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An improved Hardy type inequality on Heisenberg group

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Abstract

Motivated by the work of Ghoussoub and Moradifam, we prove some improved Hardy inequalities on the Heisenberg group $\mathbb{H}^n$ via Bessel function.

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1 Introduction

Hardy inequality in $\mathbb{R}^N$ reads, for all $u \in C_0^\infty(\mathbb{R}^N)$ and $N \geq 3$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \mathrm{d}x \tag{1.1}$$

and $\frac{(N-2)^2}{4}$ is the best constant in (1.1) and is never achieved. A similar inequality with the same best constant holds in $\mathbb{R}^N$ is replaced by an arbitrary domain $\Omega \subset \mathbb{R}^N$ and Ω contains the origin. Moreover, in case $\Omega \subset \mathbb{R}^N$ is a bounded domain, Brezis and Vázquez [1] have improved it by establishing that for $u \in C_0^\infty(\Omega)$,

$$\int_{\Omega} |\nabla u|^2 \mathrm{d}x \geq \frac{(N-2)^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} \mathrm{d}x + z_0^2 \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \int_{\Omega} u^2 \mathrm{d}x, \tag{1.2}$$

where ω_N and $|\Omega|$ denote the volume of the unit ball and Ω, respectively, and $z_0 = 2.4048...$ denotes the first zero of the Bessel function $J_0(z)$. Inequality (1.2) is optimal in case Ω is a ball centered at zero. Triggered by the work of Brezis and Vázquez (1.2), several Hardy inequalities have been established in recent years. In particular, Adimurthi et al.([2]) proved that, for $u \in C_0^\infty(\Omega)$, there exists a constant $C_{n,k}$ such that

$$\int_{\Omega} |\nabla u|^2 \mathrm{d}x \geq \frac{(N-2)^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} \mathrm{d}x + C_{n,k} \sum_{j=1}^{k} \int_{\Omega} \frac{u^2}{|x|^2} \left(\prod_{i=1}^{j} \log^{(i)} \frac{\rho}{|x|}\right)^{-2} \mathrm{d}x, \tag{1.3}$$

where

$$\rho = \left(\sup_{x\in\Omega} |x|\right) \left(e^{e^{\cdot^{e(k-times)}}}\right),$$

$\log^{(1)}(.) = \log(.)$ and $\log^{(k)}(.) = \log(\log^{(k-1)}(.))$ for $k \geq 2$. Filippas and Tertikas ([3]) proved that, for $u \in C_0^\infty(\Omega)$, there holds

$$\int_\Omega |\nabla u|^2 \geq \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} + \frac{1}{4} \sum_{k=1}^{\infty} \int_\Omega \frac{u^2}{|x|^2} X_1^2\left(\frac{|x|}{D}\right) \cdot \ldots \cdot X_k^2\left(\frac{|x|}{D}\right), \tag{1.4}$$

where $D \geq \sup_{x \in \Omega} |x|$,

$$X_1(s) = (1 - \ln s)^{-1}, \quad X_k(s) = X_1(X_{k-1}(t))$$

for $k \geq 2$ and $\frac{1}{4}$ is the best constant in (1.4) and is never achieved. More recently, Ghoussoub and Moradifam ([4]) give a necessary and sufficient condition on a radially symmetric potential $V(|x|)$ on Ω that makes it an admissible candidate for an improved Hardy inequality. It states that the following improved Hardy inequality holds for $u \in C_0^\infty(B_\rho)$, where $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$,

$$\int_\Omega |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx + \int_\Omega \frac{u^2}{|x|^2} V(|x|) dx \tag{1.5}$$

if and only if the ordinary differential equation

$$y''(r) + \frac{y'(r)}{r} + V(r) y(r) = 0$$

has a positive solution on $(0, \rho]$. These include inequalities (1.2)-(1.4).

Motivated by the work of Ghoussoub and Moradifam ([4]), in this note, we shall prove similar improved Hardy inequality on the Heisenberg group $\mathbb{H}^n$. Recall that the Heisenberg group $\mathbb{H}^n$ is the Carnot group of step two whose group structure is given by

$$(x, t) \circ (x', t') = \left(x + x', t + t' + 2 \sum_{j=1}^{n} \left(x_{2j} x'_{2j-1} - x_{2j-1} x'_{2j} \right) \right).$$

The vector fields

$$X_{2j-1} = \frac{\partial}{\partial x_{2j-1}} + 2x_{2j} \frac{\partial}{\partial t},$$
$$X_{2j} = \frac{\partial}{\partial x_{2j}} - 2x_{2j-1} \frac{\partial}{\partial t},$$

$(j = 1, \ldots, n)$ are left invariant and generate the Lie algebra of $\mathbb{H}^n$. The horizontal gradient on $\mathbb{H}^n$ is the $(2n)$ -dimensional vector given by

$$\nabla_{\mathbb{H}} = (X_1, \ldots, X_{2n}) = \nabla_x + 2\Lambda x \frac{\partial}{\partial t},$$

where $\nabla_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{2n}} \right)$, Λ is a skew symmetric and orthogonal matrix given by

$$\Lambda = \mathrm{diag}(J_1, \ldots, J_n), \quad J_1 = \cdots = J_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For more information about $\mathbb{H}^n$, we refer to [5-8]. To this end we have:

Theorem 1.1

Let $B_R = \{x \in \mathbb{R}^{2n} : |x| < R\}$ *and* $\Omega_H = B_R \times \mathbb{R} \in \mathbb{H}^n$. *Let* $V(|x|)$ *be a radially symmetric decreasing nonnegative function on* B_R. *If the ordinary differential equation*

$$y''(r) + \frac{y'(r)}{r} + V(r)y(r) = 0$$

has a positive solution on $(0, R]$, *then the following improved Hardy inequality holds for* $u \in C_0^\infty(\Omega_H)$

$$\int_{\Omega_H} |\nabla_{\mathbb{H}} u|^2 \mathrm{d}x\mathrm{d}t \geq (n-1)^2 \int_{\Omega_H} \frac{u^2}{|x|^2} \mathrm{d}x\mathrm{d}t + \int_{\Omega_H} \frac{u^2}{|x|^2} V(|x|) \mathrm{d}x\mathrm{d}t \tag{1.6}$$

and the constant $(n - 1)^2$ *in* (1.6) *is sharp in the sense of*

$$(n-1)^2 = \inf_{u \in C_0^\infty(\Omega_H)\setminus\{0\}} \frac{\int_{\Omega_H} |\nabla_{\mathbb{H}} u|^2 \mathrm{d}x\mathrm{d}t}{\int_{\Omega_H} \frac{u^2}{|x|^2} \mathrm{d}x\mathrm{d}t}.$$

Corollary 1.2

There holds, for $u \in C_0^\infty(\Omega_H)$,

$$\int_{\Omega_H} |\nabla_{\mathbb{H}} u|^2 \geq (n-1)^2 \int_{\Omega_H} \frac{u^2}{|x|^2} + \frac{1}{4} \sum_{j=1}^{k} \int_{\Omega_H} \frac{u^2}{|x|^2} \left(\prod_{i=1}^{j} \log^{(i)} \frac{R}{|x|} \right)^{-2} \tag{1.7}$$

and the constant $1/4$ *is sharp in the sense of*

$$\frac{1}{4} = \inf_{u \in C_0^\infty(\Omega_H)\setminus\{0\}} \frac{\int_{\Omega_H} |\nabla_{\mathbb{H}} u|^2 - (n-1)^2 \int_{\Omega_H} \frac{u^2}{|x|^2} - \frac{1}{4} \sum_{j=1}^{k-1} \int_{\Omega_H} \frac{u^2}{|x|^2} \left(\prod_{i=1}^{j} \log^{(i)} \frac{R}{|x|} \right)^{-2}}{\int_{\Omega_H} \frac{u^2}{|x|^2} \left(\prod_{i=1}^{k} \log^{(i)} \frac{R}{|x|} \right)^{-2}}.$$

Corollary 1.3

There holds, for $u \in C_0^\infty(\Omega_H)$ *and* $D \geq R$,

$$\int_{\Omega_H} |\nabla_{\mathbb{H}} u|^2 \geq (n-1)^2 \int_{\Omega_H} \frac{u^2}{|x|^2} + \frac{1}{4} \sum_{k=1}^{\infty} \int_{\Omega_H} \frac{u^2}{|x|^2} X_1^2 \left(\frac{|x|}{D} \right) \cdot \cdots \cdot X_k^2 \left(\frac{|x|}{D} \right), \tag{1.8}$$

and the constant $1/4$ *is sharp in the sense of*

$$\frac{1}{4} = \inf_{u \in C_0^\infty(\Omega_H)\setminus\{0\}} \frac{\int_{\Omega_H} |\nabla_{\mathbb{H}} u|^2 - (n-1)^2 \int_{\Omega_H} \frac{u^2}{|x|^2} - \frac{1}{4} \sum_{j=1}^{k-1} \int_{\Omega_H} \frac{u^2}{|x|^2} X_1^2 \left(\frac{|x|}{D} \right) \cdots X_j^2 \left(\frac{|x|}{D} \right)}{\int_{\Omega_H} \frac{u^2}{|x|^2} X_1^2 \left(\frac{|x|}{D} \right) \cdot \cdots \cdot X_k^2 \left(\frac{|x|}{D} \right)}.$$

2 Proof

To prove the main result, we first need the following preliminary result.

Lemma 2.1

Let $B_R = \{x \in \mathbb{R}^{2n} : |x| < R\}$ and $V(|x|)$ be a radially symmetric decreasing nonnegative function on B_R. If the ordinary differential equation

$$y''(r) + \frac{y'(r)}{r} + V(r)y(r) = 0$$

has a positive solution on $(0, R]$, then the following improved Hardy inequality holds for $f \in C_0^\infty(B_R)$,

$$\int_{B_R} |\partial_r f|^2 \mathrm{d}x \geq (n-1)^2 \int_{B_R} \frac{f^2}{|x|^2}\mathrm{d}x + \int_{B_R} \frac{f^2}{|x|^2} V(|x|)\mathrm{d}x, \tag{2.1}$$

where $r = |x|$ and $\partial_r = \dfrac{\langle x, \nabla\rangle}{|x|}$ is the radial derivation.

Proof

Observe that if f is radial, i.e., $f(x) = f(r)$, then $|\nabla f| = |\partial_r f|$. By inequality (1.5), inequality (2.1) holds.

Now let $f \in C_0^\infty(B_R)$. If we extend f as zero outside B_R, we may consider $f \in C_0^\infty(\mathbb{R}^{2n})$. Decomposing f into spherical harmonics we get (see e.g., [9])

$$f = \sum_{k=0}^{\infty} f_k(r)\phi_k(\sigma),$$

where $\varphi_k(\sigma)$ are the orthonormal eigenfunctions of the Laplace-Beltrami operator with responding eigenvalues

$$c_k = k(N + k - 2), \quad k \geq 0.$$

The functions $f_k(r)$ belong to $C_0^\infty(B_R)$, satisfying $f_k(r) = O(r^k)$ and $f_k'(r) = O(r^{k-1})$ as $r \to 0$. So

$$\int_{B_R} |\partial_r f|^2 \mathrm{d}x = \sum_{k=0}^{\infty} \int_{B_R} |f_k'|^2 \mathrm{d}x \tag{2.2}$$

and

$$(n-1)^2 \int_{B_R} \frac{f^2}{|x|^2}\mathrm{d}x + \int_{B_R} \frac{f^2}{|x|^2} V(|x|)\mathrm{d}x = \sum_{k=0}^{\infty} \left((n-1)^2 \int_{B_R} \frac{f_k^2}{|x|^2}\mathrm{d}x + \int_{B_R} \frac{f_k^2}{|x|^2} V(|x|)\mathrm{d}x \right). \tag{2.3}$$

Note that if f is radial, then inequality (2.1) holds. We have, since $f_k(r) \in C_0^\infty(B_R)$,

$$\int_{B_R} |f_k'|^2 \mathrm{d}x \geq (n-1)^2 \int_{B_R} \frac{f_k^2}{|x|^2}\mathrm{d}x + \int_{B_R} \frac{f_k^2}{|x|^2} V(|x|)\mathrm{d}x.$$

Therefore, by (2.2) and (2.3),

$$\begin{aligned}\int_{B_R} |\partial_r f|^2 \mathrm{d}x &= \sum_{k=0}^{\infty} \int_{B_R} |f'_k|^2 \mathrm{d}x \\ &\geq \sum_{k=0}^{\infty} \left((n-1)^2 \int_{B_R} \frac{f_k^2}{|x|^2} \mathrm{d}x + \int_{B_R} \frac{f_k^2}{|x|^2} V(|x|) \mathrm{d}x \right) \\ &= (n-1)^2 \int_{B_R} \frac{f^2}{|x|^2} \mathrm{d}x + \int_{B_R} \frac{f^2}{|x|^2} V(|x|) \mathrm{d}x.\end{aligned}$$

This completes the proof of lemma 2.1.

Proof of Theorem 1.1

Recall that the horizontal gradient on $\mathbb{H}^n$ is the $(2n)$-dimensional vector given by

$$\nabla_{\mathbb{H}} = (X_1, \ldots, X_{2n}) = \nabla_x + 2\Lambda x \frac{\partial}{\partial t},$$

where $\nabla_x = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{2n}})$, Λ is a skew symmetric and orthogonal matrix given by

$$\Lambda = \operatorname{diag}(J_1, \ldots, J_n), \quad J_1 = \cdots = J_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Therefore, for any $\phi \in C_0^\infty(\mathbb{H}^n)$,

$$\begin{aligned}\langle x, \nabla_{\mathbb{H}}\phi \rangle &= \langle x, \nabla_x \phi \rangle + 2\langle x, \Lambda x \rangle \frac{\partial \phi}{\partial t} \\ &= \langle x, \nabla_x \phi \rangle.\end{aligned} \tag{2.4}$$

Here we use the fact $\langle x, \Lambda x \rangle = 0$ since Λ is a skew symmetric matrix.
Since $u \in C_0^\infty(\Omega_H)$, for every $t \in \mathbb{R}, u(\cdot, t) \in C_0^\infty(B_R)$. By Lemma 2.1,

$$\int_{B_R} |\partial_r u|^2 \mathrm{d}x \geq (n-1)^2 \int_{B_R} \frac{u^2}{|x|^2} \mathrm{d}x + \int_{B_R} \frac{u^2}{|x|^2} V(|x|) \mathrm{d}x \tag{2.5}$$

Integrating both sides of the inequality (2.5) with respect to t, we have,

$$\int_{\Omega_H} |\partial_r u|^2 \mathrm{d}x\mathrm{d}t \geq (n-1)^2 \int_{\Omega_H} \frac{u^2}{|x|^2} \mathrm{d}x\mathrm{d}t + \int_{\Omega_H} \frac{u^2}{|x|^2} V(|x|) \mathrm{d}x\mathrm{d}t \tag{2.6}$$

By (2.4) and the pointwise Schwartz inequality, we have

$$|\partial_r u| = \frac{|\langle x, \nabla_x u \rangle|}{|x|} = \frac{|\langle x, \nabla_{\mathbb{H}} u \rangle|}{|x|} \leq |\nabla_{\mathbb{H}} u|.$$

Therefore, we obtain, by (2.6)

$$(n-1)^2 \int_{\Omega_H} \frac{u^2}{|x|^2} \mathrm{d}x\mathrm{d}t + \int_{\Omega_H} \frac{u^2}{|x|^2} V(|x|) \mathrm{d}x\mathrm{d}t \leq \int_{\Omega_H} |\nabla_{\mathbb{H}} u|^2 \mathrm{d}x\mathrm{d}t. \tag{2.7}$$

To see the constant $(n - 1)^2$ is sharp, we choose $u(x, t) = \varphi(|x|)w(t)$ with $\phi(|x|) \in C_0^\infty(B_R)$ and $w(t) \in C_0^\infty(\mathbb{R})$. Since φ is radial, we have

$$\begin{aligned}|\nabla_{\mathbb{H}} u(x,t)|^2 &= \langle w(t)\nabla_x\phi(|x|) + 2\phi(|x|)\Lambda x w'(t), w(t)\nabla_x\phi(|x|) + 2\phi(|x|)\Lambda x w'(t)\rangle \\ &= |\nabla_x\phi(|x|)|^2 w^2(t) + 4|\Lambda x|^2\phi^2(w'(t))^2 + 4\langle\nabla_x\phi(|x|), \Lambda x\rangle\phi(|x|)w'(t) \\ &= |\nabla_x\phi(|x|)|^2 w^2(t) + 4|\Lambda x|^2\phi^2(w'(t))^2 + 4\phi'(|x|)\left\langle\frac{x}{|x|}, \Lambda x\right\rangle\phi(|x|)w'(t) \\ &= |\nabla_x\phi(|x|)|^2 w^2(t) + 4|x|^2\phi^2(|x|)(w'(t))^2.\end{aligned}$$

Here we use the fact $|\Lambda x| = |x|$ since Λ is a orthogonal matrix. Therefore,

$$\begin{aligned}\frac{\int_{\Omega_H}|\nabla_{\mathbb{H}} u|^2\mathrm{d}x\mathrm{d}t}{\int_{\Omega_H}\frac{u^2}{|x|^2}\mathrm{d}x\mathrm{d}t} &= \frac{\int_{\Omega_H}|\nabla_x\phi(|x|)|^2 w^2(t)}{\int_{\Omega_H}\frac{\phi(|x|)^2 w(t)^2}{|x|^2}} + 4\frac{\int_{\Omega_H}|x|^2\phi^2(|x|)(w'(t))^2}{\int_{\Omega_H}\frac{\phi(|x|)^2 w(t)^2}{|x|^2}} \\ &= \frac{\int_{B_R}|\nabla_x\phi(|x|)|^2\mathrm{d}x\cdot\int_{-\infty}^{+\infty} w^2(t)\mathrm{d}t}{\int_{B_R}\frac{\phi(|x|)^2}{|x|^2}\mathrm{d}x\cdot\int_{-\infty}^{+\infty} w^2(t)\mathrm{d}t} + 4\frac{\int_{B_R}|x|^2\phi^2(|x|)\mathrm{d}x\cdot\int_{-\infty}^{+\infty}(w'(t))^2}{\int_{B_R}\frac{\phi(|x|)^2}{|x|^2}\mathrm{d}x\cdot\int_{-\infty}^{+\infty} w^2(t)\mathrm{d}t} \\ &= \frac{\int_{B_R}|\nabla_x\phi(|x|)|^2\mathrm{d}x}{\int_{B_R}\frac{\phi(|x|)^2}{|x|^2}\mathrm{d}x} + 4\frac{\int_{B_R}|x|^2\phi^2(|x|)\mathrm{d}x}{\int_{B_R}\frac{\phi(|x|)^2}{|x|^2}\mathrm{d}x}\cdot\frac{\int_{-\infty}^{+\infty}(w'(t))^2}{\int_{-\infty}^{+\infty} w^2(t)\mathrm{d}t}\end{aligned}$$

Since

$$\inf_{w(t)\in C_0^\infty(\mathbb{R})\setminus\{0\}}\frac{\int_{\mathbb{R}}|w'(t)|^2\mathrm{d}t}{\int_{\mathbb{R}}|w(t)|^2\mathrm{d}t} = 0,$$

we obtain

$$\inf_{u\in C_0^\infty(\Omega_H)\setminus\{0\}}\frac{\int_{\Omega_H}|\nabla_{\mathbb{H}} u|^2\mathrm{d}x\mathrm{d}t}{\int_{\Omega_H}\frac{u^2}{|x|^2}\mathrm{d}x\mathrm{d}t} \le \inf_{\phi\in C_0^\infty(B_R)\setminus\{0\}}\frac{\int_{B_R}|\nabla_x\phi(|x|)|^2\mathrm{d}x}{\int_{B_R}\frac{\phi(|x|)^2}{|x|^2}\mathrm{d}x} = (n-1)^2.$$

The proof of Theorem 1.1 is completed.

Proof of Corollary 1.2

By Theorem 1.1 and following [4], it is enough to show the constant 1/4 is sharp. Choose $u(x, t) = \varphi(|x|)w(t)$ with $\phi(|x|) \in C_0^\infty(B_R)$ and $w(t) \in C_0^\infty(\mathbb{R})$. By the proof of Theorem 1.1,

$$|\nabla_{\mathbb{H}} u(x,t)|^2 = |\nabla_x\phi(|x|)|^2 w^2(t) + 4|x|^2\phi^2(|x|)(w'(t))^2.$$

Therefore,

$$\begin{aligned}&\frac{\int_{\Omega_H}|\nabla_{\mathbb{H}} u|^2 - (n-1)^2\int_{\Omega_H}\frac{u^2}{|x|^2} - \frac{1}{4}\sum_{j=1}^{k-1}\int_{\Omega_H}\frac{u^2}{|x|^2}\left(\prod_{i=1}^{j}\log^{(i)}\frac{R}{|x|}\right)^{-2}}{\int_{\Omega_H}\frac{u^2}{|x|^2}\left(\prod_{i=1}^{k}\log^{(i)}\frac{R}{|x|}\right)^{-2}} \\ &= \frac{\int_{\Omega_H}|\nabla_x\phi(|x|)|^2 w^2(t) - (n-1)^2\int_{\Omega_H}\frac{\phi^2 w^2(t)}{|x|^2} - \frac{1}{4}\sum_{j=1}^{k-1}\int_{\Omega_H}\frac{\phi^2 w^2(t)}{|x|^2}\left(\prod_{i=1}^{j}\log^{(i)}\frac{R}{|x|}\right)^{-2}}{\int_{\Omega_H}\frac{\phi^2 w^2(t)}{|x|^2}\left(\prod_{i=1}^{k}\log^{(i)}\frac{R}{|x|}\right)^{-2}} \\ &\quad + 4\frac{\int_{\Omega_H}|x|^2\phi^2(|x|)(w'(t))^2}{\int_{\Omega_H}\frac{\phi^2 w^2(t)}{|x|^2}\left(\prod_{i=1}^{k}\log^{(i)}\frac{R}{|x|}\right)^{-2}} \\ &= \frac{\int_{B_R}|\nabla_x\phi(|x|)|^2 - (n-1)^2\int_{B_R}\frac{\phi^2}{|x|^2} - \frac{1}{4}\sum_{j=1}^{k-1}\int_{B_R}\frac{\phi^2}{|x|^2}\left(\prod_{i=1}^{j}\log^{(i)}\frac{R}{|x|}\right)^{-2}}{\int_{B_R}\frac{\phi^2}{|x|^2}\left(\prod_{i=1}^{k}\log^{(i)}\frac{R}{|x|}\right)^{-2}} \\ &\quad + 4\frac{\int_{B_R}|x|^2\phi^2(|x|)}{\int_{B_R}\frac{\phi^2}{|x|^2}\left(\prod_{i=1}^{k}\log^{(i)}\frac{R}{|x|}\right)^{-2}}\cdot\frac{\int_{-\infty}^{+\infty}(w'(t))^2}{\int_{-\infty}^{+\infty} w^2(t)\mathrm{d}t}.\end{aligned}$$

Since

$$\inf_{w(t)\in C_0^\infty(\mathbb{R})\setminus\{0\}} \frac{\int_{\mathbb{R}} |w'(t)|^2 \mathrm{d}t}{\int_{\mathbb{R}} |w(t)|^2 \mathrm{d}t} = 0,$$

we have

$$\begin{aligned}
&\inf_{u\in C_0^\infty(\Omega_H)\setminus\{0\}} \frac{\int_{\Omega_H} |\nabla_{\mathbb{H}} u|^2 - (n-1)^2 \int_{\Omega_H} \frac{u^2}{|x|^2} - \frac{1}{4}\sum_{j=1}^{k-1} \int_{\Omega_H} \frac{u^2}{|x|^2}\left(\prod_{i=1}^{j} \log^{(i)} \frac{R}{|x|}\right)^{-2}}{\int_{\Omega_H} \frac{u^2}{|x|^2}\left(\prod_{i=1}^{k} \log^{(i)} \frac{R}{|x|}\right)^{-2}} \\
&\le \inf_{\phi(|x|)\in C_0^\infty(B_R)\setminus\{0\}} \frac{\int_{B_R} |\nabla_x \phi(|x|)|^2 - (n-1)^2 \int_{B_R} \frac{\phi^2}{|x|^2} - \frac{1}{4}\sum_{j=1}^{k-1} \int_{B_R} \frac{\phi^2}{|x|^2}\left(\prod_{i=1}^{j} \log^{(i)} \frac{R}{|x|}\right)^{-2}}{\int_{B_R} \frac{\phi^2}{|x|^2}\left(\prod_{i=1}^{k} \log^{(i)} \frac{R}{|x|}\right)^{-2}} \\
&= \frac{1}{4}.
\end{aligned}$$

Here we use the fact that the sharp constant in inequality (1.3) is 1/4 (see [4]). This completes the proof of Corollary 1.2.

Proof of Corollary 1.3

The proof is similar to that of Corollary 1.2 and it is enough to show the constant 1/4 is sharp. Choose $u(x, t) = \varphi(|x|)w(t)$ with $\phi(|x|) \in C_0^\infty(B_R)$ and $w(t) \in C_0^\infty(\mathbb{R})$. Then

$$|\nabla_{\mathbb{H}} u(x,t)|^2 = |\nabla_x \phi(|x|)|^2 w^2(t) + 4|x|^2 \phi^2(|x|)(w'(t))^2.$$

Therefore,

$$\begin{aligned}
&\frac{\int_{\Omega_H} |\nabla_{\mathbb{H}} u|^2 - (n-1)^2 \int_{\Omega_H} \frac{u^2}{|x|^2} - \frac{1}{4}\sum_{j=1}^{k-1} \int_{\Omega_H} \frac{u^2}{|x|^2} X_1^2\left(\frac{|x|}{D}\right)\cdots X_j^2\left(\frac{|x|}{D}\right)}{\int_{\Omega_H} \frac{u^2}{|x|^2} X_1^2\left(\frac{|x|}{D}\right)\cdot\cdots\cdot X_k^2\left(\frac{|x|}{D}\right)} \\
&= \frac{\int_{\Omega_H} |\nabla_x \phi(|x|)|^2 w^2(t) - (n-1)^2 \int_{\Omega_H} \frac{\phi^2 w^2(t)}{|x|^2} - \frac{1}{4}\sum_{j=1}^{k-1} \int_{\Omega_H} \frac{\phi^2 w^2(t)}{|x|^2} X_1^2\left(\frac{|x|}{D}\right)\cdots X_j^2\left(\frac{|x|}{D}\right)}{\int_{\Omega_H} \frac{\phi^2 w^2(t)}{|x|^2} X_1^2\left(\frac{|x|}{D}\right)\cdots X_j^2\left(\frac{|x|}{D}\right)} \\
&\quad + 4\frac{\int_{\Omega_H} |x|^2 \phi^2(|x|)(w'(t))^2}{\int_{\Omega_H} \frac{\phi^2 w^2(t)}{|x|^2} X_1^2\left(\frac{|x|}{D}\right)\cdots X_j^2\left(\frac{|x|}{D}\right)} \\
&= \frac{\int_{B_R} |\nabla_x \phi(|x|)|^2 - (n-1)^2 \int_{B_R} \frac{\phi^2}{|x|^2} - \frac{1}{4}\sum_{j=1}^{k-1} \int_{B_R} \frac{\phi^2}{|x|^2} X_1^2\left(\frac{|x|}{D}\right)\cdots X_j^2\left(\frac{|x|}{D}\right)}{\int_{B_R} \frac{\phi^2}{|x|^2} X_1^2\left(\frac{|x|}{D}\right)\cdots X_j^2\left(\frac{|x|}{D}\right)} \\
&\quad + 4\frac{\int_{B_R} |x|^2 \phi^2(|x|)}{\int_{B_R} \frac{\phi^2}{|x|^2} X_1^2\left(\frac{|x|}{D}\right)\cdots X_j^2\left(\frac{|x|}{D}\right)} \cdot \frac{\int_{-\infty}^{+\infty} (w'(t))^2}{\int_{-\infty}^{+\infty} w^2(t)\mathrm{d}t}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\inf_{u\in C_0^\infty(\Omega_H)\setminus\{0\}} \frac{\int_{\Omega_H} |\nabla_{\mathbb{H}} u|^2 - (n-1)^2 \int_{\Omega_H} \frac{u^2}{|x|^2} - \frac{1}{4}\sum_{j=1}^{k-1} \int_{\Omega_H} \frac{u^2}{|x|^2} X_1^2\left(\frac{|x|}{D}\right)\cdots X_j^2\left(\frac{|x|}{D}\right)}{\int_{\Omega_H} \frac{u^2}{|x|^2} X_1^2\left(\frac{|x|}{D}\right)\cdot\cdots\cdot X_k^2\left(\frac{|x|}{D}\right)} \\
&\le \inf_{\phi(|x|)\in C_0^\infty(B_R)\setminus\{0\}} \frac{\int_{B_R} |\nabla_x \phi(|x|)|^2 - (n-1)^2 \int_{B_R} \frac{\phi^2}{|x|^2} - \frac{1}{4}\sum_{j=1}^{k-1} \int_{B_R} \frac{\phi^2}{|x|^2} \left(\frac{|x|}{D}\right)\cdots X_j^2\left(\frac{|x|}{D}\right)}{\int_{B_R} \frac{\phi^2}{|x|^2} \left(\frac{|x|}{D}\right)\cdots X_j^2\left(\frac{|x|}{D}\right)} \\
&= \frac{1}{4}.
\end{aligned}$$

since

$$\inf_{w(t)\in C_0^\infty(\mathbb{R})\setminus\{0\}} \frac{\int_{\mathbb{R}} |w'(t)|^2 \mathrm{d}t}{\int_{\mathbb{R}} |w(t)|^2 \mathrm{d}t} = 0.$$

This completes the proof of Corollary 1.3.

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Authors' contributions
YX designed and performed all the steps of proof in this research and also wrote the paper. All authors read and approved the final manuscript.

Competing interests
The author declares that they have no competing interests.

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