Uyanik et al. Journal of Inequalities and Applications 2011, 2011:61
http://www.journalofinequalitiesandapplications.com/content/2011/1/61

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Reciprocal classes of *p*-valently spirallike and *p*-valently Robertson functions

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Abstract

For p-valently spirallike and p-valently Robertson functions in the open unit disk $\mathbb{U}$, reciprocal classes $\mathcal{S}_p(\alpha, \beta)$, and $\mathcal{C}_p(\alpha, \beta)$ are introduced. The object of the present paper is to discuss some interesting properties for functions $f(z)$ belonging to the classes $\mathcal{S}_p(\alpha, \beta)$ and $\mathcal{C}_p(\alpha, \beta)$.

2010 Mathematics Subject Classification
Primary 30C45

Keywords: Reciprocal class, Subordination, Schwarz function, Robertson function, Miller and Mocanu lemma

1 Introduction

Let $\mathcal{A}_p$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

For $f(z) \in \mathcal{A}_p$, we say that $f(z)$ belongs to the class $\mathcal{S}_p(\alpha, \beta)$ if it satisfies

$$\operatorname{Re}\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) < \beta \qquad (z \in \mathbb{U}) \tag{1.2}$$

for some real $\alpha \left(|\alpha| < \frac{\pi}{2}\right)$ and β $(\beta > p\cos\alpha)$.

When $\alpha = 0$, the class $\mathcal{S}_p(0, \beta)$ was studied by Polatoglu et al. [1], and the classes $\mathcal{S}_1(0, \beta)$ and $\mathcal{C}_1(0, \beta)$ were introduced by Owa and Nishiwaki [2].

Further, let $\mathcal{C}_p(\alpha, \beta)$ denote the subclass of $\mathcal{A}_p$ consisting of functions $f(z)$, which satisfy

$$\operatorname{Re}\left\{e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} < \beta \qquad (z \in \mathbb{U}) \tag{1.3}$$

for some real $\alpha \left(|\alpha| < \frac{\pi}{2}\right)$ and β $(\beta > p\cos\alpha)$.

We note that $f(z) \in \mathcal{C}_p(\alpha, \beta)$ if and only if $\frac{zf'(z)}{p} \in \mathcal{S}_p(\alpha, \beta)$, and that, $f(z) \in \mathcal{S}_p(\alpha, \beta)$ if and only if $p\int_0^z \frac{f(t)}{t}dt \in \mathcal{C}_p(\alpha, \beta)$.

Remark 1 If $f(z) \in \mathcal{A}_p$ satisfies

$$\mathrm{Re}\left(\mathrm{e}^{i\alpha}\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}),$$

then we say that $f(z)$ is p-valently spirallike in $\mathbb{U}$ (cf. [1]). Also, if $f(z) \in \mathcal{A}_p$ satisfies

$$\mathrm{Re}\left\{\mathrm{e}^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0 \qquad (z \in \mathbb{U}),$$

then $f(z)$ is said to be p-valently Robertson function in $\mathbb{U}$ (cf. [3,4]). Therefore, $\mathcal{S}_p(\alpha, \beta)$ defined by (1.2) is the reciprocal class of p-valently spirallike functions in $\mathbb{U}$, and $\mathcal{C}_p(\alpha, \beta)$ defined by (1.3) is the reciprocal class of p-valently Robertson functions in $\mathbb{U}$.

Let $\mathcal{P}$ be the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \qquad (z \in \mathbb{U}) \tag{1.4}$$

that are analytic in $\mathbb{U}$ and satisfy $\mathrm{Re}p(z) > 0$ $(z \in \mathbb{U})$. A function $p(z) \in \mathcal{P}$ is called the Carathéodory function and satisfies

$$|c_n| \leqq 2 \qquad (n = 1, 2, 3, \ldots) \tag{1.5}$$

with the equality for $p(z) = \dfrac{1+z}{1-z}$ (cf. [5]).

For analytic functions $g(z)$ and $h(z)$ in $\mathbb{U}$, we say that $g(z)$ is subordinate to $h(z)$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1 (z \in \mathbb{U})$, and such that $g(z) = h(w(z))$. We denote this subordination by

$$g(z) \prec h(z) \qquad (z \in \mathbb{U}). \tag{1.6}$$

If $h(z)$ is univalent in $\mathbb{U}$, then this subordination (1.6) is equivalent to $g(0) = h(0)$ and $g(\mathbb{U}) \subset h(\mathbb{U})$ (cf. [5]).

2 Subordinations for classes

We consider subordination properties of function $f(z)$ in the classes $\mathcal{S}_p(\alpha, \beta)$ and $\mathcal{C}_p(\alpha, \beta)$.

Theorem 1 *A function $f(z)$ belongs to the class $\mathcal{S}_p(\alpha, \beta)$ if and only if*

$$e^{i\alpha}\frac{zf'(z)}{f(z)} \prec 2\beta - pe^{-i\alpha} + \frac{2(p\cos\alpha - \beta)}{1-z} \qquad (z \in \mathbb{U}) \tag{2.1}$$

for some real α $\left(|\alpha| < \dfrac{\pi}{2}\right)$ and β $(\beta > p \cos \alpha)$.

The result is sharp for $f(z)$ given by

$$f(z) = \frac{z^p}{(1-z)^{2e^{-i\alpha}(p\cos\alpha - \beta)}}. \tag{2.2}$$

Proof. Let $f(z) \in \mathcal{S}_p(\alpha, \beta)$. If we define the function $w(z)$ by

$$\frac{\beta - \mathrm{e}^{i\alpha}\dfrac{zf'(z)}{f(z)} + ip\sin\alpha}{\beta - p\cos\alpha} = \frac{1+w(z)}{1-w(z)} \qquad (w(z) \neq 1), \tag{2.3}$$

then we know that $w(z)$ is analytic in $\mathbb{U}$, $w(0) = 0$, and

$$\mathrm{Re}\left(\frac{1+w(z)}{1-w(z)}\right) > 0 \qquad (z \in \mathbb{U}). \tag{2.4}$$

Therefore, we have that $|w(z)| < 1 (z \in \mathbb{U})$. If follows from (2.3) that

$$\mathrm{e}^{i\alpha}\frac{zf'(z)}{f(z)} = 2\beta - p\mathrm{e}^{-i\alpha} + \frac{2(p\cos\alpha - \beta)}{1-w(z)} \qquad (z \in \mathbb{U}), \tag{2.5}$$

which is equivalent to the subordination (2.1).

Conversely, we suppose that the subordination (2.1) holds true. Then, we have that

$$\mathrm{e}^{i\alpha}\frac{zf'(z)}{f(z)} = 2\beta - p\mathrm{e}^{-i\alpha} + \frac{2(p\cos\alpha - \beta)}{1-w(z)} \qquad (z \in \mathbb{U}), \tag{2.6}$$

for some Shwarz function $w(z)$, which is analytic in $\mathbb{U}$, $w(0) = 0$, and $|w(z)| < 1 (z \in \mathbb{U})$. It is easy to see that the equality (2.6) is equivalent to the equality (2.3). Since

$$\mathrm{Re}\left(\frac{1+w(z)}{1-w(z)}\right) = \mathrm{Re}\left\{\frac{\beta - \mathrm{e}^{i\alpha}\dfrac{zf'(z)}{f(z)} + ip\sin\alpha}{\beta - p\cos\alpha}\right\} > 0 \qquad (z \in \mathbb{U}), \tag{2.7}$$

we conclude that

$$\mathrm{Re}\left(\beta - \mathrm{e}^{i\alpha}\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}), \tag{2.8}$$

which shows that $f(z) \in \mathcal{S}_p(\alpha, \beta)$.

Finally, we consider the function $f(z)$ given by (2.2). Then, $f(z)$ satisfies

$$\mathrm{e}^{i\alpha}\frac{zf'(z)}{f(z)} = 2\beta - p\mathrm{e}^{-i\alpha} + \frac{2(p\cos\alpha - \beta)}{1-z}. \tag{2.9}$$

This completes the proof of the theorem. □

Noting that $f(z) \in \mathcal{C}_p(\alpha, \beta)$ if and only if $\dfrac{zf'(z)}{p} \in \mathcal{S}_p(\alpha, \beta)$, we also have

Corollary 1 *A function $f(z)$ belongs to the class $\mathcal{C}_p(\alpha, \beta)$ if and only if*

$$e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec 2\beta - pe^{-i\alpha} + \frac{2(p\cos\alpha - \beta)}{1-z} \qquad (z \in \mathbb{U}) \tag{2.10}$$

for some real α $\left(|\alpha| < \frac{\pi}{2}\right)$ and β $(\beta > p\cos\alpha)$.

The result is sharp for $f(z)$ given by

$$f'(z) = \frac{pz^{p-1}}{(1-z)^{2e^{-i\alpha}(p\cos\alpha - \beta)}}. \tag{2.11}$$

3 Coefficient inequalities

Applying the properties for Carathéodory functions, we discuss the coefficient inequalities for $f(z)$ in the classes $\mathcal{S}_p(\alpha, \beta)$ and $\mathcal{C}_p(\alpha, \beta)$.

Theorem 2 *If f(z) belongs to the class $\mathcal{S}_p(\alpha, \beta)$, then*

$$|a_{p+k}| \leqq \frac{1}{k!}\prod_{j=0}^{k-1}(2(\beta - p\cos\alpha) + j) \qquad (k = 1, 2, 3, \ldots). \tag{3.1}$$

The result is sharp for

$$f(z) = \frac{z^p}{(1-z)^{2(p-\beta)}} \tag{3.2}$$

for $\alpha = 0$.

Proof. In view of Theorem 1, we can consider the function $w(z)$ given by (2.3) for $f(z) \in \mathcal{S}_p(\alpha, \beta)$. Since $w(z)$ is the Schwarz function, the function $q(z)$ defined by

$$q(z) = \frac{\beta - e^{i\alpha}\dfrac{zf'(z)}{f(z)} + ip\sin\alpha}{\beta - p\cos\alpha} \tag{3.3}$$

is the Carathéodory function. If we write that

$$q(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{3.4}$$

then we see that

$$|c_n| \leqq 2 \qquad (n = 1, 2, 3, \ldots)$$

and the equality holds true for $q(z) = \dfrac{1+z}{1-z}$ and its rotation. It is to be noted that the equation (3.3) is equivalent to

$$e^{i\alpha}\frac{zf'(z)}{f(z)} = \beta + ip\sin\alpha - (\beta - p\cos\alpha)q(z). \tag{3.5}$$

This gives us that

$$\begin{aligned} &e^{i\alpha}\left(pz^p + \sum_{n=p+1}^{\infty} na_n z^n\right) \\ &= \left\{pe^{i\alpha} - (\beta - p\cos\alpha)\left(\sum_{n=1}^{\infty} c_n z^n\right)\right\}\left(z^p + \sum_{n=p+1}^{\infty} a_n z^n\right), \end{aligned} \tag{3.6}$$

which implies that

$$e^{i\alpha}(n-p)a_n = -(\beta - p\cos\alpha)(c_{n-1} + a_2 c_{n-2} + \cdots + a_{n-1}c_1). \tag{3.7}$$

It follows from (3.7) that

$$|a_n| \leqq \frac{2(\beta - p\cos\alpha)}{n-p}(1 + |a_2| + |a_3| + \cdots + |a_{n-1}|). \tag{3.8}$$

If $n = p + 1$, then we have that

$$|a_{p+1}| \leqq 2(\beta - p\cos\alpha). \tag{3.9}$$

If $n = p + 2$, then we also have that

$$\begin{aligned} |a_{p+2}| &\leqq \frac{2(\beta - p\cos\alpha)}{2}(1 + |a_2|) \\ &\leqq (\beta - p\cos\alpha)(1 + 2(\beta - p\cos\alpha)). \end{aligned} \tag{3.10}$$

Thus, the coefficient inequality (3.1) is true for $n = p + 1$ and $n = p + 2$. Next, we suppose that (3.1) holds true for $n = p + 1, p + 2, p + 3, \ldots, p + k - 1$. Then

$$\begin{aligned} |a_{p+k}| &\leqq \frac{2(\beta - p\cos\alpha)}{k}(1 + |a_2| + |a_3| + \cdots + |a_{p+k-1}|) \\ &\leqq \frac{2(\beta - p\cos\alpha)}{k}\left\{1 + 2(\beta - p\cos\alpha) + \frac{2(\beta - p\cos\alpha)}{2}(1 + 2(\beta - p\cos\alpha))\right. \\ &\quad + \frac{2(\beta - p\cos\alpha)}{3}(1 + 2(\beta - p\cos\alpha))\left(1 + \frac{2(\beta - p\cos\alpha)}{2}\right) \\ &\quad \left. + \cdots + \frac{1}{(k-1)!}\prod_{j=0}^{k-2}(2(\beta - p\cos) + j)\right\} \\ &= \frac{2(\beta - p\cos\alpha)}{k}(1 + 2(\beta - p\cos\alpha))\left\{1 + \frac{2(\beta - p\cos\alpha)}{2}\right. \\ &\quad + \left(1 + \frac{2(\beta - p\cos\alpha)}{2}\right)\frac{2(\beta - p\cos\alpha)}{3} \\ &\quad \left. + \cdots + \frac{2(\beta - p\cos\alpha)}{(k-1)!}\prod_{j=1}^{k-2}(2(\beta - p\cos\alpha) + j)\right\} \\ &= \frac{1}{k!}\prod_{j=0}^{k-1}(2(\beta - p\cos\alpha) + j). \end{aligned} \tag{3.11}$$

This means that the inequality (3.1) holds true for $n = p + k$. Therefore, by the mathematical induction, we prove the coefficient inequality (3.1).

Finally, let us consider the function $f(z)$ given by (3.2). Then, $f(z)$ can be written by

$$\begin{aligned} f(z) &= z^p\left(\sum_{j=0}^{\infty}\binom{2(\beta - p)}{j}(-z)^j\right) \\ &= z^p + 2(\beta - p)z^{p+1} + \cdots + \left(\frac{1}{k!}\prod_{j=0}^{k-1}(2(\beta - p) + j)z^{p+k}\right) + \ldots. \end{aligned} \tag{3.12}$$

Thus, this function $f(z)$ satisfies the equality in (3.1). □

Corollary 2 *If $f(z)$ belongs to the class $\mathcal{C}_p(\alpha, \beta)$, then*

$$|a_{p+k}| \leqq \frac{1}{(k-1)!}\prod_{j=0}^{k-1}(2(\beta - p\cos\alpha) + j) \qquad (k = 1, 2, 3, \ldots). \tag{3.13}$$

The result is sharp for $f(z)$ defined by

$$f'(z) = \frac{pz^{p-1}}{(1-z)^{2(p-\beta)}} \tag{3.14}$$

for $\alpha = 0$.

Remark 2 We know that the extremal functions for $f(z) \in \mathcal{S}_p(\alpha, \beta)$ is $f(z)$ given by (2.2) and for $f(z) \in \mathcal{C}_p(\alpha, \beta)$ is $f(z)$ given by (2.11). But, we see that

$$|a_{p+k}| \leqq \frac{1}{k!} \prod_{j=0}^{k-1} \left| 2e^{-i\alpha}(\beta - p\cos\alpha) + j \right| \tag{3.15}$$

and

$$|a_{p+k}| \leqq \frac{1}{(k-1)!} \prod_{j=0}^{k-1} \left| 2e^{-i\alpha}(\beta - p\cos\alpha) + j \right| \tag{3.16}$$

for such functions.

Therefore, the extremal functions for $f(z) \in \mathcal{S}_p(\alpha, \beta)$ and $f(z) \in \mathcal{C}_p(\alpha, \beta)$ do not satisfy the equalities in (3.1) and (3.13), respectively.

Furthermore, if we consider $\alpha = 0$ in Theorem 2, then we obtain the corresponding result due to Polatoglu et al. [1].

4 Inequalities for the real parts

We discuss some problems of inequalities for the real parts of $\frac{zf'(z)}{f(z)}$.

Theorem 3 *If* $f(z) \in \mathcal{S}_p(\alpha, \beta)$, *then we have*

$$\frac{p\cos\alpha - (2\beta - p\cos\alpha)r}{1-r} \leqq \mathrm{Re}\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) \leqq \frac{p\cos\alpha + (2\beta - p\cos\alpha)r}{1+r} \tag{4.1}$$

for $|z| = r < 1$. *The equalities hold true for* $f(z)$ *given by* (2.2).

Proof. By virtue of Theorem 1, we consider the function $g(z)$ defined by

$$g(z) = 2\beta - pe^{-i\alpha} + \frac{2(p\cos\alpha - \beta)}{1-z} \qquad (z \in \mathbb{U}). \tag{4.2}$$

Letting $z = re^{i\theta}$ $(0 \leqq r < 1)$, we see that

$$\mathrm{Re}\, g(z) = 2\beta - p\cos\alpha + \frac{2(p\cos\alpha - \beta)(1 - r\cos\theta)}{1 + r^2 - 2r\cos\theta}. \tag{4.3}$$

Let us define

$$h(t) = \frac{1-rt}{1+r^2-2rt} \qquad (t = \cos\theta). \tag{4.4}$$

Then, we know that $h'(t) \geqq 0$. This implies that

$$2\beta - p\cos\alpha + \frac{2(p\cos\alpha - \beta)}{1-r} \leqq \mathrm{Re}\, g(z) \leqq 2\beta - p\cos\alpha + \frac{2(p\cos\alpha - \beta)}{1+r}, \tag{4.5}$$

which is equivalent to

$$\frac{p\cos\alpha - (2\beta - p\cos\alpha)r}{1-r} \leqq \mathrm{Re}\, g(z) \leqq \frac{p\cos\alpha + (2\beta - p\cos\alpha)r}{1+r}. \tag{4.6}$$

Noting that $e^{i\alpha}\frac{zf'(z)}{f(z)} \prec g(z)$ $(z \in \mathbb{U})$ by Theorem 1 and $g(z)$ is univalent in $\mathbb{U}$, we prove the inequality (4.1). Since the subordination (2.1) is sharp for $f(z)$ given by (2.2), we say that the equalities in (4.1) are attained by the function $f(z)$ given by (2.2). □

Taking $\alpha = 0$ in Theorem 3, we have

Corollary 3 *If* $f(z) \in \mathcal{S}_p(0, \beta)$*, then*

$$\frac{p - (2\beta - p)r}{1 - r} \leqq \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \leqq \frac{p + (2\beta - p)r}{1 + r} \tag{4.7}$$

for $|z| = r < 1$*. The equalities hold true for*

$$f(z) = \frac{z^p}{(1 - z)^{2(p-\beta)}}. \tag{4.8}$$

Corollary 4 *If* $f(z) \in \mathcal{C}_p(\alpha, \beta)$*, then we have*

$$\frac{p\cos\alpha - (2\beta - p\cos\alpha)r}{1 - r} \leqq \operatorname{Re}\left\{e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} \leqq \frac{p\cos\alpha + (2\beta - p\cos\alpha)r}{1 + r} \tag{4.9}$$

for $|z| = r < 1$*. The equalities hold true for f(z) defined by* (2.11).

Corollary 5 *If* $f(z) \in \mathcal{C}_p(0, \beta)$*, then*

$$\frac{p - (2\beta - p)r}{1 - r} \leqq \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \leqq \frac{p + (2\beta - p)r}{1 + r} \tag{4.10}$$

for $|z| = r < 1$*. The equalities hold true for f(z) defined by*

$$f'(z) = \frac{pz^{p-1}}{(1 - z)^{2(p-\beta)}}. \tag{4.11}$$

5 Sufficient conditions

We consider some sufficient conditions for $f(z)$ to be in the classes $\mathcal{S}_p(0, \beta)$ and $\mathcal{C}_p(0, \beta)$.

To discuss our sufficient conditions, we have to recall here the following lemma by Miller and Mocanu [6] (also due to Jack [7]).

Lemma 1 *Let* $w(z)$ *be analytic in* $\mathbb{U}$ *with* $w(0) = 0$*. If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)|, \tag{5.1}$$

then we can write

$$z_0 w'(z_0) = k w(z_0), \tag{5.2}$$

where k is real and $k \geqq 1$.

Applying Lemma 1, we derive

Theorem 4 *If* $f(z) \in \mathcal{A}_p$ *satisfies*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right) > \frac{p + \beta}{2\beta} \qquad (z \in \mathbb{U}) \tag{5.3}$$

for some real $\beta > p$*, then* $f(z) \in \mathcal{S}_p(0, \beta)$*.*

Proof. Let us define the function $w(z)$ by

$$\frac{zf'(z)}{f(z)} = \frac{p + (p - 2\beta)w(z)}{1 - w(z)} \qquad (w(z) \neq 1). \tag{5.4}$$

Then we see that $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$.

It follows from (5.4) that

$$\begin{aligned} \mathrm{Re}\left(\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right) &= \mathrm{Re}\left(1 - \frac{(p-2\beta)zw'(z)}{p+(p-2\beta)w(z)} - \frac{zw'(z)}{1-w(z)}\right) \\ &> \frac{p+\beta}{2\beta} \qquad (z \in \mathbb{U}). \end{aligned} \tag{5.5}$$

We suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, Lemma 1 gives us that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = ke^{i\theta}$. For such a point z_0, we have that

$$\begin{aligned} \mathrm{Re}\left(\frac{z_0 f'(z_0)}{f(z_0)} - \frac{z_0 f''(z_0)}{f'(z_0)}\right) &= \mathrm{Re}\left(1 - \frac{(p-2\beta)ke^{i\theta}}{p+(p-2\beta)e^{i\theta}} - \frac{ke^{i\theta}}{1-e^{i\theta}}\right) \\ &= 1 + \frac{(2\beta-p)k(p\cos\theta + p - 2\beta)}{p^2 + (p-2\beta)^2 + 2p(p-2\beta)cos\theta} + \frac{k}{2} \\ &\leqq 1 - \frac{(2\beta-p)k}{2\beta} + \frac{k}{2} \\ &= 1 - \frac{(\beta-p)k}{2\beta} \leqq \frac{p+\beta}{2\beta}. \end{aligned} \tag{5.6}$$

This contradicts our condition (5.3). Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This implies that $|w(z)| < 1 (z \in \mathbb{U})$, that is, that

$$\left| \frac{\frac{zf'(z)}{f(z)} - p}{\frac{zf'(z)}{f(z)} + (p-2\beta)} \right| < 1 \qquad (z \in \mathbb{U}). \tag{5.7}$$

Thus, we observe that $f(z) \in \mathcal{S}_p(0, \beta)$. □

Further, we derive

Theorem 5 *If* $f(z) \in \mathcal{S}_p(0, \beta)$ *for some real* $\beta \geqq p + \frac{1}{2}$, *then*

$$\mathrm{Re}\left(\frac{z^p}{f(z)}\right) > \frac{1}{2\beta - 2p + 1} \qquad (z \in \mathbb{U}). \tag{5.8}$$

Proof. We consider the function $w(z)$ such that

$$\frac{z^p}{f(z)} = \frac{1 + (1-2\gamma)w(z)}{1 - w(z)} \qquad (w(z) \neq 1) \tag{5.9}$$

for $\gamma = \frac{1}{2\beta - 2p + 1}$ and for $f(z) \in \mathcal{S}_p(0, \beta)$.

Then, we know that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) = \mathrm{Re}\left(p - \frac{(1-2\gamma)zw'(z)}{1+(1-2\gamma)w(z)} - \frac{zw'(z)}{1-w(z)}\right) < \beta \tag{5.10}$$

for $z \in \mathbb{U}$.

Since $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$, we suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, applying Lemma 1, we can write that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = ke^{i\theta}$ $(k \geqq 1)$. This gives us that

$$\begin{aligned} \mathrm{Re}\left(\frac{z_0 f'(z_0)}{f(z_0)}\right) &= \mathrm{Re}\left(p - \frac{(1-2\gamma)ke^{i\theta}}{1+(1-2\gamma)e^{i\theta}} - \frac{ke^{i\theta}}{1-e^{i\theta}}\right) \\ &= p + \frac{(1-2\gamma)k}{2\gamma} + \frac{k}{2} \\ &\geqq p + \frac{1-\gamma}{2\gamma} = \beta, \end{aligned} \tag{5.11}$$

which contradicts the inequality (5.10). Thus, there is no point $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This means that $|w(z)| < 1 (z \in \mathbb{U})$, and that,

$$\mathrm{Re}\left(\frac{z^p}{f(z)}\right) > \frac{1}{2\beta - 2p + 1} \qquad (z \in \mathbb{U}).$$

This completes the proof of the theorem. □

Letting $\frac{zf'(z)}{p}$ instead of $f(z)$ in Theorem 5, we have

Corollary 6 *If* $f(z) \in \mathcal{C}_p(\alpha, \beta)$ *for some* $\beta \geqq p + \frac{1}{2}$, *Then*

$$\mathrm{Re}\left(\frac{pz^{p-1}}{f'(z)}\right) > \frac{1}{2\beta - 2p + 1} \qquad (z \in \mathbb{U}). \tag{5.12}$$

Finally, we consider the coefficient estimates for functions $f(z)$ to be in the classes $\mathcal{S}_p(\alpha, \beta)$ and $\mathcal{C}_p(\alpha, \beta)$.

Theorem 6 *If* $f(z) \in \mathcal{A}_p$ *satisfies*

$$\sum_{n=p+1}^{\infty} \left(|ne^{i\alpha} - k| + |ne^{i\alpha} - (2\beta - k)|\right)|a_n| \leqq |pe^{i\alpha} - (2\beta - k)| - |pe^{i\alpha} - k| \tag{5.13}$$

for some real $\alpha \left(|\alpha| < \frac{\pi}{2}\right)$, β $(\beta > p \cos \alpha)$, *and* k $(0 \leqq k \leqq p \cos \alpha)$, *then* $f(z) \in \mathcal{S}_p(\alpha, \beta)$

Proof. It is to be noted that if $f(z) \in \mathcal{A}_p$ satisfies

$$\left|\frac{e^{i\alpha}\frac{zf'(z)}{f(z)} - k}{e^{i\alpha}\frac{zf'(z)}{f(z)} - (2\beta - k)}\right| < 1 \qquad (z \in \mathbb{U}), \tag{5.14}$$

Then $f(z) \in \mathcal{S}_p(\alpha, \beta)$. It follows that

$$\left|\frac{e^{i\alpha}\frac{zf'(z)}{f(z)} - k}{e^{i\alpha}\frac{zf'(z)}{f(z)} - (2\beta - k)}\right| = \left|\frac{e^{i\alpha}zf'(z) - kf(z)}{e^{i\alpha} - (2\beta - k)f(z)}\right|$$

$$< \frac{|pe^{i\alpha} - k| + \sum_{n=p+1}^{\infty} |ne^{i\alpha} - k||a_n|}{|pe^{i\alpha} - (2\beta - k)| - \sum_{n=p+1}^{\infty} |ne^{i\alpha} - (2\beta - k)||a_n|}.$$

Therefore, if $f(z)$ satisfies the coefficient estimate (5.13), then we know that $f(z)$ satisfies the inequality (5.14). This completes the proof of the theorem. □

Letting $\alpha = 0$ and $k = p$ in Theorem 6, we have

Corollary 7 *If* $f(z) \in \mathcal{A}_p$ *satisfies*

$$\sum_{n=p+1}^{\infty} (n-\beta)|a_n| \leqq (\beta - p)$$

for some real $\beta \left(p < \beta < p + \frac{1}{2}\right)$, *then* $f(z) \in \mathcal{S}_p(0,\beta)$.

Further, we have

Theorem 7 *If* $f(z) \in \mathcal{A}_p$*satisfies*

$$\sum_{n=p+1}^{\infty} n\left(|ne^{i\alpha} - k| + |ne^{i\alpha} - (2\beta - k)|\right)|a_n| \leqq p\left(|pe^{i\alpha} - (2\beta - k)| - |pe^{i\alpha} - k|\right)$$

for some real $\alpha \left(|\alpha| < \frac{\pi}{2}\right)$, β $(\beta > p \cos\alpha)$ *and* k $(0 \leqq k \leqq p\cos\alpha)$, *then* $f(z) \in \mathcal{C}_p(\alpha,\beta)$

Corollary 8 *If* $f(z) \in \mathcal{A}_p$*satisfies*

$$\sum_{n=p+1}^{\infty} n(n-\beta)|a_n| \leqq p(\beta - p)$$

for some real $\beta \left(p < \beta < p + \frac{1}{2}\right)$, *then* $f(z) \in \mathcal{C}_p(\alpha,\beta)$.

Acknowledgements

This paper was completed when the first author was visiting Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan, between February 17 and February 26, 2011.

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Received: 10 April 2011 Accepted: 18 September 2011 Published: 18 September 2011

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doi:10.1186/1029-242X-2011-61
Cite this article as: Uyanik *et al.*: Reciprocal classes of *p*-valently spirallike and *p*-valently Robertson functions. *Journal of Inequalities and Applications* 2011 **2011**:61.