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Some new identities on the twisted carlitz's *q*-bernoulli numbers and *q*-bernstein polynomials

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Abstract

In this paper, we consider the twisted Carlitz's q-Bernoulli numbers using p-adic q-integral on $\mathbb{Z}_p$. From the construction of the twisted Carlitz's q-Bernoulli numbers, we investigate some properties for the twisted Carlitz's q-Bernoulli numbers. Finally, we give some relations between the twisted Carlitz's q-Bernoulli numbers and q-Bernstein polynomials.

Keywords: q-Bernoulli numbers, p-adic q-integral, twisted

1. Introduction and preliminaries

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers, and let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$. In this paper, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. The q-number is defined by $[x]_q = \dfrac{1-q^x}{1-q}$. Note that $\lim_{q \to 1} [x]_q = x$.

We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotient $F_f(x, y) = \frac{f(x)-f(y)}{x-y}$ has a limit $f'(a)$ as $(x, y) \to (a, a)$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$, which is called the q-Volkenborn integral, is defined by Kim as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \quad (\text{see [1]}). \tag{1}$$

In [2], Carlitz defined q-Bernoulli numbers, which are called the Carlitz's q-Bernoulli numbers, by

$$\beta_{0,q} = 1, \quad \text{and} \quad q(q\beta+1)^n - \beta_{n,q} = \begin{cases} 1 \text{ if } & n = 1, \\ 0 \text{ if } & n > 1, \end{cases} \tag{2}$$

with the usual convention about replacing β^n by $\beta_{n,q}$.

In [2,3], Carlitz also considered the expansion of q-Bernoulli numbers as follows:

$$\beta_{0,q}^{(h)} = \frac{h}{[h]_q}, \quad \text{and} \quad q^h(q\beta^{(h)}+1)^n - \beta_{n,q}^{(h)} = \begin{cases} 1 \text{ if } & n = 1, \\ 0 \text{ if } & n > 1, \end{cases} \tag{3}$$

with the usual convention about replacing $(\beta^{(h)})^n$ by $\beta_{n,q}^{(h)}$.

Let $C_{p^n} = \{\xi \mid \xi^{p^n} = 1\}$ be the cyclic group of order p^n, and let $T_p = \lim_{n\to\infty} C_{p^n} = C_{p^\infty} = \bigcup_{n\geq 0} C_{p^n}$ (see [1-16]). Note that T_p is a locally constant space.

For $\xi \in T_p$, the twisted q-Bernoulli numbers are defined by

$$\frac{t}{\xi e^t - 1} = e^{B_\xi t} = \sum_{n=0}^{\infty} B_{n,\xi} \frac{t^n}{n!}, \tag{4}$$

(see [1-19]). From (4), we note that

$$B_{0,q} = 0, \quad \text{and} \quad \xi (B_\xi + 1)^n - B_{n,\xi} = \begin{cases} 1 \text{ if } & n = 1, \\ 0 \text{ if } & n > 1, \end{cases} \tag{5}$$

with the usual convention about replacing B_ξ^n by $B_{n,\xi}$ (see [17-19]). Recently, several authors have studied the twisted Bernoulli numbers and q-Bernoulli numbers in the area of number theory(see [17-19]).

In the viewpoint of (5), it seems to be interesting to investigate the twisted properties of (3). Using p-adic q-integral equation on $\mathbb{Z}_p$, we investigate the properties of the twisted q-Bernoulli numbers and polynomials related to q-Bernstein polynomials. From these properties, we derive some new identities for the twisted q-Bernoulli numbers and polynomials. Final purpose of this paper is to give some relations between the twisted Carlitz's q-Bernoulli numbers and q-Bernstein polynomials.

2. On the twisted Carlitz 's *q*-Bernoulli numbers

In this section, we assume that $n \in \mathbb{Z}_+$, $\xi \in T_p$ and $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$.

Let us consider the nth twisted Carlitz's q-Bernoulli polynomials using p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned} \beta_{n,\xi,q}(x) &= \int_{\mathbb{Z}_p} [y + x]_q^n \xi^y d\mu_q(y) \\ &= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \int_{\mathbb{Z}_p} \xi^y q^{ly} d\mu_q(y) \\ &= \frac{1}{(1-q)^{n-1}} \sum_{l=0}^{n} \binom{n}{l} \left(\frac{l+1}{1-\xi q^{l+1}}\right) (-1)^l q^{lx}. \end{aligned} \tag{6}$$

In the special case, $x = 0$, $\beta_{n,\xi,q}(0) = \beta_{n,\xi,q}$ are called the nth twisted Carlitz's q-Bernoulli numbers.

From (6), we note that

$$\begin{aligned} \beta_{n,\xi,q}(x) &= \frac{1}{(1-q)^{n-1}} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l q^{lx} \left(\frac{1}{1-\xi q^{l+1}}\right) \\ &\quad + \frac{1}{(1-q)^{n-1}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \left(\frac{1}{1-\xi q^{l+1}}\right) \\ &= -n \sum_{m=0}^{\infty} \xi^m q^{2m+x} [x+m]_q^{n-1} + \sum_{m=0}^{\infty} \xi^m q^m (1-q) [x+m]_q^n. \end{aligned} \tag{7}$$

Therefore, by (7), we obtain the following theorem.

Theorem 1. *For* $n \in \mathbb{Z}_+$, *we have*

$$\beta_{n,\xi,q}(x) = -n\sum_{m=0}^{\infty}\xi^m q^m [x+m]_q^{n-1} + (1-q)(n+1)\sum_{m=0}^{\infty}\xi^m q^m [x+m]_q^n.$$

Let $F_{q,\ \xi}\ (t,\ x)$ be the generating function of the twisted Carlitz's q-Bernoulli polynomials, which are given by

$$F_{q,\xi}(t,x) = e^{\beta_{\xi,q}(x)t} = \sum_{n=0}^{\infty}\beta_{n,\xi,q}(x)\frac{t^n}{n!}, \tag{8}$$

with the usual convention about replacing $(\beta_{\xi,q}(x))^n$ by $\beta_{n,\xi,q}(x)$.

By (8) and Theorem 1, we get

$$\begin{aligned} F_{q,\xi}(t,x) &= \sum_{n=0}^{\infty}\beta_{n,\xi,q}(x)\frac{t^n}{n!} \\ &= -t\sum_{m=0}^{\infty}\xi^m q^{2m+x}e^{[x+m]_q t} + (1-q)\sum_{m=0}^{\infty}\xi^m q^m e^{[x+m]_q t}. \end{aligned} \tag{9}$$

Let $F_{q,\xi}(t,\ 0) = F_{q,\xi}(t)$. Then, we have

$$q\xi F_{q,\xi}(t,1) - F_{q,\xi}(t) = t + (q-1). \tag{10}$$

Therefore, by (9) and (10), we obtain the following theorem.

Theorem 2. *For* $n \in \mathbb{Z}_+$, *we have*

$$\beta_{0,\xi,q}(x) = \frac{q-1}{q\xi-1}, \quad and \quad q\xi\beta_{n,\xi,q}(1) - \beta_{n,\xi,q} = \begin{cases} 1 \text{ if} & n=1, \\ 0 \text{ if} & n>1. \end{cases}$$

From (6), we note that

$$\begin{aligned} \beta_{n,\xi,q}(x) &= \sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{lx}\int_{\mathbb{Z}_p}\xi^y[y]_q^l d\mu_q(y) \\ &= \sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{lx}\beta_{l,\xi,q} \\ &= \left([x]_q + q^x\beta_{\xi,q}\right)^n, \end{aligned} \tag{11}$$

with the usual convention about replacing $(\beta_{\xi,q})^n$ by $\beta_{n,\xi,q}$. By (11) and Theorem 2, we get

$$q\xi(q\beta_{\xi,q}+1)^n - \beta_{n,\xi,q} = \begin{cases} q-1 \text{ if} & n=0, \\ 1 \quad\ \text{if} & n=1, \\ 0 \quad\ \text{if} & n>1. \end{cases} \tag{12}$$

It is easy to show that

$$\begin{aligned} \beta_{n,\xi^{-1},q^{-1}}(1-x) &= \int_{\mathbb{Z}_p}\xi^{-y}[1-x+y]_{q^{-1}}^n d\mu_{q^{-1}}(y) \\ &= \frac{(-1)^n q^n}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{-l+lx}\int_{\mathbb{Z}_p}\xi^{-y}q^{-ly}d\mu_{q^{-1}}(y) \\ &= \xi q^n(-1)^n\left(\frac{1}{(1-q)^{n-1}}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\left(\frac{l+1}{1-\xi q^{l+1}}\right)\right) \\ &= \xi q^n(-1)^n\beta_{n,\xi,q}(x). \end{aligned} \tag{13}$$

Therefore, by (13), we obtain the following theorem.

Theorem 3. *For $n \in \mathbb{Z}_+$, we have*

$$\beta_{n,\xi^{-1},q^{-1}}(1-x) = \xi q^n (-1)^n \beta_{n,\xi,q}(x).$$

From Theorem 3, we can derive the following functional equation:

$$F_{q^{-1},\xi^{-1}}(t, 1-x) = \xi F_{q,\xi}(-qt, x). \tag{14}$$

Therefore, by (14), we obtain the following corollary.

Corollary 4. *Let $F_{q,\xi}(t,x) = \sum_{n=0}^{\infty} \beta_{n,\xi,q}(x)\frac{t^n}{n!}$. Then we have*

$$F_{q^{-1},\xi^{-1}}(t, 1-x) = \xi F_{q,\xi}(-qt, x).$$

By (11), we get that

$$\begin{aligned} q^2\xi^2\beta_{n,\xi,q}(2) &= q^2\xi^2 \sum_{l=0}^{n} \binom{n}{l} q^l (1+q\beta_{\xi,q})^l \\ &= q^2\xi^2\left(\frac{1-q}{1-q\xi}\right) + \binom{n}{1} q^2\xi(1+\beta_{1,\xi,q}) + q^2\xi^2 \sum_{l=0}^{n} \binom{n}{l} q^l \beta_{l,\xi,q}(1) \\ &= (1-q)\frac{q^2\xi^2}{1-q\xi} + \binom{n}{1} q^2\xi + q\xi \sum_{l=0}^{n} \binom{n}{l} q^l \beta_{l,\xi,q} \\ &= \frac{1-q}{1-q\xi} q^2\xi^2 + nq^2\xi - q\xi\frac{1-q}{1-q\xi} + \beta_{n,\xi,q}, \quad \text{if} \quad n > 1. \end{aligned} \tag{15}$$

Therefore, by (15), we obtain the following theorem.

Theorem 5. *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$\beta_{n,\xi,q}(2) = \frac{1-q}{1-q\xi} + \frac{n}{\xi} - \frac{1}{q\xi}\left(\frac{1-q}{1-q\xi}\right) + \left(\frac{1}{q\xi}\right)^2 \beta_{n,\xi,q}.$$

By a simple calculation, we easily set

$$\begin{aligned} \xi \int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^n \xi^x d\mu_q(x) &= \xi(-1)^n q^n \int_{\mathbb{Z}_p} [x-1]_q^n \xi^x d\mu_q(x) \\ &= \xi(-1)^n q^n \beta_{n,\xi,q}(-1) = \beta_{n,\xi^{-1},q^{-1}}(2). \end{aligned} \tag{16}$$

For $n \in \mathbb{Z}_+$ with $n > 1$, we have

$$\begin{aligned} \xi \int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^n \xi^x d\mu_q(x) &= \beta_{n,\xi^{-1},q^{-1}}(2) \\ &= \xi\left(\frac{1-q}{1-q\xi}\right) + n\xi - q\xi^2\left(\frac{1-q}{1-q\xi}\right) + (q\xi)^2 \beta_{n,\xi^{-1},q^{-1}} \\ &= \xi(1-q) + n\xi + (q\xi)^2 \beta_{n,\xi^{-1},q^{-1}}. \end{aligned} \tag{17}$$

Therefore, by (16) and (17), we obtain the following theorem.

Theorem 6. *For $n \in \mathbb{Z}_+$ with* n > 1, *we have*

$$\int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^n \xi^x d\mu_q(x) = (1-q) + n + q^2\xi\beta_{n,\xi^{-1},q^{-1}}.$$

For $x \in \mathbb{Z}_p$ and $n, k \in \mathbb{Z}_+$, the p-adic q-Bernstein polynomials are given by

$$B_{k,n}(x,q) = \binom{n}{k} [x]_q^k [1-x]_{q^{-1}}^{n-k}, \tag{18}$$

(see [8,20]).

In [8], the q-Bernstein operator of order n is given by

$$\mathbb{B}_{n,q}(f|x) = \sum_{k=0}^{n} f(\frac{n}{k}) B_{k,n}(x,q) = \sum_{k=0}^{n} f(\frac{n}{k}) \binom{n}{k} [x]_q^k [1-x]_{q^{-1}}^{n-k}.$$

Let f be continuous function on $\mathbb{Z}_p$. Then, the sequence $\mathbb{B}_{n,q}(f|x)$ converges uniformly to f on $\mathbb{Z}_p$ (see [8]). If q is same version in (18), we cannot say that the sequence $\mathbb{B}_{n,q}(f|x)$ converges uniformly to f on $\mathbb{Z}_p$.

Let $s \in \mathbb{N}$ with $s \geq 2$. For $n_1, ..., n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk + 1$, we take the p-adic q-integral on $\mathbb{Z}_p$ for the multiple product of q-Bernstein polynomials as follows:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \xi^x B_{k,n_1}(x,q) \cdots B_{k,n_s}(x,q) d\mu_q(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \int_{\mathbb{Z}_p} [x]_q^k [1-x]_{q^{-1}}^{n_1+\cdots+n_s-sk} \xi^x d\mu_q(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^{n_1+\cdots+n_s-l} \xi^x d\mu_q(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \\
&\quad \times (q^2 \xi \beta_{n_1+\cdots+n_s-l, \xi^{-1}, q^{-1}} + n_1 + \cdots + n_s - l + 1 - q) d\mu_q(x) \\
&= \begin{cases} q^2 \xi \beta_{n_1+\cdots+n_s, \xi^{-1}, q^{-1}} + n_1 + \cdots + n_s + (1-q) & \text{if } k = 0, \\ q^2 \xi \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \beta_{n_1+\cdots+n_s-l, \xi^{-1}. q^{-1}} & \text{if } k > 0, \end{cases}
\end{aligned} \tag{19}$$

and we also have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \xi^x B_{k,n_1}(x,q) \cdots B_{k,n_s}(x,q) d\mu_q(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1 + \cdots + n_s - sk}{l} (-1)^l \beta_{l+sk, \xi, q}.
\end{aligned} \tag{20}$$

By comparing the coefficients on the both sides of (19) and (20), we obtain the following theorem.

Theorem 7. *Let* $s \in \mathbb{N}$ *with* $s \geq 2$. *For* $n_1, ..., n_s, k \in \mathbb{Z}_+$ *with* $n_1 + ... + n_s > sk + 1$, *we have*

$$\begin{aligned}
&\sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1 + \cdots + n_s - sk}{l} (-1)^l \beta_{l+sk, \xi, q} \\
&= \begin{cases} q^2 \xi \beta_{n_1+\cdots+n_s, \xi^{-1}, q^{-1}} + n_1 + \cdots + n_s + (1-q) & \text{if } k = 0, \\ q^2 \xi \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \beta_{n_1+\cdots+n_s-l, \xi^{-1}. q^{-1}} & \text{if } k > 0. \end{cases}
\end{aligned}$$

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Competing interests
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