \varphi \rangle}{||\nabla u - \nabla \varphi||_p} \to \infty.$$

Using (4.9), we conclude

$$\frac{\langle \mathscr{L}u - \mathscr{L}\varphi, u - \varphi \rangle}{||\nabla u - \nabla \varphi||} \geq \frac{\langle \mathscr{L}u - \mathscr{L}\varphi, u - \varphi \rangle}{(C\,\mathtt{diam}\Omega + 1)||\nabla u - \nabla \varphi||_p} \to \infty.$$

It follows that $\mathscr{L}$ is coercive on K.

Lemma 4.5 *$\mathscr{L}$ is weakly continuous on K, that means $\mathscr{L}u_j$ converges to $\mathscr{L}u$ weakly in X', i.e.*

$$\langle \mathscr{L}u_j, v \rangle \to \langle \mathscr{L}u, v \rangle \quad \mathtt{for\ all}\ v = (v_1, v_2) \in X, \tag{4.11}$$

whenever $(u_j, \nabla u_j) \in K$ converges to $(u, \nabla u) \in K$ in X.

Proof. Let $(u_j, \nabla u_j) \in K$ be any sequence that converges to an element $(u, \nabla u) \in K$ in X. It suffices to prove that $\mathscr{L}u_j$ converges to $\mathscr{L}u$ weakly in X', i.e.

$$\langle \mathscr{L}u_j - \mathscr{L}u, v \rangle \to 0 \text{ for all } v = (v_1, v_2) \in X.$$

By the definition of $\mathscr{L}$,

$$\langle \mathscr{L}u_j - \mathscr{L}u, v \rangle = \int_\Omega (\mathcal{A}(x, \nabla u_j) - \mathcal{A}(x, \nabla u)) \cdot v_2 \mathrm{d}x.$$

By the definition of X and $(u_j, \nabla u_j) \to (u, \nabla u)$ in X, $\nabla u_j \to \nabla u$ in $L^p(\Omega; \mathbb{R}^n)$. There exists a subsequence u_{j_k} such that $\nabla u_{j_k} \to \nabla u$ a.e. in Ω.

Since $\mathcal{A}$ satisfies the assumption (I), $\mathcal{A}(x, \nabla u_{j_k}(x)) \to \mathcal{A}(x, \nabla u(x))$ a.e. in Ω.

By the assumption (III), we obtain

$$\int_\Omega |\mathcal{A}(x, \nabla u_j)|^{p/(p-1)} \mathrm{d}x \leq \int_\Omega (\beta |\nabla u_j|^{p-1})^{p/(p-1)} \mathrm{d}x = \beta^{p/(p-1)} \int_\Omega |\nabla u_j|^p \mathrm{d}x. \tag{4.12}$$

Since $\nabla u_j \to \nabla u$ in $L^p(\Omega; \mathbb{R}^n)$, (4.12) shows $L^{p/(p-1)}(\Omega; \mathbb{R}^n)$ - norms of $\mathcal{A}(x, \nabla u_j)$ are uniformly bounded. By Lemma 1.1, $\mathcal{A}(x, \nabla u_{j_k})$ converges to $\mathcal{A}(x, \nabla u)$ weakly in $L^{p/(p-1)}(\Omega; \mathbb{R}^n)$.

By the same discussion, we know that, for any subsequence ∇u_{j_k} of ∇u_j, there exists a subsequence $\nabla u_{j_{k_l}}$ of ∇u_{j_k}, such that $\mathcal{A}(x, \nabla u_{j_{k_l}})$ converges to $\mathcal{A}(x, \nabla u)$ weakly in $L^{p/(p-1)}(\Omega; \mathbb{R}^n)$.

Since the weak limit $\mathcal{A}(x, \nabla u)$ is independent of the choice of the subsequence and by Lemma 4.1, it follows that $\mathcal{A}(x, \nabla u_j)$ converges to $\mathcal{A}(x, \nabla u)$ weakly in $L^{p/(p-1)}(\Omega; \mathbb{R}^n)$.

Consequently, for all $v = (v_1, v_2) \in X$,

$$\langle \mathscr{L}u_j - \mathscr{L}u, v \rangle = \int_\Omega (\mathcal{A}(x, \nabla u_j) - \mathcal{A}(x, \nabla u)) \cdot v_2 \mathrm{d}x \to 0.$$

Then, $\langle \mathscr{L}u_j, v \rangle \to \langle \mathscr{L}u, v \rangle \mathtt{for\ all} v \in X$. Hence, $\mathscr{L}$ is weakly continuous on K.

Based on the above lemmas, we can prove our main results.

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^n$ is a bounded open set, $\vartheta \in H^{1,p}(\Omega)$ and $\psi: \Omega \to [-\infty, \infty]$ be any function. If $\mathcal{K}_{\psi,\vartheta} \neq \emptyset$, then there is a unique function u in $\mathcal{K}_{\psi,\vartheta}$, such that*

$$\int_\Omega (\mathcal{A}(x,\nabla u)\cdot(\nabla v-\nabla u)-f(v-u))\mathrm{d}x \geq 0$$

whenever $v \in \mathcal{K}_{\psi,\vartheta}$. *That is, if* $\mathcal{K}_{\psi,\vartheta} \neq \emptyset$, *there is a unique solution u to the obstacle problem in* $\mathcal{K}_{\psi,\vartheta}$.

Proof. (i) Construct X, K and $\mathscr{L}$ as Lemmas 4.2, 4.3, 4.4 and 4.5. By the proposition (4.1) and Lemmas 4.2, 4.3, 4.4 and 4.5, there exists an element u in K such that

$$\langle \mathscr{L}u, v-u\rangle \geq 0$$

whenever $v \in K$.

This means that there is a function u in $\mathcal{K}_{\psi,\vartheta}$, such that

$$\int_\Omega (\mathcal{A}(x,\nabla u)\cdot(\nabla v-\nabla u)-f(v-u))\mathrm{d}x \geq 0$$

Whenever $v \in \mathcal{K}_{\psi,\vartheta}$.

(ii) Suppose that u_1 and u_2 are two solutions to the obstacle problem in $\mathcal{K}_{\psi,\vartheta}$. Then $\min\{u_1,u_2\} \in \mathcal{K}_{\psi,\vartheta}$ and both u_1 and u_2 are supersolutions to (2.1) in Ω. By Lemma 3.1, $u_1 \geq u_2$ a.e. in Ω and $u_2 \geq u_1$ a.e. in Ω. Thus, $u_1 = u_2$ a.e. in Ω and the uniqueness is proved.

Theorem 4.2 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set and* $\vartheta \in H^{1,p}(\Omega)$. *There is a unique function* $u \in H^{1,p}(\Omega)$ *with* $u-\vartheta \in H_0^{1,p}(\Omega)$ *such that*

$$\int_\Omega (\mathcal{A}(x,\nabla u)\cdot(\nabla\varphi)-f\varphi)\mathrm{d}x = 0$$

whenever $\varphi \in H_0^{1,p}(\Omega)$. *That is u is the unique solution to (2.1) with* $u-\vartheta \in H_0^{1,p}(\Omega)$.

Proof. (i) Choose $\psi \equiv -\infty$. Since $\vartheta \geq -\infty = \psi$ and $\vartheta-\vartheta = 0 \in H_0^{1,p}(\Omega)$, $\vartheta \in \mathcal{K}_{\psi,\vartheta} \neq \emptyset$.

By Theorem 4.1, there is a function u in $\mathcal{K}_{\psi,\vartheta}$ such that

$$\int_\Omega (\mathcal{A}(x,\nabla u)\cdot(\nabla v-\nabla u)-f(v-u))\mathrm{d}x \geq 0$$

whenever $v \in \mathcal{K}_{\psi,\vartheta}$.

For each $\varphi \in H_0^{1,p}(\Omega)$,

$$u+\varphi \geq -\infty = \psi,$$
$$u-\varphi \geq -\infty = \psi,$$
$$u+\varphi-\vartheta = (u-\vartheta)+\varphi \in H_0^{1,p}(\Omega),$$
$$u-\varphi-\vartheta = (u-\vartheta)-\varphi \in H_0^{1,p}(\Omega).$$

Therefore, both $u + \phi$ and $u - \vartheta$ are in $\mathcal{K}_{\psi,\vartheta}$ for all $\varphi \in H_0^{1,p}(\Omega)$ Then,

$$\int_\Omega (\mathcal{A}(x,\nabla u)\cdot\nabla\varphi-f\varphi)\mathrm{d}x \geq 0$$

and

$$\int_{\Omega} (\mathcal{A}(x, \nabla u) \cdot \nabla(-\varphi) - f(-\varphi))\mathrm{d}x \geq 0.$$

Thus,

$$\int_{\Omega} (\mathcal{A}(x, \nabla u) \cdot \nabla\varphi - f\varphi)\mathrm{d}x = 0.$$

(ii) Let u_1 and u_2 are two solutions to (2.1) with $u_i - \vartheta \in H_0^{1,p}(\Omega)$, $i = 1, 2$. Since $\vartheta \in H^{1,p}(\Omega)$, u_1, $u_2 \in H^{1,p}(\Omega)$ and $u_1 - u_2 = (u_1 - \vartheta) - (u_2 - \vartheta) \in H_0^{1,p}(\Omega)$. Then, $\eta_1 = \min\{u_1 - u_2, 0\} \in H_0^{1,p}(\Omega)$. Since u_1 and u_2 are two solutions and by Proposition 2.2, u_1 is a supersolution and u_2 is a subsolution. By Lemma 2.2, $u_1 \geq u_2$ a.e in Ω. Similarly, $u_2 \geq u_1$ a.e in Ω. Thus, $u_1 = u_2$ a.e. in Ω and the uniqueness is proved.

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Authors' contributions
All authors contributed equally in this paper. They read and approved the final manuscript.

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