Fang and Ding *Journal of Inequalities and Applications* 2011, **2011**:74
http://www.journalofinequalitiesandapplications.com/content/2011/1/74

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Singular integrals of the compositions of Laplace-Beltrami and Green's operators

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Abstract

We establish the Poincaré-type inequalities for the composition of the Laplace-Beltrami operator and the Green's operator applied to the solutions of the non-homogeneous *A*-harmonic equation in the John domain. We also obtain some estimates for the integrals of the composite operator with a singular density.

Keywords: Poincaré-type inequalities, differential forms, *A*-harmonic equations, the Laplace-Beltrami operator, Green's operator

1 Introduction

The purpose of the article is to develop the Poincaré-type inequalities for the composition of the Laplace-Beltrami operator $\Delta = dd^* + d^*d$ and Green's operator G over the δ-John domain. Both operators play an important role in many fields, including partial differential equations, harmonic analysis, quasiconformal mappings and physics [1-6]. We first give a general estimate of the composite operator $\Delta \circ G$. Then, we consider the composite operator with a singular factor. The consideration was motivated from physics. For instance, when calculating an electric field, we will deal with the integral $E(r) = \frac{1}{4\pi\varepsilon_0}\int_D \rho(x)\frac{r-x}{\|r-x\|^3}\mathrm{d}x$, where $\rho(x)$ is a charge density and x is the integral variable. It is singular if $r \in D$. Obviously, the singular integrals are more interesting to us because of their wide applications in different fields of mathematics and physics.

In this article, we assume that M is a bounded, convex domain and B is a ball in $\mathbb{R}^n$, $n \geq 2$. We use σB to denote the ball with the same center as B and with diam $(\sigma B) = \sigma \mathrm{diam}(B)$, $\sigma > 0$. We do not distinguish the balls from cubes in this article. We use $|E|$ to denote the Lebesgue measure of a set $E \subset \mathbb{R}^n$. We call ω a weight if $\omega \in L^1_{loc}(\mathbb{R}^n)$ and $\omega > 0$ a.e. Differential forms are extensions of functions in $\mathbb{R}^n$. For example, the function $u(x_1, x_2,..., x_n)$ is called a 0-form. Moreover, if $u(x_1, x_2,..., x_n)$ is differentiable, then it is called a differential 0-form. The 1-form $u(x)$ in $\mathbb{R}^n$ can be written as $u(x) = \sum_{i=1}^n u_i(x_1, x_2, \ldots, x_n)\mathrm{d}x_i$. If the coefficient functions $u_i(x_1, x_2,..., x_n)$, $i = 1, 2,..., n$, are differentiable, then $u(x)$ is called a differential l-form. Similarly, a differential k-form $u(x)$ is generated by $\{\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_k}\}$, $k = 1, 2,..., n$, that is, $u(x) = \sum_I u_I(x)\mathrm{d}x_I = \sum u_{i_1 i_2 \ldots i_k}(x)\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \ldots \wedge \mathrm{d}x_{i_k}$, where $I = (i_1, i_2,..., i_k)$, $1 \leq i_1 < i_2 < ... < i_k \leq n$. Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l-forms in $\mathbb{R}^n$, $D'(M, \wedge^l)$ be the space of all differential l-forms on M and $L^p(M, \wedge^l)$ be the l-forms $u(x) = \sum_I u_I(x)\mathrm{d}x_I$ on M satisfying $\int_M |u_I|^p < \infty$ for all ordered l-tuples I, $l = 1, 2,..., n$. We denote the exterior derivative by $d : D'(M, \wedge^l) \to D'(M, \wedge^{l+1})$ for $l = 0, 1,..., n - 1$, and define the Hodge star operator $* : \wedge^k \to \wedge^{n-k}$ as follows, If $u = u_{i_1 i_2 \ldots i_k}(x_1, x_2, \ldots, x_n)\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_k} = u_I \mathrm{d}x_I$, $i_1 < i_2 < \ldots < i_k$, is a differential k-form, then $*u = *(u_{i_1 i_2 \ldots i_k}\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_k}) = (-1)^{\sum(I)} u_I \mathrm{d}x_J$, where $I = (i_1, i_2, \ldots i_k)$, $J = \{1, 2,..., n\} - I$, and $\sum(I) = \frac{k(k+1)}{2} + \sum_{i=1}^{k} i_j$. The Hodge codifferential operator d^* : $D'(M, \wedge^{l+1}) \to D'(M, \wedge^l)$ is given by $d^* = (-1)^{nl+1} * \mathrm{d}*$ on $D'(M, \wedge^{l+1})$, $l = 0, 1,..., n - 1$. and the Laplace-Beltrami operator Δ is defined by $\Delta = dd^* + d^* d$. We write $\| u\|_{s,M} = (\int_M |u|^s)^{1/s}$ and $\| u\|_{s,M,\omega} = (\int_M |u|^s \omega(x)\mathrm{d}x)^{1/s}$, where $\omega(x)$ is a weight. Let $\wedge^l M$ be the l-th exterior power of the cotangent bundle, $C^\infty(\wedge^l M)$ be the space of smooth l-forms on M and $\mathcal{W}(\wedge^l M) = \{u \in L^1_{loc}(\wedge^l M) : u \text{ has generalized gradient}\}$. The harmonic l-fields are defined by $\mathcal{H}(\wedge^l M) = \{u \in \mathcal{W}(\wedge^l M) : du = d^*u = 0, u \in L^p \text{ for some } 1 < p < \infty\}$. The orthogonal complement of $\mathcal{H}$ in L^1 is defined by $\mathcal{H}^\perp = \{u \in L^1 : \langle u, h \rangle = 0 \text{ for all } h \in \mathcal{H}\}$. Then, the Green's operator G is defined as $G : C^\infty(\wedge^l M) \to \mathcal{H}^\perp \cap C^\infty(\wedge^l M)$ by assigning $G(u)$ be the unique element of $\mathcal{H}^\perp \cap C^\infty(\wedge^l M)$ satisfying Poisson's equation $\Delta G(u) = u - H(u)$, where H is the harmonic projection operator that maps $C^\infty(\wedge^l M)$ onto $\mathcal{H}$ so that $H(u)$ is the harmonic part of u [[7,8], for more properties of these operators]. The differential forms can be used to describe various systems of PDEs and to express different geometric structures on manifolds. For instance, some kinds of differential forms are often utilized in studying deformations of elastic bodies, the related extrema for variational integrals, and certain geometric invariance [9,10].

We are particularly interested in a class of differential forms satisfying the well known non-homogeneous A-harmonic equation

$$d^*A(x, du) = B(x, du), \tag{1.1}$$

where $A : M \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ and $B : M \times \wedge^l(\mathbb{R}^n) \to \wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions:

$$|A(x, \xi)| \le a|\xi|^{p-1}, \quad A(x, \xi) \cdot \xi \ge |\xi|^p, \quad |B(x, \xi)| \le b|\xi|^{p-1} \tag{1.2}$$

for almost every $x \in M$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a > 0$ and $b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with the Equation (1.1). If the operator $B = 0$, Equation (1.1) becomes $d^* A(x, du) = 0$, which is called the homogeneous A-harmonic equation. A solution to (1.1) is an element of the Sobolev space $W^{1,p}_{loc}(M, \wedge^{l-1})$ such that $\int_M A(x, du) \cdot \mathrm{d}\varphi + B(x, du) \cdot \varphi = 0$ for all $\varphi \in W^{1,p}_{loc}(M, \wedge^{l-1})$ with compact support. Let $A : M \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ be defined by $A(x, \xi) = \xi|\xi|^{p-2}$ with $p > 1$. Then, A satisfies the required conditions and $d^* A(x, du) = 0$ becomes the p-harmonic equation

$$d^*(du|du|^{p-2}) = 0 \tag{1.3}$$

for differential forms. If u is a function (0-form), the equation (1.3) reduces to the usual p-harmonic equation $\mathrm{div}(\nabla u|\nabla u|^{p-2}) = 0$ for functions. Some results have been obtained in recent years about different versions of the A-harmonic equation [8,11-16].

2 Main results and proofs

We first introduce the following definition and lemmas that will be used in this article.

Definition 2.1 A proper subdomain $\Omega \subset \mathbb{R}^n$ is called a δ-John domain, $\delta > 0$, if there exists a point $x_0 \in \Omega$ which can be joined with any other point $x \in \Omega$ by a continuous curve $\gamma \subset \Omega$ so that

$$d(\xi, \partial\Omega) \ge \delta|x - \xi|$$

for each $\xi \in \gamma$. Here $d(\xi, \partial\Omega)$ is the Euclidean distance between ξ and $\partial\Omega$.

Lemma 2.1 [17]*Let φ be a strictly increasing convex function on $[0, \infty)$ with $\varphi(0) = 0$, and D be a domain in $\mathbb{R}^n$. Assume that u is a function in D such that $\varphi(|u|) \in L^1(D, \mu)$ and $\mu(\{x \in D : |u - c| > 0\}) > 0$ for any constant c, where μ is a Radon measure defined by $d\mu(x) = \omega(x)dx$ for a weight $\omega(x)$. Then, we have*

$$\int_D \phi(\frac{a}{2}|u - u_{D,\mu}|)d\mu \le \int_D \phi(a|u|)d\mu$$

for any positive constant a, *where* $u_{D,\mu} = \dfrac{1}{\mu(D)} \int_D u d\mu$.

Lemma 2.2 [3] *Let $u \in C^\infty(\Lambda^l M)$ and $l = 1, 2,..., n$, $1 < s < \infty$. Then, there exists a positive constant C, independent of u, such that*

$$\| dd^*G(u) \|_{s,M} + \| d^*dG(u) \|_{s,M} + \| dG(u) \|_{s,M} + \| d^*G(u) \|_{s,M} + \| G(u) \|_{s,M} \le C \| u\|_{s,M}$$

Lemma 2.3 [18]*Each Ω has a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ such that $\cup_i Q_i = \Omega$, $\sum_{Q_i \in \mathcal{V}} \chi_{\sqrt{\frac{5}{4}}Q_i} \le N\chi_\Omega$ and some $N > 1$, and if $Q_i \cap Q_j \ne \emptyset$, then there exists a cube R (this cube need not be a member of $\mathcal{V}$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, if Ω is δ-John, then there is a distinguished cube $Q_0 \in \mathcal{V}$ which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0, Q_1,..., Q_k = Q$ from $\mathcal{V}$ and such that $Q \subset \rho Q_i$, $i = 0, 1, 2,..., k$, for some $\rho = \rho(n, \delta)$.*

Lemma 2.4 *Let $u \in L^s_{loc}(M, \Lambda^l)$, $l = 1, 2,..., n$, $1 < s < \infty$, G be the Green's operator and Δ be the Laplace-Beltrami operator. Then, there exists a constant C, independent of u, such that*

$$\| \Delta G(u) \|_{s,B} \le C \| u\|_{s,B} \tag{2.1}$$

for all balls $B \subset M$.

Proof By using Lemma 2.2, we have

$$\| \Delta G(u) \|_{s,B} = \| (dd^* + d^*d)G(u) \|_{s,B} \le \| dd^*G(u) \|_{s,B} + \| d^*dG(u) \|_{s,B} \le C \| u\|_{s,B}. \tag{2.2}$$

This ends the proof of Lemma 2.4. □

Lemma 2.5 *Let $u \in L^s_{loc}(M, \Lambda^l)$, $l = 1, 2,..., n$, $1 < s < \infty$, be a solution of the non-homogeneous A-harmonic equation in a bound and convex domain M, G be the Green's operator and Δ be the Laplace-Beltrami operator. Then, there exists a constant C independent of u, such that*

$$\left(\int_B |\Delta G(u)|^s \frac{1}{d(x, \partial M)^\alpha} dx\right)^{1/s} \le C\left(\int_{\sigma B} |u|^s \frac{1}{|x - x_B|^\lambda} dx\right)^{1/s} \tag{2.3}$$

for all balls B with $\sigma B \subset M$ and $diam(B) \ge d_0 > 0$, where d_0 is a constant, $\sigma > 1$, and any real number α and λ with $\alpha > \lambda \ge 0$. Here x_B is the center of the ball B.

Proof Let $\varepsilon \in (0, 1)$ be small enough such that $\varepsilon n < \alpha - \lambda$ and $B \subset M$ be any ball with center x_B and radius r_B. Also, let $\delta > 0$ be small enough, $B_\delta = \{x \in B : |x - x_B| \le$

$\delta\}$ and $D_\delta = B \setminus B_\delta$. Choose $t = s/(1 - \varepsilon)$, then, $t > s$. Write $\beta = t/(t - s)$. Using the Hölder inequality and Lemma 2.4, we have

$$\begin{aligned}
\left(\int_{D_\delta} |\Delta G(u)|^s \frac{1}{\mathrm{d}(x,\partial M)^\alpha}\mathrm{d}x\right)^{1/s} &= \left(\int_{D_\delta} \left(|\Delta G(u)| \frac{1}{\mathrm{d}(x,\partial M)^{\alpha/s}}\right)^s \mathrm{d}x\right)^{1/s}\\
&\leq \| \Delta G(u)\|_{t,D_\delta} \left(\int_{D_\delta} \left(\frac{1}{\mathrm{d}(x,\partial M)}\right)^{t\alpha/(t-s)} \mathrm{d}x\right)^{(t-s)/st}\\
&= \| \Delta G(u)\|_{t,D_\delta} \left(\int_{D_\delta} \left(\frac{1}{\mathrm{d}(x,\partial M)}\right)^{\alpha\beta} \mathrm{d}x\right)^{1/\beta s}\\
&\leq \| \Delta G(u)\|_{t,B} \left(\int_{D_\delta} \left(\frac{1}{\mathrm{d}(x,\partial M)}\right)^{\alpha\beta} \mathrm{d}x\right)^{1/\beta s}\\
&\leq C_1 \| u\|_{t,B} \left\| \frac{1}{\mathrm{d}(x,\partial M)^\alpha} \right\|_{\beta,D_\delta}^{1/s}.
\end{aligned} \tag{2.4}$$

We may assume that $x_B = 0$. Otherwise, we can move the center to the origin by a simple transformation. Then, $\frac{1}{\mathrm{d}(x,\partial M)} \leq \frac{1}{r_B - |x|}$ for any $x \in B$, we have

$$\left(\int_{D_\delta} \left(\frac{1}{\mathrm{d}(x,\partial M)}\right)^{\alpha\beta} \mathrm{d}x\right)^{1/\beta s} \leq \left(\int_{D_\delta} \frac{1}{|x - x_B|^{\alpha\beta}} \mathrm{d}x\right)^{1/\beta s}. \tag{2.5}$$

Therefore, for any $x \in B$, $|x - x_B| \geq |x| - |x_B| = |x|$. By using the polar coordinate substitution, we have

$$\begin{aligned}
\left(\int_{D_\delta} \frac{1}{|x - x_B|^{\alpha\beta}} \mathrm{d}x\right)^{1/\beta s} &\leq \left(C_2 \int_\delta^{r_B} \rho^{-\alpha\beta}\rho^{n-1}\mathrm{d}\rho\right)^{1/\beta s}\\
&= \left|\frac{C_2}{n - \alpha\beta}({r_B}^{n-\alpha\beta} - \delta^{n-\alpha\beta})\right|^{1/\beta s}\\
&\leq C_3 |{r_B}^{n-\alpha\beta} - \delta^{n-\alpha\beta}|^{1/\beta s}.
\end{aligned} \tag{2.6}$$

Choose $m = nst/(ns + \alpha t - \lambda t)$, then $0 < m < s$. By the reverse Hölder inequality, we find that

$$\| u\|_{t,B} \leq C_4 |B|^{\frac{m-t}{mt}} \| u\|_{m,\sigma B}, \tag{2.7}$$

where $\sigma > 1$ is a constant. By the Hölder inequality again, we obtain

$$\begin{aligned}
\| u\|_{m,\sigma B} &= \left(\int_{\sigma B} (|u||x - x_B|^{-\lambda/s}|x - x_B|^{\lambda/s})^m \mathrm{d}x\right)^{1/m}\\
&\leq \left(\int_{\sigma B} (|u||x - x_B|^{-\lambda/s})^s \mathrm{d}x\right)^{1/s} \left(\int_{\sigma B} (|x - x_B|^{\lambda/s})^{\frac{ms}{s-m}} \mathrm{d}x\right)^{\frac{s-m}{ms}}\\
&\leq \left(\int_{\sigma B} |u|^s |x - x_B|^{-\lambda} \mathrm{d}x\right)^{1/s} C_5(\sigma r_B)^{\lambda/s + n(s-m)/ms}\\
&\leq C_6 \left(\int_{\sigma B} |u|^s |x - x_B|^{-\lambda} \mathrm{d}x\right)^{1/s} (r_B)^{\lambda/s + n(s-m)/ms}.
\end{aligned} \tag{2.8}$$

By a simple calculation, we find that $n - \alpha\beta + \lambda\beta + n\beta(s - m)/m = 0$. Substituting (2.6)-(2.8) in (2.4), we have

$$
\begin{aligned}
&\left(\int_{D_\delta} |\Delta G(u)|^s \frac{1}{\mathrm{d}(x,\partial M)^\alpha}\mathrm{d}x\right)^{1/s} \\
&\le C_7|B|^{\frac{m-t}{mt}}\left(\int_{\sigma B}|u|^s|x-x_B|^{-\lambda}\mathrm{d}x\right)^{1/s}(r_B)^{\frac{\lambda}{s}+\frac{n(s-m)}{ms}}|r_B^{n-\alpha\beta}-\delta^{n-\alpha\beta}|^{1/\beta s} \\
&= C_7|B|^{\frac{m-t}{mt}}\left(\int_{\sigma B}|u|^s|x-x_B|^{-\lambda}\mathrm{d}x\right)^{1/s}\left[r_B^{(\frac{\lambda}{s}+\frac{n(s-m)}{ms})\beta_s}\left|r_B^{n-\alpha\beta}-\delta^{n-\alpha\beta}\right|\right]^{1/\beta s} \\
&= C_7|B|^{\frac{m-t}{mt}}\left(\int_{\sigma B}|u|^s|x-x_B|^{-\lambda}\mathrm{d}x\right)^{1/s}\left|C_8 r_B^{n-\alpha\beta+\lambda\beta+\frac{n\beta(s-m)}{m}}-\delta^{n-\alpha\beta}r_B^{\lambda\beta+\frac{n\beta(s-m)}{m}}\right|^{1/\beta s} \\
&\le C_7|B|^{\frac{m-t}{mt}}\left(\int_{\sigma B}|u|^s|x-x_B|^{-\lambda}\mathrm{d}x\right)^{1/s}\left[C_8 r_B^{n-\alpha\beta+\lambda\beta+\frac{n\beta(s-m)}{m}}-\delta^{n-\alpha\beta}\delta^{\lambda\beta+\frac{n\beta(s-m)}{m}}\right]^{1/\beta s} \\
&\le C_7|B|^{\frac{m-t}{mt}}\left(\int_{\sigma B}|u|^s|x-x_B|^{-\lambda}\mathrm{d}x\right)^{1/s}\left[C_8 r_B^{n-\alpha\beta+\lambda\beta+\frac{n\beta(s-m)}{m}}+\delta^{n-\alpha\beta+\lambda\beta+\frac{n\beta(s-m)}{m}}\right]^{1/\beta s} \\
&\le C_9|B|^{\frac{\lambda-\alpha}{ns}}\left(\int_{\sigma B}|u|^s|x-x_B|^{-\lambda}\mathrm{d}x\right)^{1/s} \\
&\le C_{10}\left(\int_{\sigma B}|u|^s|x-x_B|^{-\lambda}\mathrm{d}x\right)^{1/s},
\end{aligned}
\tag{2.9}
$$

thus is,

$$\left(\int_{D_\delta}|\Delta G(u)|^s\frac{1}{d(x,\partial M)^\alpha}\mathrm{d}x\right)^{1/s} \le C_{10}\left(\int_{\sigma B}|u|^s\frac{1}{|x-x_B|^\lambda}\mathrm{d}x\right)^{1/s}. \tag{2.10}$$

Notice that $\lim_{\delta\to 0}\left(\int_{D_\delta}|\Delta G(u)|^s\frac{1}{\mathrm{d}(x,\partial M)^\alpha}\mathrm{d}x\right)^{1/s} = \left(\int_B|\Delta G(u)|^s\frac{1}{\mathrm{d}(x,\partial M)^\alpha}\mathrm{d}x\right)^{1/s}$. letting $\delta \to 0$ in (2.10), we obtain (2.3). we have completed the proof of Lemma 2.5. □

Theorem 2.6 *Let $u \in D'(\Omega, \Lambda^l)$ be a solution of the A-harmonic equation (1.1), G be the Green's operator and Δ be the Laplace-Beltrami operator. Assume that s is a fixed exponent associated with the non-homogeneous A-harmonic equation. Then, there exists a constant C, independent of u, such that*

$$\left(\int_\Omega|\Delta G(u)-(\Delta G(u))_{Q_0}|^s\frac{1}{\mathrm{d}(x,\partial\Omega)^\alpha}\mathrm{d}x\right)^{1/s} \le C\left(\int_\Omega|u|^s g(x)\mathrm{d}x\right)^{1/s} \tag{2.11}$$

for any bounded and convex δ-John domain $\Omega \in \mathbb{R}^n$, where $g(x) = \sum_i \chi_{Q_i}\frac{1}{|x-x_{Q_i}|^\lambda}$, $x_{Q_i}$is the center of Q_i with $\Omega = \cup_i Q_i$. Here α and λ are constants with $0 \le \lambda < \alpha < n$, and the fixed cube $Q_0 \in \Omega$, the constant $N > 1$ and the cubes $Q_i \in \Omega$ appeared in Lemma 2.3, $x_{Q_i}$is the center of Q_i.

Proof We use the notation appearing in Lemma 2.3. There is a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ for Ω such that $\Omega = \cup Q_i$, and $\sum_{Q_i \in \mathcal{V}} \chi_{\sqrt{\frac{5}{4}} Q_i} \leq N \chi_{\Omega}$ for some $N > 1$. For each $Q_i \in \mathcal{V}$, if $\operatorname{diam}(Q_i) \geq d_0$ (where d_0 is the constant appearing in Lemma 2.5), it is fine and we keep Q_i in the collection $\mathcal{V}$. Otherwise, if $\operatorname{diam}(Q_i) < d_0$, we replace Q_i by a new cube Q_i^* with the same center as Q_i and $\operatorname{diam}(Q_i^*) = d_0$. Thus, we obtain a modified collection $\mathcal{V}^*$ consisting of all cubes Q_i^*, and $\mathcal{V}^*$ has the same properties as $\mathcal{V}$. Moreover, diam $(Q_i^*) \geq d_0$ for any $Q_i^* \in \mathcal{V}^*$. Let $\Omega^* = \cup Q_i^*$. Also, we may extend the definition of u to Ω^* such that $u(x) = 0$ if $x \in \Omega^* - \Omega$. Hence, without loss of generality, we assume that $\operatorname{diam}(Q_i) \geq d_0$ for any $Q_i \in \mathcal{V}$. Thus, $|Q_i| \geq K d_0^n$ for any $Q_i \in \mathcal{V}$ and some constant $K > 0$. Since $\Omega = \cup Q_i$, for any $x \in \Omega$, it follows that $x \in Q_i$ for some i. Applying Lemma 2.5 to Q_i, we have

$$\left(\int_{Q_i} |\Delta G(u)|^s \frac{1}{d(x, \partial\Omega)^\alpha} dx \right)^{1/s} \leq C_1 \left(\int_{\sigma Q_i} |u|^s \frac{1}{d(x, x_{Q_i})^\lambda} dx \right)^{1/s}, \tag{2.12}$$

where $\sigma > 1$ is a constant. Let $\mu(x)$ and $\mu_1(x)$ be the Radon measure defined by $d\mu = \frac{1}{d(x,\partial\Omega)^\alpha} dx$ and $d\mu_1(x) = g(x)dx$, respectively. Then,

$$\mu(Q) = \int_Q \frac{1}{d(x, \partial\Omega)^\alpha} dx \geq \int_Q \frac{1}{(\operatorname{diam}(\Omega))^\alpha} dx = P|Q|, \tag{2.13}$$

where P is a positive constant. Then, by the elementary in equality $(a + b)^s \leq 2^s(|a|^s + |b|^s)$, $s \geq 0$, we have

$$\begin{aligned}
&\left(\int_\Omega |\Delta G(u) - (\Delta G(u))_{Q_0}|^s \frac{1}{d(x, \partial\Omega)^\alpha} dx \right)^{1/s} \\
&= \left(\int_{\cup Q_i} |\Delta G(u) - (\Delta G(u))_{Q_0}|^s d\mu \right)^{1/s} \\
&\leq \left(\sum_{Q_i \in \mathcal{V}} \left(2^s \int_{Q_i} |\Delta G(u) - (\Delta G(u))_{Q_i}|^s d\mu + 2^s \int_{Q_i} |(\Delta G(u))_{Q_i} - (\Delta G(u))_{Q_0}|^s d\mu \right) \right)^{1/s} \\
&\leq C_2 \left(\left(\sum_{Q_i \in \mathcal{V}} \int_{Q_i} |\Delta G(u) - (\Delta G(u))_{Q_i}|^s d\mu \right)^{1/s} \right. \\
&\quad \left. + \left(\sum_{Q_i \in \mathcal{V}} \int_{Q_i} |(\Delta G(u))_{Q_i} - (\Delta G(u))_{Q_0}|^s d\mu \right)^{1/s} \right)
\end{aligned} \tag{2.14}$$

for a fixed $Q_0 \subset \Omega$. The first sum in (2.14) can be estimated by using Lemma 2.1 with $\phi = t^s$, $a = 2$, and Lemma 2.5

$$\begin{aligned}\sum_{Q_i\in\mathcal{V}}\int_{Q_i}|\Delta G(u)-(\Delta G(u))_{Q_i}|^s\mathrm{d}\mu &\le \sum_{Q_i\in\mathcal{V}}\int_{Q_i}2^s|\Delta G(u)|^s\mathrm{d}\mu\\ &\le C_3\sum_{Q_i\in\mathcal{V}}\int_{\sigma Q_i}|u|^s\mathrm{d}\mu_1\\ &\le C_4\sum_{Q_i\in\mathcal{V}}\int_{\Omega}(|u|^s\mathrm{d}\mu_1)\chi_{\sigma Q_i}\\ &\le C_5\int_{\Omega}|u|^s\mathrm{d}\mu_1\\ &= C_5\int_{\Omega}|u|^s g(x)dx.\end{aligned}\tag{2.15}$$

To estimate the second sum in (2.14), we need to use the property of δ-John domain. Fix a cube $Q\in\mathcal{V}$ and let $Q_0, Q_1,\ldots, Q_k = Q$ be the chain in Lemma 2.3.

$$|(\Delta G(u))_Q-(\Delta G(u))_{Q_0}|\le\sum_{i=0}^{k-1}|(\Delta G(u)_{Q_i}-(\Delta G(u))_{Q_{i+1}}|.\tag{2.16}$$

The chain $\{Q_i\}$ also has property that, for each i, $i = 0, 1,\ldots, k - 1$, with $Q_i\cap Q_{i+1}\neq\emptyset$, there exists a cube D_i such that $D_i\subset Q_i\cap Q_{i+1}$ and $Q_i\cup Q_{i+1}\subset ND_i$, $N>1$.

$$\frac{\max\{|Q_i|,|Q_{i+1}|\}}{|Q_i\cap Q_{i+1}|}\le\frac{\max\{|Q_i|,|Q_{i+1}|\}}{|D_i|}\le C_6.$$

For such D_j, $j = 0, 1,\ldots, k - 1$, Let $|D^*| = \min\{|D_0|, |D_1|,\ldots, |D_k - 1|\}$ then

$$\frac{\max\{|Q_i|,|Q_{i+1}|\}}{|Q_i\cap Q_{i+1}|}\le\frac{\max\{|Q_i|,|Q_{i+1}|\}}{|D^*|}\le C_7.\tag{2.17}$$

By (2.13), (2.17) and Lemma 2.5, we have

$$\begin{aligned}|(\Delta G(u))_{Q_i}-(\Delta G(u))_{Q_{i+1}}|^s &= \frac{1}{\mu(Q_i\cap Q_{i+1})}\int_{Q_i\cap Q_{i+1}}|(\Delta G(u))_{Q_i}-(\Delta G(u))_{Q_{i+1}}|^s\frac{\mathrm{d}x}{\mathrm{d}(x,\partial\Omega)^\alpha}\\ &\le\frac{C_8}{|Q_i\cap Q_{i+1}|}\int_{Q_i\cap Q_{i+1}}|(\Delta G(u))_{Q_i}-(\Delta G(u))_{Q_{i+1}}|^s\frac{\mathrm{d}x}{\mathrm{d}(x,\partial\Omega)^\alpha}\\ &\le\frac{C_8C_7}{\max\{|Q_i|,|Q_{i+1}|\}}\int_{Q_i\cap Q_{i+1}}|(\Delta G(u))_{Q_i}-(\Delta G(u))_{Q_{i+1}}|^s\mathrm{d}\mu\\ &\le C_9\sum_{j=i}^{i+1}\frac{1}{|Q_j|}\int_{Q_j}|\Delta G(u)-(\Delta G(u))_{Q_j}|^s\mathrm{d}\mu\\ &\le C_{10}\sum_{j=i}^{i+1}\frac{1}{|Q_j|}\int_{\sigma Q_j}|u|^s\mathrm{d}\mu_1\\ &= C_{10}\sum_{j=i}^{i+1}|Q_j|^{-1}\int_{\sigma Q_j}|u|^s\mathrm{d}\mu_1.\end{aligned}\tag{2.18}$$

Since $Q \subset NQ_j$ for $j = i,\ i + 1,\ 0 \leq i \leq k - 1$, from (2.18)

$$\begin{aligned} |(\Delta G(u))_{Q_i} - (\Delta G(u))_{Q_{i+1}}|^s \chi_Q(x) &\leq C_{11} \sum_{j=i}^{i+1} \chi_{NQ_j}(x) |Q_j|^{-1} \int_{\sigma Q_j} |u|^s \mathrm{d}\mu_1 \\ &\leq C_{12} \sum_{j=i}^{i+1} \chi_{NQ_j}(x) \frac{1}{d_0^n} \int_{\sigma Q_j} |u|^s \mathrm{d}\mu_1 \\ &\leq C_{13} \sum_{j=i}^{i+1} \chi_{NQ_j}(x) \int_{\sigma Q_j} |u|^s \mathrm{d}\mu_1. \end{aligned} \tag{2.19}$$

Using $(a + b)^{1/s} \leq 2^{1/s}(|a|^{1/s} + |b|^{1/s})$, (2.16) and (2.19), we obtain

$$|(\Delta G(u))_Q - (\Delta G(u))_{Q_0}| \chi_Q(x) \leq C_{14} \sum_{D_i \in \mathcal{V}} \left(\int_{\sigma D_i} |u|^s \mathrm{d}\mu_1 \right)^{1/s} \cdot \chi_{ND_i}(x)$$

for every $x \in \mathbb{R}^n$. Then

$$\sum_{Q \in \mathcal{V}} \int_Q |(\Delta G(u))_Q - (\Delta G(u))_{Q_0}|^s \mathrm{d}\mu \leq C_{14} \int_{\mathbb{R}^n} | \sum_{D_i \in \mathcal{V}} \left(\int_{\sigma D_i} |u|^s \mathrm{d}\mu_1 \right)^{1/s} \chi_{ND_i}(x)|^s \mathrm{d}\mu.$$

Notice that

$$\sum_{D_i \in \mathcal{V}} \chi_{ND_i}(x) \leq \sum_{D_i \in \mathcal{V}} \chi_{\sigma ND_i}(x) \leq N \chi_\Omega(x).$$

Using elementary inequality $| \sum_{i=1}^M t_i|^s \leq M^{s-1} \sum_{i=1}^M |t_i|^s$ for $s > 1$, we finally have

$$\begin{aligned} \sum_{Q \in \mathcal{V}} \int_Q |(\Delta G(u))_Q - (\Delta G(u))_{Q_0}|^s \mathrm{d}\mu &\leq C_{15} \int_{\mathbb{R}^n} \left(\sum_{D_i \in \mathcal{V}} \left(\int_{\sigma D_i} |u|^s \mathrm{d}\mu_1 \right) \chi_{ND_i}(x) \right) \mathrm{d}\mu \\ &= C_{15} \sum_{D_i \in \mathcal{V}} \left(\int_{\sigma D_i} |u|^s \mathrm{d}\mu_1 \right) \\ &\leq C_{16} \int_\Omega |u|^s g(x) \mathrm{d}x. \end{aligned} \tag{2.20}$$

Substituting (2.15) and (2.20) in (2.14), we have completed the proof of Theorem 2.6. Using Lemma 2.2, we obtain

$$\begin{aligned} \| \nabla(\Delta G(u) \|_{s,B} &= \| \mathrm{d}(\Delta G(u)) \|_{s,B} \\ &= \| \Delta G(du) \|_{s,B} \\ &= \| (dd^* + d^* d)(G(du)) \|_{s,B} \\ &\leq \| dd^*(G(du)) \|_{s,B} + \| d^* \mathrm{d}(G(du)) \|_{s,B} \\ &\leq C_1 \| du\|_{s,B} + C_2 \| du\|_{s,B} \\ &\leq C_3 \| du\|_{s,B} \\ &\leq C_4 (diam(B))^{-1} u\|_{s,\sigma B} \\ &\leq C_5 \| u\|_{s,\sigma B}, \end{aligned} \tag{2.21}$$

where $\sigma > 1$ is a constant. Using (2.21), we have the following Lemma 2.7 whose proof is similar to the proof of Lemma 2.5. □

Lemma 2.7 *Let $u \in L^s_{loc}(M, \Lambda^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the non-homogeneous A-harmonic equation in a bounded and convex domain M, G be the Green's operator and Δ be the Laplace-Beltrami operator. Then, there exists a constant C independent of u, such that*

$$\left(\int_B |\nabla(\Delta G(u))|^s \frac{1}{\mathrm{d}(x, \partial M)^\alpha} \mathrm{d}x\right)^{1/s} \le C \left(\int_{\rho B} |u|^s \frac{1}{|x - x_B|^\lambda} \mathrm{d}x\right)^{1/s} \tag{2.22}$$

for all balls B with $\rho B \subset M$ and $diam(B) \ge d_0 > 0$, where d_0 is a constant, $\rho > 1$, any real number α and λ with $\alpha > \lambda \ge 0$. Here, x_B is the center of the ball.

Notice that (2.22) can also be written as

$$\| \nabla(\Delta G(u)) \|_{s,B,\omega_1} \le C \| u\|_{s,\rho B,\omega_2}. \tag{2.22a}$$

Next, we prove the imbedding inequality with a singular factor in the John domain.

Theorem 2.8 *Let $u \in D'(\Omega, \Lambda^l)$ be a solution of the A-harmonic equation (1.1), G be the Green's operator and Δ be the Laplace-Beltrami operator. Assume that s is a fixed exponent associated with the non-homogeneous A-harmonic equation. Then, there exists a constant C, independent of u, such that*

$$\| \nabla(\Delta G(u)) \|_{s,\Omega,\omega_1} \le C \| u\|_{s,\Omega,\omega_2}, \tag{2.23}$$

$$\| \Delta G(u) \|_{W^{1,s}(\Omega),\omega_1} \le C \| u\|_{s,\Omega,\omega_2} \tag{2.24}$$

for any bounded and convex δ-John domain $\Omega \in \mathbb{R}^n$. Here, the weights are defined by $\omega_2(x) = \sum_i \chi_{Q_i} \frac{1}{|x - x_{Q_i}|^\lambda}$ and $\omega_2(x) = \sum_i \chi_{Q_i} \frac{1}{|x - x_{Q_i}|^\lambda}$, respectively, α and λ are constants with $0 \le \lambda < \alpha$.

Proof Applying the Covering Lemma 2.3 and Lemma 2.7, we have (2.23) immediately. For inequality (2.24), using Lemma 2.5 and the Covering Lemma 2.3, we have

$$\| \Delta G(u) \|_{s,\Omega,\omega_1} \le C_1 \| u\|_{s,\Omega,\omega_2}. \tag{2.25}$$

By the definition of the $\| \cdot \|_{W^{1,s}(\Omega),\omega_1}$ norm, we know that

$$\| \Delta G(u) \|_{W^{1,s}(\Omega),\omega_1} = \mathrm{diam}\ (\Omega)^{-1} \| \Delta G(u) \|_{s,\Omega,\omega_1} + \| \mathrm{d}(\Delta G(u) \|_{s,\Omega,\omega_1}. \tag{2.26}$$

Substituting (2.23) and (2.25) into (2.26) yields

$$\| \Delta G(u) \|_{W^{1,s}(\Omega),\omega_1} \le C_2 \| u\|_{s,\Omega,\omega_2}.$$

We have completed the proof of the Theorem 2.8. □

Theorem 2.9 *Let $u \in D'(\Omega, \Lambda^l)$ be a solution of the A-harmonic equation (1.1), G be the Green's operator and Δ be the Laplace-Beltrami operator. Assume that s is a fixed exponent associated with the non-homogeneous A-harmonic equation. Then, there exists a constant C, independent of u, such that*

$$\| \Delta G(u) - (\Delta G(u))_{Q_0} \|_{W^{1,s}(\Omega),\omega_1} \le C \| u\|_{s,\Omega,\omega_2} \tag{2.27}$$

for any bounded and convex δ-John domain $\Omega \in \mathbb{R}^n$. *Here the weights are defined by* $\omega_2(x) = \sum_i \chi_{Q_i} \frac{1}{|x-x_{Q_i}|^\lambda}$*and* $\omega_2(x) = \sum_i \chi_{Q_i} \frac{1}{|x-x_{Q_i}|^\lambda}$, *α and λ are constants with* $0 \le \lambda < \alpha$, *and the fixed cube* $Q_0 \subset \Omega$ *and the constant* $N > 1$ *appeared in Lemma 2.3.*

Proof Since $(\Delta G(u))_{Q_0}$ is a closed form, $\nabla((\Delta G(u))_{Q_0}) = d((\Delta G(u))_{Q_0}) = 0$. Thus, by using Theorem 2.6 and (2.23), we have

$$\begin{aligned}
&\| \Delta G(u) - (\Delta G(u))_{Q_0} \|_{W^{1,s}(\Omega),\omega_1} \\
&= \operatorname{diam}(\Omega)^{-1} \| \Delta G(u) - (\Delta G(u))_{Q_0} \|_{s,\Omega,\omega_1} + \| \nabla(\Delta G(u) - (\Delta G(u))_{Q_0}) \|_{s,\Omega,\omega_1} \\
&= \operatorname{diam}(\Omega)^{-1} \| \Delta G(u) - (\Delta G(u))_{Q_0} \|_{s,\Omega,\omega_1} + \| \nabla(\Delta G(u)) \|_{s,\Omega,\omega_1} \\
&\le C_1 \| u \|_{s,\Omega,\omega_2} + C_2 \| u \|_{s,\Omega,\omega_2} \\
&\le C_3 \| u \|_{s,\Omega,\omega_2}.
\end{aligned}$$

Thus, (2.27) holds. The proof of Theorem 2.9 has been completed. □

As applications of our main results, we consider the following example.

Example 1 Let $B = 0$, $A(x, \xi) = \xi|\xi|^{p-2}$, $p > 1$, and u be a function(0-form) in (1.1). Then, the operator A satisfies the required conditions and the non-homogeneous A-harmonic equation(1.1) reduces to the usual p-harmonic equation

$$\operatorname{div}(\nabla u|\nabla u|^{p-2}) = 0 \tag{2.28}$$

which is equivalent to

$$(p-2)\sum_{k=1}^{n}\sum_{i=1}^{n} u_{x_k}u_{x_i}u_{x_k x_i} + |\nabla u|^2 \Delta u = 0. \tag{2.29}$$

If we choose $p = 2$ in (2.28), we have Laplace equation $\Delta u = 0$ for functions. Hence, the Equations (2.28), (2.29) and the $\Delta u = 0$ are the special cases of the non-homogeneous A-harmonic equation (1.1). Therefore, all results proved in Theorem 2.6, 2.8, and 2.9 are still true for u that satisfies one of the above three equations.

Example 2 Let $f : \Omega \to \mathbb{R}^n$, $f = (f^1,\dots, f^n)$, be a mapping of the Sobolev class $W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$, $1 < p < \infty$, whose distributional differential $Df = [\partial f^i/\partial x_j] : \Omega \to GL(n)$ is a locally integrable function in Ω with values in the space $GL(n)$ of all $n \times n$-matrices, i, $j = 1, 2,\dots, n$. we use

$$J(x,f) = \det Df(x) = \begin{vmatrix} f^1_{x_1} & f^1_{x_2} & f^1_{x_3} & \cdots & f^1_{x_n} \\ f^2_{x_1} & f^2_{x_2} & f^2_{x_3} & \cdots & f^2_{x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f^n_{x_1} & f^n_{x_2} & f^n_{x_3} & \cdots & f^n_{x_n} \end{vmatrix}$$

to denote the Jacobian determinant of f. A homeomorphism $f : \Omega \to \mathbb{R}^n$ of the Sobolev class $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ is said to be K-quasiconformal, $1 \le K < \infty$, if its differential matrix $Df(x)$ and the Jacobian determinant $J(x, f)$ satisfy

$$|Df(x)|^n \le KJ(x,f), \tag{2.30}$$

where $|Df(x)| = \max |Df(x)h| : |h| = 1$ denotes the norm of the Jacobi matrix $Df(x)$. It is well known that if the differential matrix $Df(x) = [\partial f^i / \partial x_j]$, $i, j = 1, 2,\dots, n$, of a homeomorphism $f(x) = (f^1, f^2,\dots, f^n) : \Omega \to \mathbb{R}^n$ satisfies (2.30), then, each of the functions

$$u = f^i(x), \quad i = 1, 2, ..., n, \quad \text{or} \quad u = \log|f(x)|, \tag{2.31}$$

is a generalized solution of the quasilinear elliptic equation

$$\operatorname{div} A(x, \nabla u) = 0, \tag{2.32}$$

in Ω - $f^{1}(0)$, where

$$A = (A_1, A_2, \ldots, A_n), A(x, \xi) = \frac{\partial}{\partial \xi_i}\left(\sum_{i,j=1}^{n} \theta_{i,j}(x)\xi_i\xi_j\right)^{n/2}$$

and $\theta_{i,j}$ are some functions, which can be expressed in terms of the differential matrix $Df(x)$ and satisfy

$$C_1(K)|\xi|^2 \le \sum_{i,j=1}^{n} \theta_{i,j}(x)\xi_i\xi_j \le C_2(K)|\xi|^2 \tag{2.33}$$

for some constants $C_1(K)$, $C_2(K)$ >0. Choosing u is defined in (2.31) and applying Theorems (2.6), (2.8) and (2.9) to u, respectively, we have the following theorems.

Theorem 3.0 *Let $u = f^i(x)$ or $u = \log |f(x)| \in D'(\Omega, \Lambda^l)$, $i = 1, 2,..., n$, be a solution of the quasilinear elliptic equation (2.32), where $f : \Omega \to \mathbb{R}^n$, $f = (f^1,..., f^n)$ be a K-quasiconformal mapping of the Sobolev class $W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$, $1 < p <\infty$, G be the Green's operator and Δ be the Laplace-Beltrami operator. Assume that s is a fixed exponent associated with the non-homogeneous A-harmonic equation. Then, there exists a constant C, independent of u, such that*

$$\left(\int_{\Omega} |\Delta G(u) - (\Delta G(u))_{Q_0}|^s \frac{1}{d(x, \partial\Omega)^{\alpha}} dx\right)^{1/s} \le C\left(\int_{\Omega} |u|^s g(x) dx\right)^{1/s} \tag{2.34}$$

for any bounded and convex δ-John domain $\Omega \in \mathbb{R}^n$, where $g(x) = \sum_i \chi_{Q_i} \frac{1}{|x - x_{Q_i}|^{\lambda}}$, $x_{Q_i}$is the center of Q_i with $\Omega = \cup_i Q_i$. Here α and λ are constants with $0 \le \lambda < \alpha <n$, and the fixed cube $Q_0 \in \Omega$, the constant $N >1$ and the cubes $Q_i \in \Omega$ appeared in Lemma 2.3, $x_{Q_i}$is the center of Q_i.

Theorem 3.1 *Let $u = f^i(x)$ or $u = \log |f(x)| \in D'(\Omega, \Lambda^l)$, $i = 1, 2,..., n$, be a solution of the quasilinear elliptic equation (2.32), where $f : \Omega \to \mathbb{R}^n$, $f = (f^1,..., f^n)$ be a K-quasiconformal mapping of the Sobolev class $W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$, $1 <p < \infty$, G be the Green's operator and Δ be the Laplace-Beltrami operator. Assume that s is a fixed exponent associated with the non-homogeneous A-harmonic equation. Then, there exists a constant C, independent of u, such that*

$$\| \nabla(\Delta G(u)) \|_{s,\Omega,\omega_1} \le C \| u\|_{s,\Omega,\omega_2}, \tag{2.35}$$

$$\| \Delta G(u) \|_{W^{1,s}(\Omega),\omega_1} \le C \| u\|_{s,\Omega,\omega_2} \tag{2.36}$$

for any bounded and convex δ-John domain $\Omega \in \mathbb{R}^n$. Here, the weights are defined by $\omega_2(x) = \sum_i \chi_{Q_i} \frac{1}{|x - x_{Q_i}|^{\lambda}}$and $\omega_2(x) = \sum_i \chi_{Q_i} \frac{1}{|x - x_{Q_i}|^{\lambda}}$, respectively, α and λ are constants with $0 \le \lambda < \alpha < n$.

Theorem 3.2 *Let $u = f^i(x)$ or $u = \log |f(x)| \in D'(\Omega, \Lambda^l)$, $i = 1, 2,..., n$, be a solution of the quasilinear elliptic equation (2.32), where $f : \Omega \to \mathbb{R}^n$, $f = (f^1,..., f^n)$ be a K-quasiconformal mapping of the Sobolev class $W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$, $1 < p < \infty$, G be the Green's operator and Δ be the Laplace-Beltrami operator. Assume that s is a fixed exponent associated with the non-homogeneous A-harmonic equation. Then, there exists a constant C, independent of u, such that*

$$\| \Delta G(u) - (\Delta G(u))_{Q_0} \|_{W^{1,s}(\Omega),\omega_1} \leq C \| u \|_{s,\Omega,\omega_2} \tag{2.37}$$

for any bounded and convex δ-John domain $\Omega \in \mathbb{R}^n$. Here, the weights are defined by $\omega_2(x) = \sum_i \chi_{Q_i} \frac{1}{|x-x_{Q_i}|^\lambda}$and $\omega_2(x) = \sum_i \chi_{Q_i} \frac{1}{|x-x_{Q_i}|^\lambda}$, α and λ are constants with $0 \leq \lambda < \alpha < n$, and the fixed cube $Q_0 \subset \Omega$ and the constant $N > 1$ appeared in Lemma 2.3.

Our results can be applied to all differential forms or functions satisfying some version of the A-harmonic equation, the usual p-harmonic equation or the Laplace equation [1, 9, 10, for more applications].

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Authors' contributions
RF and SD jointly contributed to the main results and RF drafted the manuscript. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 21 April 2011 Accepted: 30 September 2011 Published: 30 September 2011

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doi:10.1186/1029-242X-2011-74
Cite this article as: Fang and Ding: **Singular integrals of the compositions of Laplace-Beltrami and Green's operators.** *Journal of Inequalities and Applications* 2011 **2011**:74.