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On some Opial-type inequalities

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Abstract

In the present paper we establish some new Opial-type inequalities involving higher-order partial derivatives. Our results in special cases yield some of the recent results on Opial's inequality and also provide new estimates on inequalities of this type.

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1 Introduction

In the year 1960, Opial [1] established the following integral inequality:

Theorem 1.1. *Suppose $f \in C^1[0, h]$ satisfies $f(0) = f(h) = 0$ and $f(x) > 0$ for all $x \in (0, h)$. Then the integral inequality holds*

$$\int_0^h |f(x)f'(x)|\, dx \le \frac{h}{4} \int_0^h (f'(x))^2\, dx, \tag{1.1}$$

where this constant $\frac{h}{4}$ is best possible.

Opial's inequality and its generalizations, extensions and discretizations play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [2-6]. The inequality (1.1) has received considerable attention, and a large number of papers dealing with new proofs, extensions, generalizations, variants and discrete analogues of Opial's inequality have appeared in the literature [7-22]. For an extensive survey on these inequalities, see [2,6]. For Opial-type integral inequalities involving high-order partial derivatives see [23-27]. The main purpose of the present paper is to establish some new Opial-type inequalities involving higher-order partial derivatives by an extension of Das's idea [28]. Our results in special cases yield some of the recent results on Opial's-type inequalities and provide some new estimates on such types of inequalities.

2 Main results

Let $n \ge 1$, $k \ge 1$. Our main results are given in the following theorems.

Theorem 2.1 *Let $x(s, t) \in C^{(n-1)}[0, a] \times C^{(k-1)}[0, b]$ be such that $\frac{\partial^i}{\partial \sigma^i} x(0, \tau) = 0$, $\frac{\partial^j}{\partial \tau^j} x(\sigma, 0) = 0$, $\sigma \in [0, s]$, $\tau \in [0, t]$, $0 \le i \le n - 1$, $0 \le j \le k - 1$. Further, let $\frac{\partial^{n-1}}{\partial s^{n-1}} x(s, t)$, $\frac{\partial^{k-1}}{\partial t^{k-1}} x(s, t)$ be absolutely continuous, and $\int_0^a \int_0^b |x^{(n,k)}(s, t)|^2\, ds\, dt < \infty$. Then*

$$\int_0^a \int_0^b \left| x(s,t) \cdot x^{(n,k)}(s,t) \right| \, ds \, dt \leq c_{n,k} \cdot a^n b^k \cdot \int_0^a \int_0^b \left| x^{(n,k)}(s,t) \right|^2 \, ds \, dt, \tag{2.1}$$

where

$$x^{(n,k)}(s,t) = \frac{\partial^n}{\partial s^n} \left(\frac{\partial^k}{\partial t^k} x(s,t) \right),$$

and

$$c_{n,k} = \frac{1}{4n!k!} \left(\frac{2nk}{(2n-1)(2k-1)} \right)^{\frac{1}{2}}.$$

Proof. For σ integration by parts $(n-1)$-times and in view of $\frac{\partial^i}{\partial \sigma^i} x(0, \tau) = 0$, $\frac{\partial^j}{\partial \tau^j} x(\sigma, 0) = 0$, $0 \leq i \leq n-1$, $0 \leq j \leq k-1$ we have

$$x(s,t) = \frac{(-1)^n}{(n-1)!} \int_s^0 (\sigma - s)^{n-1} \frac{\partial^n}{\partial \sigma^n} x(\sigma, t) \, d\sigma$$

$$= \frac{(-1)^{2n-1}}{(n-1)!} \int_s^0 (s-\sigma)^{n-1} \frac{\partial^n}{\partial \sigma^n} x(\sigma, t) \, d\sigma = \frac{1}{(n-1)!} \int_0^s (s-\sigma)^{n-1} \frac{\partial^n}{\partial \sigma^n} x(\sigma, t) \, d\sigma$$

$$= \frac{1}{(n-1)!(k-1)!} \int_0^s (s-\sigma)^{n-1} \frac{\partial^n}{\partial \sigma^n} \left(\int_0^t (t-\tau)^{k-1} \frac{\partial^k}{\partial \tau^k} x(\sigma, \tau) d\tau \right) d\sigma$$

$$= \frac{1}{(n-1)!(k-1)!} \int_0^s \int_0^t (s-\sigma)^{n-1} (t-\tau)^{k-1} \cdot x^{(n,k)}(\sigma, \tau) \, d\sigma \, d\tau. \tag{2.2}$$

Multiplying both sides of (2.2) by $x^{(n,k)}(s,t)$ and using the Cauchy-Schwarz inequality, we have

$$\left| x(s,t) \cdot x^{(n,k)}(s,t) \right| \leq \frac{\left| x^{(n,k)}(s,t) \right|}{(n-1)!(k-1)!} \left(\int_0^s \int_0^t (s-\sigma)^{2n-2} (t-\tau)^{2k-2} \, d\sigma \, d\tau \right)^{\frac{1}{2}}$$

$$\times \left(\int_0^s \int_0^t \left| x^{(n,k)}(\sigma, \tau) \right|^2 d\sigma \, d\tau \right)^{\frac{1}{2}} \tag{2.3}$$

$$= \frac{1}{(n-1)!(k-1)!\sqrt{(2n-1)(2k-1)}} \cdot s^{n-\frac{1}{2}} t^{k-\frac{1}{2}} \left| x^{(n,k)}(s,t) \right| \left(\int_0^s \int_0^t \left| x^{(n,k)}(\sigma, \tau) \right|^2 d\sigma \, d\tau \right)^{\frac{1}{2}}.$$

Thus, integrating both sides of (2.3) over t from 0 to b first and then integrating the resulting inequality over s from 0 to a and applying the Cauchy-Schwarz inequality again, we obtain

$$\int_0^a \int_0^b \left| x(s,t) \cdot x^{(n,k)}(s,t) \right| \, ds \, dt$$

$$\leq \frac{1}{(n-1)!(k-1)!\sqrt{(2n-1)(2k-1)}} \left(\int_0^a \int_0^b s^{2n-1} t^{2k-1} \, ds \, dt \right)^{\frac{1}{2}}$$

$$\times \left(\int_0^a \int_0^b \left| x^{(n,k)}(s,t) \right|^2 \left(\int_0^s \int_0^t \left| x^{(n,k)}(\sigma, \tau) \right|^2 d\sigma \, d\tau \right) ds \, dt \right)^{\frac{1}{2}}$$

$$= \frac{1}{2n!} \left(\frac{2n}{2n-1} \right)^{\frac{1}{2}} \frac{1}{2k!} \left(\frac{2k}{2k-1} \right)^{\frac{1}{2}} a^n b^k$$

$$\times \left(\frac{1}{2} \int_0^a \int_0^b \frac{\partial^2}{\partial s \partial t} \left\{ \left(\int_0^s \int_0^t \left| x^{(n,k)}(\sigma, \tau) \right|^2 d\sigma \, d\tau \right)^2 \right\} ds \, dt \right)^{\frac{1}{2}}$$

$$= c_{n,k} a^n b^k \int_0^a \int_0^b \left| x^{(n,k)}(s,t) \right|^2 ds \, dt.$$

This completes the proof.

Remark 2.1. Let $x(s, t)$ reduce to $s(t)$ and with suitable modifications, Then (2.1) becomes the following inequality:

$$\int_0^a |x(t)x^{(n)}(t)|\mathrm{d}t \leq \frac{1}{2n!} \cdot \left(\frac{n}{2n-1}\right)^{\frac{1}{2}} a^n \int_0^a |x^{(n)}(t)|^2 \mathrm{d}t. \tag{2.4}$$

This is just an inequality established by Das [28]. Obviously, for $n \geq 2$, (2.4) is sharper than the following inequality established by Willett [29].

$$\int_0^a |x(t)x^n(t)|\mathrm{d}t \leq \frac{1}{2}a^n \int_0^a |x^n(t)|^2 \mathrm{d}t. \tag{2.5}$$

Remark 2.2. Taking for $n = k = 1$ in (2.1), (2.1) reduces to

$$\int_0^a \int_0^b \left| x(s, t) \cdot \frac{\partial^2}{\partial s \partial t} x(s, t) \right| \mathrm{d}s\, \mathrm{d}t \leq \frac{\sqrt{2}}{4} ab \int_0^a \int_0^b \left| \frac{\partial^2}{\partial s \partial t} x(s, t) \right|^2 \mathrm{d}s\, \mathrm{d}t. \tag{2.6}$$

Let $x(s, t)$ reduce to $s(t)$ and with suitable modifications. Then (2.6) becomes the following inequality: If $x(t)$ is absolutely continuous in $[0, a]$ and $x(0) = 0$, then

$$\int_0^a |x(t)x'(t)|\mathrm{d}t \leq \frac{a}{2} \int_0^a |x'(t)|^2 \mathrm{d}t.$$

This is just an inequality established by Beesack [30].

Remark 2.3. Let $0 \leq \alpha, \beta < n$, but fixed, and let $g(s, t) \in C^{(n-\alpha-1)}[0, a] \times C^{(k-\beta-1)}[0, b]$ be such that $\frac{\partial^i}{\partial s^i}g(0, t) = \frac{\partial^i}{\partial t^i}g(s, 0) = 0$, $0 \leq i \leq n - \alpha - 1$, $0 \leq i \leq k - \beta -1$ and suppose that $\frac{\partial^{n-\alpha-1}}{\partial s^{n-\alpha-1}}g(s, t)$, $\frac{\partial^{k-\beta-1}}{\partial t^{k-\beta-1}}g(s, t)$ are absolutely continuous, and $\int_0^a \int_0^b \left|x^{(n-\alpha, k-\beta)}(s, t)\right|^2 \mathrm{d}s\, \mathrm{d}t < \infty$.

Then from (2.1) it follows that

$$\int_0^a \int_0^b \left| g(s, t) \cdot g^{(n-\alpha, k-\beta)}(s, t) \right| \mathrm{d}s\, \mathrm{d}t \leq c_{n-\alpha, k-\beta} a^{n-\alpha} b^{k-\beta} \int_0^a \int_0^b \left| g^{(n-\alpha, k-\beta)}(s, t) \right|^2 \mathrm{d}s\, \mathrm{d}t.$$

Thus, for $g(s, t) = x^{(\alpha, \beta)}(s, t)$, where $x(s, t) \in C^{(n-1)}[0, a] \times C^{(k-1)}[0, b]$, $\frac{\partial^i}{\partial s^i}x(0, t) = 0$, $\frac{\partial^j}{\partial t^j}x(s, 0) = 0$, $\alpha \leq i \leq n - 1$, $\beta \leq j \leq k - 1$, and $x^{(n-1, k-1)}(s, t)$ are absolutely continuous, and $\int_0^a \int_0^b \left|x^{(n,k)}(s, t)\right|^2 \mathrm{d}s\, \mathrm{d}t < \infty$, then

$$\int_0^a \int_0^b \left| x^{(\alpha, \beta)}(s, t) \cdot x^{(n,k)}(s, t) \right| \mathrm{d}s\, \mathrm{d}t \leq c_{n-\alpha, k-\beta} a^{n-\alpha} b^{k-\beta} \int_0^a \int_0^b \left| x^{(n,k)}(s, t) \right|^2 \mathrm{d}s\, \mathrm{d}t. \tag{2.7}$$

Obviously, a special case of (2.7) is the following inequality:

$$\int_0^a \int_0^b \left| x^{(k,k)}(s, t) \cdot x^{(n,n)}(s, t) \right| \mathrm{d}s\, \mathrm{d}t \leq c_{n-k, n-k} (ab)^{n-k} \int_0^a \int_0^b \left| x^{(n,n)}(s, t) \right|^2 \mathrm{d}s\, \mathrm{d}t \tag{2.8}$$

Let $x(s, t)$ reduce to $s(t)$ and with suitable modifications. Then (2.8) becomes the following inequality:

$$\int_0^a |x^{(k)}(t)x^{(n)}(t)| \mathrm{d}t \le \frac{1}{2(n-k)!} \cdot \left(\frac{n-k}{2(n-k)-1}\right)^{\frac{1}{2}} a^{n-k} \int_0^a |x^{(n)}(t)|^2 \mathrm{d}t.$$

This is just an inequality established by Agarwal and Thandapani [31].

Theorem 2.2. *Let l and m be positive numbers satisfying $l + m > 1$. Further, let $x(s, t)$ $\in C^{(n-1)}[0, a] \times C^{(k-1)}[0, b]$ be such that $\frac{\partial^i}{\partial \sigma^i} x(0, \tau) = 0$, $\frac{\partial^j}{\partial \tau^j} x(\sigma, 0) = 0$, $\sigma \in [0, s]$, $\tau \in [0, t]$, $0 \le i \le n - 1$, $0 \le j \le k - 1$ and assume that $\frac{\partial^{n-1}}{\partial s^{n-1}} x(s, t)$, $\frac{\partial^{k-1}}{\partial t^{k-1}} x(s, t)$ are absolutely continuous, and $\int_0^a \int_0^b |x^{(n,k)}(s, t)|^{l+m} \mathrm{d}s\,\mathrm{d}t < \infty$. Then*

$$\int_0^a \int_0^b |x(s, t)|^l \left|x^{(n,k)}(s, t)\right|^m \mathrm{d}s\,\mathrm{d}t \le c_{n,k}^* a^{nl} b^{kl} \int_0^a \int_0^b \left|x^{(n,k)}(s, t)\right|^{l+m} \mathrm{d}s\,\mathrm{d}t, \qquad (2.9)$$

where

$$c_{n,k}^* = \xi^{l\xi+1} m^{\xi m} \left(\frac{kn(1-\xi)^2}{(n-\xi)(k-\xi)}\right)^{l(1-\xi)} \cdot (n!k!)^{-l}, \quad \xi = \frac{1}{l+m}.$$

Proof. From (2.2), we have

$$|x(s, t)| \le \frac{1}{(n-1)!(k-1)!} \int_0^s \int_0^t (s-\sigma)^{n-1}(t-\tau)^{k-1} \left|x^{(n,k)}(\sigma, \tau)\right| \mathrm{d}\sigma\,\mathrm{d}\tau,$$

by Hölder's inequality with indices $l + m$ and $\frac{l+m}{l+m-1}$, it follows that

$$|x(s, t)| \le \frac{1}{(n-1)!(k-1)!} \left(\int_0^s \int_0^t [(s-\sigma)^{n-1}(t-\tau)^{k-1}]^{\frac{l+m}{l+m-1}} \mathrm{d}\sigma\,\mathrm{d}\tau\right)^{\frac{l+m-1}{l+m}}$$

$$\times \left(\int_0^s \int_0^t \left|x^{(n,k)}(\sigma, \tau)\right|^{l+m} \mathrm{d}\sigma\,\mathrm{d}\tau\right)^{\frac{1}{l+m}}$$

$$= As^{n-\xi} t^{k-\xi} \left(\int_0^s \int_0^t \left|x^{(n,k)}(\sigma, \tau)\right|^{l+m} \mathrm{d}\sigma\,\mathrm{d}\tau\right)^{\xi},$$

where

$$A = \left(\frac{(1-\xi)^2}{(n-\xi)(k-\xi)}\right)^{1-\xi} \frac{1}{(n-1)!(k-1)!}.$$

Multiplying the both sides of above inequality by $|x^{(n,k)}(s, t)|^m$ and integrating both sides over t from 0 to b first and then integrating the resulting inequality over s from 0 to a, we obtain

$$\int_0^a \int_0^b |x(s, t)|^l \left|x^{(n,k)}(s, t)\right|^m \mathrm{d}s\,\mathrm{d}t$$

$$\le A^l \int_0^a \int_0^b s^{l(n-\xi)} t^{l(k-\xi)} \left|x^{(n,k)}(s, t)\right|^m \left(\int_0^s \int_0^t \left|x^{(n,k)}(\sigma, \tau)\right|^{l+m} \mathrm{d}\sigma\,\mathrm{d}\tau\right)^{l\xi} \mathrm{d}s\,\mathrm{d}t.$$

Now, applying Hölder's inequality with indices $\dfrac{l+m}{l}$ and $\dfrac{l+m}{m}$ to the integral on the right-side, we obtain

$$\int_0^a \int_0^b |x(s,t)|^l \left|x^{(n,k)}(s,t)\right|^m ds\,dt \le A^l \left(\int_0^a \int_0^b s^{(n-\xi)(l+m)} t^{(k-\xi)(l+m)} ds\,dt\right)^{\frac{l}{l+m}}$$

$$\times \left(\int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right|^{m+l} \left(\int_0^s \int_0^t \left|x^{(n,k)}(\sigma,\tau)\right|^{l+m} d\sigma\,d\tau\right)^{\frac{l}{m}} ds\,dt\right)^{\frac{m}{l+m}}$$

$$= A^l \left(\int_0^a \int_0^b s^{(n-\xi)(l+m)} t^{(k-\xi)(l+m)} ds\,dt\right)^{\frac{l}{l+m}}$$

$$\times \left(\frac{m}{l+m} \int_0^a \int_0^b \frac{\partial^2}{\partial s \partial t} \left(\int_0^s \int_0^t \left|x^{(n,k)}(\sigma,\tau)\right|^{l+m} d\sigma\,d\tau\right)^{\frac{l}{m}+1} ds\,dt\right)^{\frac{m}{l+m}}$$

$$= A^l \left(\frac{\xi^2}{kn}\right)^{\xi l} (m\xi)^{m\xi} a^{nl} b^{kl} \int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right|^{l+m} ds\,dt$$

$$= c_{n,k}^* a^{nl} b^{kl} \int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right|^{l+m} ds\,dt.$$

This completes the proof.

Remark 2.4. Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications. Then (2.9) becomes the following inequality:

$$\int_0^a |x(t)|^l \left|x^{(n)}(t)\right|^m dt \le \xi m^{m\xi} \left(\frac{n(1-\xi)}{n-\xi}\right)^{l(1-\xi)} (n!)^{-l} a^{nl} \int_0^a \left|x^{(n)}(t)\right|^{l+m} dt. \quad (2.10)$$

This is an inequality given by Das [28]. Taking for $n = 1$ in (2.10), we have

$$\int_0^a |x(t)|^l |x'(t)|^m dt \le \frac{m^{m/(l+m)}}{l+m} a^l \int_0^a |x'(t)|^{m+l} dt. \quad (2.11)$$

For $m, l \ge 1$ Yang [32] established the following inequality:

$$\int_0^a |x(t)|^l |x'(t)|^m dt \le \frac{m}{l+m} a^l \int_0^a |x'(t)|^{m+l} dt. \quad (2.12)$$

Obviously, for $m, l \ge 1$, (2.11) is sharper than (2.12).

Remark 2.5. For $n = k = 1$; (2.9) reduces to

$$\int_0^a \int_0^b |x(s,t)|^l \left|\frac{\partial^2}{\partial s \partial t} x(s,t)\right|^m ds\,dt \le c_{1,1}^* (ab)^l \int_0^a \int_0^b \left|\frac{\partial^2}{\partial s \partial t} x(s,t)\right|^{m+l} ds\,dt.$$

Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications. Then above inequality becomes the following inequality:

$$\int_0^a |x(t)|^l |x'(t)|^m dt \le \xi m^{m\xi} a^l \int_0^a |x'(t)|^{m+l} dt, \quad \xi = (l+m)^{-1}.$$

This is just an inequality established by Yang [32].

Remark 2.6. Following Remark 2.3, for $x(s, t) \in C^{(n-1)}[0, a] \times C^{(k-1)}[0, b]$, $\frac{\partial^j}{\partial t^j}x(s, 0) = 0$, $\frac{\partial^j}{\partial t^j}x(s, 0) = 0$, $\alpha \le i \le n - 1$, $\beta \le j \le k - 1$ and $x^{(n-1, k-1)}(s, t)$ are absolutely continuous, and $\int_0^a \int_0^b |x^{(n,k)}(s, t)|^{l+m} ds\, dt < \infty$, it is easy to obtain that

$$\int_0^a \int_0^b \left|x^{(\alpha,\beta)}(s, t)\right|^l \cdot \left|x^{(n,k)}(s, t)\right|^m ds\, dt \le c_{n-\alpha, k-\beta}^* a^{l(n-\alpha)} b^{l(k-\beta)} \int_0^a \int_0^b \left|x^{(n,k)}(s, t)\right|^{l+m} ds dt \qquad (2.13)$$

Obviously, a special case of (2.14) is the following inequality:

$$\int_0^a \int_0^b \left|x^{(k,k)}(s, t)\right|^l \cdot \left|x^{(n,n)}(s, t)\right|^m ds\, dt \le c_{n-k, n-k}^* (ab)^{l(n-k)} \int_0^a \int_0^b \left|x^{(n,n)}(s, t)\right|^{l+m} ds\, dt \qquad (2.14)$$

Let $x(s, t)$ reduce to $s(t)$ and with suitable modifications, then (2.14) becomes the following inequality:

$$\int_0^a \left|x^{(k)}(t)\right|^l \left|x^{(n)}(t)\right|^m dt$$

$$\le \xi m^{m\xi} \left(\frac{(n-k)(1-\xi)}{n-k-\xi}\right)^{l(1-\xi)} ((n-k)!)^{-l} a^{(n-k)l} \int_0^a \left|x^{(n)}(t)\right|^{l+m} dt, \quad \xi = (l+m)^{-1}.$$

This is just an inequality established by Agarwal and Thandapani [31].

Theorem 2.3. *Let l and m be positive numbers satisfying $l + m = 1$. Further, let $x(s, t) \in C^{(n-1)}[0, a] \times C^{(k-1)}[0, b]$ be such that $\frac{\partial^i}{\partial \sigma^i}x(0, \tau) = 0$, $\frac{\partial^j}{\partial \tau^j}x(\sigma, 0) = 0$, $\sigma \in [0, s]$, $\tau \in [0, t]$, $0 \le i \le n - 1$, $0 \le j \le k - 1$ and assume that $\frac{\partial^{n-1}}{\partial s^{n-1}}x(s, t)$, $\frac{\partial^{k-1}}{\partial t^{k-1}}x(s, t)$ are absolutely continuous, and $\int_0^a \int_0^b |x^{(n,k)}(s, t)| ds\, dt < \infty$. Then*

$$\int_0^a \int_0^b |x(s, t)|^l \left|x^{(n,k)}(s, t)\right|^m ds\, dt \le \frac{m^m}{(n!k!)^l} a^{nl} b^{kl} \int_0^a \int_0^b \left|x^{(n,k)}(s, t)\right| ds\, dt. \qquad (2.15)$$

Proof. It is clear that

$$|x(s, t)| \le \frac{1}{(n-1)!(k-1)!} \int_0^s \int_0^t (s - \sigma)^{n-1}(t - \tau)^{k-1} \left|x^{(n,k)}(\sigma, \tau)\right| d\sigma\, d\tau$$

$$\le \frac{1}{(n-1)!(k-1)!} s^{n-1} t^{k-1} \int_0^s \int_0^t \left|x^{(n,k)}(\sigma, \tau)\right| d\sigma\, d\tau$$

and hence

$$\int_0^a \int_0^b |x(s, t)|^l \left|x^{(n,k)}(s, t)\right|^m ds\, dt$$

$$\le \frac{1}{[(n-1)!(k-1)!]^l} \int_0^a \int_0^b s^{(n-1)l} t^{(k-1)l} \left|x^{(n,k)}(s, t)\right|^m \left(\int_0^s \int_0^t \left|x^{(n,k)}(s, t)\right| ds\, dt\right)^l ds\, dt.$$

Now applying Hölder inequality with indices $\frac{1}{l}$ and $\frac{1}{m}$, we obtain

$$\int_0^a \int_0^b |x(s,t)|^l \left|x^{(n,k)}(s,t)\right|^m ds\,dt \leq \frac{1}{[(n-1)!(k-1)!]^l} \left(\int_0^a \int_0^b s^{n-1}t^{k-1}ds\,dt\right)^l$$

$$\times \left(\int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right| \left(\int_0^s \int_0^t \left|x^{(n,k)}(\sigma,\tau)\right| d\sigma\,d\tau\right)^{\frac{l}{m}} ds\,dt\right)^m$$

$$= \frac{1}{[(n-1)!(k-1)!]^l} \left(\frac{1}{n!k!}\right)^l a^{nl}b^{kl}$$

$$\times \left(m \int_0^a \int_0^b \frac{\partial^2}{\partial s \partial t} \left\{\left(\int_0^s \int_0^t \left|x^{(n,k)}(\sigma,\tau)\right| d\sigma\,d\tau\right)^{\frac{l}{m}+1}\right\} ds\,dt\right)^m$$

$$= \frac{m^m}{(n!k!)^l} a^{nl}b^{kl} \int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right| ds\,dt.$$

This completes the proof.

Remark 2.7. Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications. Then (2.16) becomes the following inequality:

$$\int_0^a |x^{(k)}(t)|^l |x^{(n)}(t)|^m dt \leq \frac{m^m}{(n!)^l} a^{nl} \int_0^a |x^n(t)| dt.$$

This is an inequality given by Das [28].

Remark 2.8. Following Remark 2.3, for $x(s,t) \in C^{(n-1)}[0,a] \times C^{(k-1)}[0,b]$, $\frac{\partial^j}{\partial t^j}x(s,0) = 0$, $\frac{\partial^j}{\partial t^j}x(s,0) = 0$, $\alpha \leq i \leq n-1$, $\beta \leq j \leq k-1$, and $x^{(n-1,k-1)}(s,t)$ are absolutely continuous, and $\int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right| ds\,dt < \infty$, from (2.16), it is easy to obtain that

$$\int_0^a \int_0^b \left|x^{(\alpha,\beta)}(s,t)\right|^l \cdot \left|x^{(n,k)}(s,t)\right|^m ds\,dt$$

$$\leq \frac{m^m}{[(n-\alpha)!(k-\beta)!]^l} a^{l(n-\alpha)}b^{l(k-\beta)} \int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right| ds\,dt. \tag{2.16}$$

Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications, then (2.16) becomes the following inequality:

$$\int_0^a |x^{(k)}(t)|^l |x^{(n)}(t)|^m dt \leq \frac{m^m}{((n-k)!)^l} a^{(n-k)l} \int_0^a |x^n(t)| dt, \ l+m=1.$$

This is an inequality given by Das [28].

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Authors' contributions

C-JZ and W-SC jointly contributed to the main results Theorems 2.1, 2.2, and 2.3. Both authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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