Li and Gao *Journal of Inequalities and Applications* 2011, **2011**:23
http://www.journalofinequalitiesandapplications.com/content/2011/1/23

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Non-differentiable multiobjective mixed symmetric duality under generalized convexity

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Abstract

The objective of this paper is to obtain a mixed symmetric dual model for a class of non-differentiable multiobjective nonlinear programming problems where each of the objective functions contains a pair of support functions. Weak, strong and converse duality theorems are established for the model under some suitable assumptions of generalized convexity. Several special cases are also obtained.

MS Classification: 90C32; 90C46.

Keywords: symmetric duality, non-differentiable nonlinear programming, generalized convexity, support function

1 Introduction

Dorn [1] introduced symmetric duality in nonlinear programming by defining a program and its dual to be symmetric if the dual of the dual is the original problem. The symmetric duality for scalar programming has been studied extensively in the literature, one can refer to Dantzig et al. [2], Bazaraa and Goode [3], Devi [4], Mond and Weir [5,6]. Mond and Schechter [7] studied non-differentiable symmetric duality for a class of optimization problems in which the objective functions consist of support functions. Following Mond and Schechter [7], Hou and Yang [8], Yang et al. [9], Mishra et al. [10] and Bector et al. [11] studied symmetric duality for such problems. Weir and Mond [6] presented two models for multiobjective symmetric duality. Several authors, such as the ones of [12-14], studied multiobjective second and higher order symmetric duality, motivated by Weir and Mond [6].

Very recently, Mishra et al. [10] presented a mixed symmetric dual formulation for a non-differentiable nonlinear programming problem. Bector et al. [11] introduced a mixed symmetric dual model for a class of nonlinear multiobjective programming problems. However, the models given by Bector et al. [11] as well as by Mishra et al. [10] do not allow the further weakening of generalized convexity assumptions on a part of the objective functions. Mishra et al [10] gave the weak and strong duality theorems for mixed dual model under the sublinearity. However, we note that they did not discuss the converse duality theorem for the mixed dual model.

In this paper, we introduce a model of mixed symmetric duality for a class of non-differentiable multiobjective programming problems with multiple arguments. We also establish weak, strong and converse duality theorems for the model and discuss several special cases of the model. The results of Mishra et al. [10] as well as that of Bector et al. [11] are particular cases of the results obtained in the present paper.

2 Preliminaries

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space and let $\mathbf{R}^n_+$ be its non-negative orthant. The following convention will be used: if x, $y \in \mathbf{R}^n$, then $x \leqq y \Leftrightarrow y - x \in \mathbf{R}^n_+$; $x < y \Leftrightarrow y - x \in int\mathbf{R}^n_+$; $x < y \Leftrightarrow y - x \in int\mathbf{R}^n_+$; $x \nleq y$ is the negation of $x \nleq y$.

Let $f(x, y)$ be a real valued twice differentiable function defined on $\mathbf{R}^n \times \mathbf{R}^m$. Let $\nabla_1 f(\bar{x}, \bar{y})$ and $\nabla_2 f(\bar{x}, \bar{y})$ denote the gradient vector of f with respect to x and y at $(\bar{x}, \bar{y})$. Also let $\nabla_{11} f(\bar{x}, \bar{y})$ denote the Hessian matrix of $f(x, y)$ with respect to the first variable x at $(\bar{x}, \bar{y})$. The symbols $\nabla_{22} f(\bar{x}, \bar{y})$, $\nabla_{12} f(\bar{x}, \bar{y})$ and $\nabla_{21} f(\bar{x}, \bar{y})$ are defined similarly. Consider the following multiobjective programming problem (VP):

$$\begin{aligned} &\min f(x) = (f_1(x), f_2(x), \ldots, f_p(x)) \\ &s.t. \quad h(x) \leqq 0, \quad x \in X, \end{aligned}$$

where X is an open set of $\mathbf{R}^n$, $f_i : X \to \mathbf{R}$, $i = 1, 2, \ldots, p$ and $h : X \to \mathbf{R}^m$.

Definition 2.1 A feasible solution $\bar{x}$ is said to be an efficient solution for (VP) if there exists no other $x \in X$ such that $f(x) \leq f(\bar{x})$.

Let C be a compact convex set in $\mathbf{R}^n$. The support function of C is defined by

$$s(x|C) := \max\{x^T y : y \in C\}.$$

A support function, being convex and everywhere finite, has a subdifferential [7], that is, there exists $z \in \mathbf{R}^n$ such that

$$s(y|C) \geqq s(x|C) + z^T(y - x) \quad \forall y \in C.$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) := \{z \in C : z^T x = s(x|C).$$

For any set $D \subset \mathbf{R}^n$, the normal cone to D at a point $x \in D$ is defined by

$$N_D(x) := \{y \in \mathbf{R}^n : y^T(z - x) \leqq 0 \quad \forall z \in D\}.$$

It is obvious that for a compact convex set C, $y \in N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, $x \in \partial s(y|C)$.

Let us consider a function $F : X \times X \times \mathbf{R}^n \to \mathbf{R}$ (where $X \subset \mathbf{R}^n$) with the properties that for all $(x, y) \in X \times X$, we have

(i)$F(x, y; \cdot)$ is a convex function, (ii)$F(x, y; 0) \geqq 0$.

If F satisfies (i) and (ii), we obviously have $F(x, y; -a) \geqq - F(x, y; a)$ for any $a \in \mathbf{R}^n$.

For example, $F(x, y; a) = M_1||a|| + M_2||a||2$, where a depends on x and y, M_1, M_2 are positive constants. This function satisfies (i) and (ii), but it is neither subadditive, nor positive homogeneous, that is, the relations

(i')$F(x, y; a + b) \leqq F(x, y; a) + F(x, y; b)$, (ii')$F(x, y; ra) = rF(x, y; a)$ are not fulfilled for any a, $b \in \mathbf{R}^n$ and $r \in \mathbf{R}_+$. We may conclude that the class of functions that verify (i) and (ii) is more general than the class of sublinear functions with respect the third argument, i.e. those which satisfy (I') and (ii'). We notice that till now, most results in optimization theory were stated under generalized convexity assumptions involving the functions F which are sublinear. The results of this paper are obtained by using weaker assumptions with respect to the above function F.

Throughout the paper, we always assume that $F, G : X \times X \times \mathbf{R}^n \to \mathbf{R}$ satisfy (i) and (ii).

Definition 2.2 Let $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$. $f(\cdot, y)$ is said to be F-convex at $\bar{x} \in X$, for fixed $y \in Y$, if

$$f(x,y) - f(\bar{x},y) \geqq F(x,\bar{x};\nabla_1 f(\bar{x},y)) \quad \forall x \in X.$$

Definition 2.3 Let $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$. $f(x,\cdot)$ is said to be F-concave at $\bar{y} \in Y$, for fixed $x \in X$, if

$$f(x,\bar{y}) - f(x,y) \geqq F(y,\bar{y};-\nabla_2 f(x,\bar{y})) \quad \forall y \in Y.$$

Definition 2.4 Let $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$. $f(\cdot, y)$ is said to be F-pseudoconvex at $\bar{x} \in X$, for fixed $y \in Y$, if

$$F(x,\bar{x};\nabla_1 f(\bar{x},y)) \geqq 0 \Rightarrow f(x,y) \geqq f(\bar{x},y) \quad \forall x \in X.$$

Definition 2.5 Let $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$. $f(x,\cdot)$ is said to be F-pseudoconcave at $\bar{y} \in Y$, for fixed $x \in X$, if

$$F(y,\bar{y};\nabla_2 f(x,\bar{y})) \geqq 0 \Rightarrow f(x,\bar{y}) \geqq f(x,y) \quad \forall y \in Y.$$

3 Mixed type multiobjective symmetric duality

For $N = \{1, 2, ..., n\}$ and $M = \{1, 2, ..., m\}$, let $J_1 \subset N$, $K_1 \subset M$ and $J_2 = N \backslash J_1$ and $K_2 = M \backslash K_1$. Let $|J_1|$ denote the number of elements in the set J_1. The other numbers $|J_2|$, $|K_1|$ and $|K_2|$ are defined similarly. Notice that if $J_1 = \emptyset$, then $J_2 = N$, that is, $|J_1| = 0$ and $|J_2| = n$. Hence, $\mathbf{R}^{|J_1|}$ is zero-dimensional Euclidean space and $\mathbf{R}^{|J_2|}$ is n-dimensional Euclidean space. It is clear that any $x \in \mathbf{R}^n$ can be written as $x = (x^1, x^2)$, $x^1 \in \mathbf{R}^{|J_1|}$, $x^2 \in \mathbf{R}^{|J_2|}$. Similarly, any $y \in \mathbf{R}^m$ can be written as $y = (y^1, y^2)$, $y^1 \in \mathbf{R}^{|K_1|}$, $y^2 \in \mathbf{R}^{|K_2|}$. Let $f : \mathbf{R}^{|J_1|} \times \mathbf{R}^{|K_1|} \to \mathbf{R}^l$ and $g : \mathbf{R}^{|J_2|} \times \mathbf{R}^{|K_2|} \to \mathbf{R}^l$ be twice continuously differentiable functions and $e = (1, 1, ..., 1) \in \mathbf{R}^l$.

Now we can introduce the following pair of non-differentiable multiobjective programs and discuss their duality theorems under some mild assumptions of generalized convexity.

Primal problem (MP):

$$\begin{aligned} &\textbf{Min } H(x^1,x^2,y^1,y^2,z^1,z^2,\lambda) = (H_1(x^1,x^2,y^1,y^2,z^1,z^2,\lambda),\ldots,H_l(x^1,x^2,y^1,y^2,z^1,z^2,\lambda)) \\ &\textbf{s.t. } (x^1,x^2,y^1,y^2,z^1,z^2,\lambda) \in \mathbf{R}^{|J_1|} \times \mathbf{R}^{|J_2|} \times \mathbf{R}^{|K_1|} \times \mathbf{R}^{|K_2|} \times \mathbf{R}^{|K_1|} \times \mathbf{R}^{|K_2|} \times \mathbf{R}^l_+, \end{aligned}$$

$$\sum_{i=1}^{l} \lambda_i [\nabla_2 f_i(x^1,y^1) - z_i^1] \leqq 0, \tag{1}$$

$$\sum_{i=1}^{l} \lambda_i [\nabla_2 g_i(x^2,y^2) - z_i^2] \leqq 0, \tag{2}$$

$$(y^1)^T \sum_{i=1}^{l} \lambda_i [\nabla_2 f_i(x^1, y^1) - z_i^1] \geqq 0, \tag{3}$$

$$(y^2)^T \sum_{i=1}^{l} \lambda_i [\nabla_2 g_i(x^2, y^2) - z_i^2] \geqq 0, \tag{4}$$

$$(x^1, x^2) \geqq 0, \tag{5}$$

$$z_i^1 \in D_i^1,\ z_i^2 \in D_i^2,\ i = 1, 2, \ldots, l, \tag{6}$$

$$\lambda > 0,\quad \lambda^T e = 1. \tag{7}$$

Dual problem (MD):

$$\textbf{Max } G(u^1, u^2, v^1, v^2, w^1, w^2, \lambda) = (G_1(u^1, u^2, v^1, v^2, w^1, w^2, \lambda), \ldots, G_l(u^1, u^2, v^1, v^2, w^1, w^2, \lambda))$$
$$\textbf{s.t. } (u^1, u^2, v^1, v^2, w^1, w^2, \lambda) \in \mathbf{R}^{|J_1|} \times \mathbf{R}^{|J_2|} \times \mathbf{R}^{|K_1|} \times \mathbf{R}^{|K_2|} \times \mathbf{R}^{|K_1|} \times \mathbf{R}^{|K_2|} \times \mathbf{R}_+^l,$$

$$\sum_{i=1}^{l} \lambda_i [\nabla_1 f_i(u^1, v^1) + w_i^1] \geqq 0, \tag{8}$$

$$\sum_{i=1}^{l} \lambda_i [\nabla_1 g_i(u^2, v^2) + w_i^2] \geqq 0, \tag{9}$$

$$(u^1)^T \sum_{i=1}^{l} \lambda_i [\nabla_1 f_i(u^1, v^1) + w_i^1] \leqq 0, \tag{10}$$

$$(u^2)^T \sum_{i=1}^{l} \lambda_i [\nabla_1 g_i(u^2, v^2) + w_i^2] \leqq 0, \tag{11}$$

$$(v^1, v^2) \geqq 0, \tag{12}$$

$$w_i^1 \in C_i^1,\ w_i^2 \in C_i^2,\ i = 1, 2, \ldots, l, \tag{13}$$

$$\lambda > 0,\ \lambda^T e = 1. \tag{14}$$

where
$H_i(x^1, x^2, y^1, y^2, z, \lambda) = f_i(x^1, y^1) + g_i(x^2, y^2) + s(x^1|C_i^1) + s(x^2|C_i^2) - (y^1)^T z_i^1 - (y^2)^T z_i^2,$
$G_i(u^1, u^2, v^1, v^2, w, \lambda) = f_i(u^1, v^1) + g_i(u^2, v^2) - s(v^1|D_i^1) - s(v^2|D_i^2) + (u^1)^T w_i^1 + (u^2)^T w_i^2,$ and C_i^1 is a compact and convex subset of $\mathbf{R}^{|J_1|}$ for $i = i = 1, 2,..., l$ and C_i^2 is a compact and convex subset of $\mathbf{R}^{|J_2|}$ for $i = 1, 2,..., l$. Similarly, D_i^1 is a compact and convex subset of $\mathbf{R}^{|K_1|}$ for $i = 1, 2,..., l$ and D_i^2 is a compact and convex subset of $\mathbf{R}^{|K_2|}$ for $i = 1, 2,..., l$.

Theorem 3.1(Weak duality). Let $(x^1, x^2, y^1, y^2, z^1, z^2, \lambda)$ be feasible for (MP) and $(u^1, u^2, v^1, v^2, w^1, w^2, \lambda)$ be feasible for (MD). Suppose that for $i = 1, 2,..., l$, $f_i(\cdot, v^1) + (\cdot)^T w_i^1$ is F_1-convex for fixed v^1, $f_i(x^1, \cdot) - (\cdot)^T z_i^1$ is F_2-concave for fixed x_1, $g_i(\cdot, v^2) + (\cdot)^T w_i^2$ is G_1-convex for fixed v^2 and $g_i(x^2, \cdot) - (\cdot)^T z_i^2$ is G_2-concave for fixed x^2, and the following conditions are satisfied:

(I) $F_1(x^1, u^1; a) + (u^1)^T a \geqq 0$ if $a \geqq 0$;
(II) $G_1(x^2, u^2; b) + (u^2)^T b \geqq 0$ if $b \geqq 0$;
(III) $F_2(v^1, y^1; c) + (y^1)^T c \geqq 0$ if $c \geqq 0$; and
(IV) $G_2(v^2, y^2; d) + (y^2)^T d \geqq 0$ if $d \geqq 0$.

Then $H(x^1, x^2, y^1, y^2, z^1, z^2, \lambda) \nleq G(u^1, u^2, v^1, v^2, w^1, w^2, \lambda)$.

Proof. Assume that the result is not true, that is $H(x^1, x^2, y^1, y^2, z^1, z^2, \lambda) \leq G(u^1, u^2, v^1, v^2, w^1, w^2, \lambda)$. Then, since $\lambda > 0$, we have

$$\begin{aligned}&\sum_{i=1}^{n}\lambda_i[f_i(x^1,y^1)+g_i(x^2,y^2)+s(x^1|C_i^1)+s(x^2|C_i^2)-(y^1)^Tz_i^1-(y^2)^Tz_i^2]\\&<\sum_{i=1}^{n}\lambda_i[f_i(u^1,v^1)+g_i(u^2,v^2)-s(v^1|D_i^1)-s(v^2|D_i^2)+(u^1)^Tw_i^1+(u^2)^Tw_i^2].\end{aligned} \tag{15}$$

By the $F1$-convexity of $f_i(\cdot, v^1) + (\cdot)^T w_i^1$, we have
$(f_i(x^1,v^1)+(x^1)^Tw_i^1)-(f_i(u^1,v^1)+(u^1)^Tw_i^1) \geqq F_1(x^1,u^1;\nabla_1 f_i(u^1,v^1)+w_i^1)$, for $i = 1,2,\ldots, l$.
From (7), (14) and F_1 satisfying (i) and (ii), the above inequality yields

$$\sum_{i=1}^{l}\lambda_i[(f_i(x^1,v^1)+(x^1)^Tw_i^1)-(f_i(u^1,v^1)+(u^1)^Tw_i^1)] \geqq F_1\left(x^1,u^1;\sum_{i=1}^{l}\lambda_i[\nabla_1 f_i(u^1,v^1)+w_i^1]\right). \tag{16}$$

By the duality constraint (8) and conditions (I), we get

$$F_1(x^1,u^1;\sum_{i=1}^{l}\lambda_i[\nabla_1 f_i(u^1,v^1)+w_i^1]) \geqq -(u_1)^T\sum_{i=1}^{l}\lambda_i[\nabla_1 f_i(u^1,v^1)+w_i^1].$$

From (10), (16) and the above inequality, we obtain

$$\sum_{i=1}^{l}\lambda_i[(f_i(x^1,v^1)+(x^1)^Tw_i^1)-(f_i(u^1,v^1)+(u^1)^Tw_i^1)] \geqq 0. \tag{17}$$

By the F_2-concavity of $f_i(x^1,\cdot)-(\cdot)^Tz_i^1$, we have, for $i = 1, 2,\ldots, l$,

$$(f_i(x^1,y^1)-(y^1)^Tz_i^1)-(f_i(x^1,v^1)-(v^1)^Tz_i^1) \geqq F_2(v^1,y^1;-[\nabla_2 f_i(x^1,y^1)-z_i^1]).$$

From (7), (14) and F_2 satisfying (i) and (ii), the above inequality yields

$$\sum_{i=1}^{l}\lambda_i[(f_i(x^1,y^1)-(y^1)^Tz_i^1)-(f_i(x^1,v^1)-(v^1)^Tz_i^1)] \geqq F_2(v^1,y^1;-\sum_{i=1}^{l}\lambda_i[\nabla_2 f_i(v^1,y^1)-z_i^1]). \tag{18}$$

By the primal constraint (1) and conditions (III), we get

$$F_2(v^1,y^1;-\sum_{i=1}^{l}\lambda_i[\nabla_2 f_i(x^1,y^1)-z_i^1]) \geqq (y_1)^T\sum_{i=1}^{l}\lambda_i[\nabla_1 f_i(x^1,y^1)-z_i^1].$$

From (3), (18) and the above inequality, we obtain

$$\sum_{i=1}^{l}\lambda_i[(f_i(x^1,y^1)-(y^1)^Tz_i^1)-(f_i(x^1,v^1)-(v^1)^Tz_i^1)] \geqq 0. \tag{19}$$

Using $(v^1)^Tz_i^1 \leqq s(v^1|D_i^1)$ and $(x^1)^Tw_i^1 \leqq s(x^1|C_i^1)$ for $i = 1, 2,\ldots, l$, it follows from (17) and (19), that

$$\sum_{i=1}^{l} \lambda_i [f_i(x^1, y^1) + s(x^1|C_i^1)) - (u^1)^T w_i^1) - f_i(u^1, v^1) + s(v^1|D_i^1) - (y^1)^T z_i^1] \geqq 0. \quad (20)$$

Similarly, by the G_1-convexity of $g_i(\cdot, v^2) + (\cdot)^T w_i^2$ and G_2-concavity of $g_i(x^2, \cdot) - (\cdot)^T z_i^2$, for $i = 1, 2, ..., l$, and condition (II) and (IV), we get

$$\sum_{i=1}^{l} \lambda_i [g_i(x^2, y^2) + s(x^2|C_i^2) - (y^2)^T z_i^2 - g_i(u^2, v^2) + s(v^2|D_i^2) - (u^2)^T w_i^2] \geqq 0. \quad (21)$$

From (20) and (21), we have

$$\begin{aligned} &\sum_{i=1}^{n} \lambda_i [f_i(x^1, y^1) + g_i(x^2, y^2) + s(x^1|C_i^1) + s(x^2|C_i^2) - (y^1)^T z_i^1 - (y^2)^T z_i^2] \\ \geqq &\sum_{i=1}^{n} \lambda_i [f_i(u^1, v^1) + g_i(u^2, v^2) - s(v^1|D_i^1) - s(v^2|D_i^2) + (u^1)^T w_i^1 + (u^2)^T w_i^2], \end{aligned}$$

which is a contradiction to (15). Hence $H(x^1, x^2, y^1, y^2, z^1, z^2, \lambda) \nleq G(u^1, u^2, v^1, v^2, w^1, w^2, \lambda)$.

Remark 3.1. Theorem 3.1 can be established for more general classes of functions such as F_1-pseudoconvexity and F_2-pseudoconcavity, and G_1-pseudoconvexity and G_2-pseudoconcavity on the functions involved in the above theorem. The proofs will follow the same lines as that of Theorem 3.1.

Strong duality theorem for the given model can be established on the lines of the proof of Theorem 2 of Yang et al. [9].

Theorem 3.2(Strong duality). Let $(\overline{x^1}, \overline{x^2}, \overline{y^1}, \overline{y^2}, \overline{z^1}, \overline{z^2}, \bar{\lambda})$ be an efficient solution for (MP), fix $\lambda = \bar{\lambda}$ in (MD), and suppose that

(A1) either the matrices $\sum_{i=1}^{l} \overline{\lambda_i} \nabla_{22} f_i(\overline{x^1}, \overline{y^1})$ and $\sum_{i=1}^{l} \overline{\lambda_i} \nabla_{22} g_i(\overline{x^2}, \overline{y^2})$ are positive definite; or $\sum_{i=1}^{l} \overline{\lambda_i} \nabla_{22} f_i(\overline{x^1}, \overline{y^1})$ and $\sum_{i=1}^{l} \overline{\lambda_i} \nabla_{22} g_i(\overline{x^2}, \overline{y^2})$ are negative definite; and

(A2) the sets $\{\nabla_2 f_1(\overline{x^1}, \overline{y^1}) - \overline{z_1^1}, \ldots, \nabla_2 f_l(\overline{x^1}, \overline{y^1}) - \overline{z_l^1}\}$ and $\{\nabla_2 g_1(\overline{x^2}, \overline{y^2}) - \overline{z_1^2}, \ldots, \nabla_2 g_l(\overline{x^2}, \overline{y^2}) - \overline{z_l^2}\}$ are linearly independent.

Then $(\overline{x^1}, \overline{x^2}, \overline{y^1}, \overline{y^2}, \overline{z^1}, \overline{z^2}, \bar{\lambda})$ is feasible for (MD) and the corresponding objective function values are equal. If in addition the hypotheses of Theorem 3.1 hold, then there exist $\overline{w^1}$, $\overline{w^2}$ such that $(u^1, u^2, v^1, v^2, w^1, w^2, \lambda) = (\overline{x^1}, \overline{x^2}, \overline{y^1}, \overline{y^2}, \overline{w^1}, \overline{w^2}, \bar{\lambda})$ is an efficient solution for (MD).

Mishra et al. [10] gave weak and strong duality theorems for the mixed model. However, we note that they did not discuss the converse duality theorem for the mixed dual model. Here, we will give a converse duality theorem for the model under some weaker assumptions.

Theorem 3.3(Converse duality). Let $(\overline{x^1}, \overline{x^2}, \overline{y^1}, \overline{y^2}, \overline{w^1}, \overline{w^2}, \bar{\lambda})$ be an efficient solution for (MD), $\lambda = \bar{\lambda}$ in (MP), and suppose that

(B1) either the matrices $\sum_{i=1}^{l} \overline{\lambda_i} \nabla_{11} f_i(\overline{x^1}, \overline{y^1})$ and $\sum_{i=1}^{l} \overline{\lambda_i} \nabla_{11} g_i(\overline{x^2}, \overline{y^2})$ are positive definite; or $\sum_{i=1}^{l} \overline{\lambda_i} \nabla_{11} f_i(\overline{x^1}, \overline{y^1})$ and $\sum_{i=1}^{l} \overline{\lambda_i} \nabla_{11} g_i(\overline{x^2}, \overline{y^2})$ are negative definite; and

(B2) the sets $\{\nabla_1 f_1(\overline{x^1},\overline{y^1})-\overline{w_1^1},\ldots,\nabla_1 f_l(\overline{x^1},\overline{y^1})-\overline{w_l^1}\}$ and $\{\nabla_1 g_1(\overline{x^2},\overline{y^2})-\overline{w_1^2},\ldots,\nabla_1 g_l(\overline{x^2},\overline{y^2})-\overline{w_l^2}\}$ are linearly independent.

Then $(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{w^1},\overline{w^2},\bar{\lambda})$ is feasible for (MP) and the corresponding objective function values are equal. If in addition the hypotheses of Theorem 3.1 hold, then there exist $\overline{z^1}$, $\overline{z^2}$ such that $(x^1,x^2,y^1,y^2,z^1,z^2,\lambda)=(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2},\bar{\lambda})$ is an efficient solution for (MP).

Proof. Since $(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{w^1},\overline{w^2},\bar{\lambda})$ be an efficient solution for (MD), by the modifying Fritz-John conditions [7], there exist $\alpha\in\mathbf{R}^l$, $\alpha_1\in\mathbf{R}^{|J_1|}$, $\alpha_2\in\mathbf{R}^{|J_2|}$, $\beta_1\in\mathbf{R}$, $\beta_2\in\mathbf{R}$, $\mu_2\in\mathbf{R}^{|K_2|}$, $\mu_2\in\mathbf{R}^{|K_2|}$, $\delta\in\mathbf{R}^l$ such that

$$\sum_{i=1}^{l}(-\alpha_i+\beta_1\overline{\lambda_i})[\nabla_1 f_i(\overline{x^1},\overline{y^1})+\overline{w_i^1}]^T+(\beta_1\overline{x^1}-\alpha_1)^T\sum_{i=1}^{l}\overline{\lambda_i}\nabla_{11}f_i(\overline{x^1},\overline{y^1})=0, \tag{22}$$

$$\sum_{i=1}^{l}(-\alpha_i+\beta_2\overline{\lambda_i})[\nabla_1 g_i(\overline{x^2},\overline{y^2})+\overline{w_i^2}]^T+(\beta_2\overline{x^2}-\alpha_2)^T\sum_{i=1}^{l}\overline{\lambda_i}\nabla_{11}g_i(\overline{x^2},\overline{y^2})=0, \tag{23}$$

$$-\sum_{i=1}^{l}\alpha_i[\nabla_2 f_i(\overline{x^1},\overline{y^1})-\overline{z_i^1}]+(\beta_1\overline{x^1}-\alpha_1)^T\sum_{i=1}^{l}\overline{\lambda_i}\nabla_{12}f_i(\overline{x^1},\overline{y^1})-\mu_1=0, \tag{24}$$

$$\overline{z_i^1}\in D_i^1,\quad (\overline{z_i^1})^T\overline{y^1}=s(\overline{y^1}|D_i^1),\quad i=1,2,\ldots,l, \tag{25}$$

$$-\sum_{i=1}^{l}\alpha_i[\nabla_2 g_i(\overline{x^2},\overline{y^2})-\overline{z_i^2}]+(\beta_2\overline{x^2}-\alpha_2)^T\sum_{i=1}^{l}\overline{\lambda_i}\nabla_{12}g_i(\overline{x^2},\overline{y^2})-\mu_2=0, \tag{26}$$

$$\overline{z_i^2}\in D_i^2,\quad (\overline{z_i^2})^T\overline{y^2}=s(\overline{y^2}|D_i^2),\quad i=1,2,\ldots,l, \tag{27}$$

$$(\alpha^T e)\overline{x^1}+\overline{\lambda_i}(\beta_1\overline{x^1}-\alpha_1)\in N_{C_i^1}(\overline{w_i^1}),\quad i=1,2,\ldots,l, \tag{28}$$

$$(\alpha^T e)\overline{x^2}+\overline{\lambda_i}(\beta_2\overline{x^2}-\alpha_2)\in N_{C_i^2}(\overline{w_i^2}),\quad i=1,2,\ldots,l, \tag{29}$$

$$(\beta_1\overline{x^1}-\alpha_1)^T[\nabla_1 f_i(\overline{x^1},\overline{y^1})+\overline{w_i^1}]+(\beta_2\overline{x^2}-\alpha_2)^T[\nabla_1 g_i(\overline{x^2},\overline{y^2})+\overline{w_i^2}]-\delta_i=0,\quad i=1,2,\ldots,l, \tag{30}$$

$$\alpha_1^T\sum_{i=1}^{l}\overline{\lambda_i}[\nabla_1 f_i(\overline{x^1},\overline{y^1})+\overline{w_i^1}]=0, \tag{31}$$

$$\alpha_2^T\sum_{i=1}^{l}\overline{\lambda_i}[\nabla_1 g_i(\overline{x^2},\overline{y^2})+\overline{w_i^2}]=0, \tag{32}$$

$$\beta_1(\overline{x^1})^T\sum_{i=1}^{l}\overline{\lambda_i}[\nabla_1 f_i(\overline{x^1},\overline{y^1})+\overline{w_i^1}]=0, \tag{33}$$

$$\beta_2(\overline{x^2})^T \sum_{i=1}^{l} \overline{\lambda_i}[\nabla_1 g_i(\overline{x^2}, \overline{y^2}) + \overline{w_i^2}] = 0, \tag{34}$$

$$\mu_1^T \overline{y^1} = 0, \tag{35}$$

$$\mu_2^T \overline{y^2} = 0, \tag{36}$$

$$\delta^T \bar{\lambda} = 0, \tag{37}$$

$$(\alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \mu_1, \mu_2, \delta) \geqq 0 \text{ and } (\alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \mu_1, \mu_2, \delta) \neq 0. \tag{38}$$

From (22) and (23), we get

$$\begin{aligned} &\sum_{i=1}^{l} (-\alpha_i + \beta_1 \overline{\lambda_i})[\nabla_1 f_i + \overline{w_i^1}]^T (\beta_1 \overline{x^1} - \alpha_1) + (\beta_1 \overline{x^1} - \alpha_1)^T \sum_{i=1}^{l} \overline{\lambda_i} \nabla_{11} f_i (\beta_1 \overline{x^1} - \alpha_1) \\ &+ \sum_{i=1}^{l} (-\alpha_i + \beta_2 \overline{\lambda_i})[\nabla_1 g_i + \overline{w_i^2}]^T (\beta_2 \overline{x^2} - \alpha_2) + (\beta_2 \overline{x^2} - \alpha_2)^T \sum_{i=1}^{l} \overline{\lambda_i} \nabla_{11} g_i (\beta_2 \overline{x^2} - \alpha_2) = 0. \end{aligned} \tag{39}$$

From (31)-(34), we have

$$(\beta_1 \overline{x^1} - \alpha_1)^T \sum_{i=1}^{l} \overline{\lambda_i}[\nabla_1 f_i + \overline{w_i^1}] + (\beta_2 \overline{x^2} - \alpha_2)^T \sum_{i=1}^{l} \overline{\lambda_i}[\nabla_1 g_i + \overline{w_i^2}] = 0, \tag{40}$$

Substituting (40) into (39), we obtain

$$\begin{aligned} &- \sum_{i=1}^{l} \alpha_i \{[\nabla_1 f_i + \overline{w_i^1}]^T (\beta_1 \overline{x^1} - \alpha_1) + [\nabla_1 g_i + \overline{w_i^2}]^T (\beta_2 \overline{x^2} - \alpha_2)\} \\ &+ (\beta_1 \overline{x^1} - \alpha_1)^T \sum_{i=1}^{l} \overline{\lambda_i} \nabla_{11} f_i (\beta_1 \overline{x^1} - \alpha_1) + (\beta_2 \overline{x^2} - \alpha_2)^T \sum_{i=1}^{l} \overline{\lambda_i} \nabla_{11} g_i (\beta_2 \overline{x^2} - \alpha_2) = 0. \end{aligned}$$

Since $\lambda > 0$, it follows from (37), that $\delta = 0$. From $\delta = 0$ and (30), the above equation yields

$$(\beta_1 \overline{x^1} - \alpha_1)^T \sum_{i=1}^{l} \overline{\lambda_i} \nabla_{11} f_i (\beta_1 \overline{x^1} - \alpha_1) + (\beta_2 \overline{x^2} - \alpha_2)^T \sum_{i=1}^{l} \overline{\lambda_i} \nabla_{11} g_i (\beta_2 \overline{x^2} - \alpha_2) = 0. \tag{41}$$

From (A1) and (41), we obtain

$$\alpha_1 = \beta_1 \overline{x^1} \text{ and } \alpha_2 = \beta_2 \overline{x^2}. \tag{42}$$

From (22), (23), (42) and (A2), we get

$$\alpha_i = \beta_1 \overline{\lambda_i} \text{ and } \alpha_i = \beta_2 \overline{\lambda_i}, \ i = 1, 2, \ldots, l. \tag{43}$$

If $\beta_1 = 0$, then from (43) and (42), $\beta_2 = 0$, $\alpha = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, and from (24) and (26), $\mu_1 = 0$, $\mu_2 = 0$. This contradicts (38). Hence $\beta_1 = \beta_2 > 0$ and $\alpha > 0$.

From (38) and (42), we have

$$(\overline{x^1}, \overline{x^2}) \geqq 0. \tag{44}$$

By (24), (38) and (43), we have

$$\sum_{i=1}^{l} \overline{\lambda_i}[\nabla_2 f_i(\overline{x^1}, \overline{y^1}) - \overline{z_i^1}] \leqq 0. \tag{45}$$

By (26), (38) and (43), we have

$$\sum_{i=1}^{l} \overline{\lambda_i}[\nabla_2 g_i(\overline{x^2}, \overline{y^2}) - \overline{z_i^2}] \leqq 0. \tag{46}$$

From (24), (35), (42) and (43), we have

$$(\overline{y^1})^T \sum_{i=1}^{l} \overline{\lambda_i}[\nabla_2 f_i(\overline{x^1}, \overline{y^1}) - \overline{z_i^1}] = 0. \tag{47}$$

From (26), (36), (42) and (43), we have

$$(\overline{y^2})^T \sum_{i=1}^{l} \overline{\lambda_i}[\nabla_2 g_i(\overline{x^2}, \overline{y^2}) - \overline{z_i^2}] = 0. \tag{48}$$

Hence from (12)-(14) and (44)-(48), $(\overline{x^1}, \overline{x^2}, \overline{y^1}, \overline{y^2}, \overline{w^1}, \overline{w^2}, \bar{\lambda})$ is feasible for (MP). Now from (28), (42) and $\alpha > 0$, we have $\overline{x^1} \in N_{C_i^1}(\overline{w_i^1})$, $i = 1, 2,..., l$, that is

$$s(\overline{x^1}|C_i^1) = (\overline{w_i^1})^T\overline{x^1}, \quad i = 1, 2, \ldots, l. \tag{49}$$

From (29), (42) and $\alpha > 0$, we have

$$s(\overline{x^2}|C_i^2) = (\overline{w_i^2})^T\overline{x^2}, \quad i = 1, 2, \ldots, l. \tag{50}$$

Finally, from (25), (27), (49) and (50), for all $i = 1, 2,..., l$, we give,

$$\begin{aligned} &f_i(\overline{x^1}, \overline{y^1}) + g_i(\overline{x^2}, \overline{y^2}) - s(\overline{y^1}|D_i^1) - s(\overline{y^2}|D_i^2) + (\overline{x^1})^T\overline{w_i^1} + (\overline{x^2})^T\overline{w_i^2} \\ = &f_i(\overline{x^1}, \overline{y^1}) + g_i(\overline{x^2}, \overline{y^2}) + s(\overline{x^1}|C_i^1) + s(\overline{x^2}|C_i^2) - (\overline{y^1})^T\overline{z_i^1} - (\overline{y^2})^T\overline{z_i^2}. \end{aligned} \tag{51}$$

Thus $G(\overline{x^1}, \overline{x^2}, \overline{y^1}, \overline{y^2}, \overline{w^1}, \overline{w^2}, \bar{\lambda}) = H(\overline{x^1}, \overline{x^2}, \overline{y^1}, \overline{y^2}, \overline{z^1}, \overline{z^2}, \bar{\lambda})$. By the weak duality and (51), $(\overline{x^1}, \overline{x^2}, \overline{y^1}, \overline{y^2}, \overline{z^1}, \overline{z^2}, \bar{\lambda})$ is an efficient solution for (MD).

4 Special cases

In this section, we consider some special cases of problems (MP) and (MD) by choosing particular forms of compact convex sets, and the number of objective and constraint functions:

(i) If $F(x, y; \cdot)$ is sublinear, then (MP) and (MD) reduce to the pair of problems (MP2) and (MD2) studied in Mishra et al. [10].
(ii) If $F(x, y; \cdot)$ is sublinear, $|J_2| = 0$, $|K_2| = 0$ and $l = 1$, then (MP) and (MD) reduce to the pair of problems (P1) and (D1) of Mond and Schechter [7]. Thus (MP) and (MD) become multiobjective extension of the pair of problems (P1) and (D1) in [7].
(iii) If $F(x, y; \cdot)$ is sublinear and $l = 1$, then (MP) and (MD) are an extension of the pair of problems studied in Yang et al. [9].
(iv) From the symmetry of primal and dual problems (MP) and (MD), we can construct other new symmetric dual pairs. For example, if we take $C_i^1 = \{A_i^1 y : y^T A_i^1 y \leqq 1\}$

and $C_i^2 = \{A_i^2 y : y^T A_i^2 y \leqq 1\}$, where A_i^1, A_i^2, $i = 1,2,,..., l$, are positive semi definite matrices, then it can be easily verified that $s(x^1|C_i^1) = (x^T A_i^1 x)^{\frac{1}{2}}$, and $s(x^1|D_i^1) = (x^T B_i^1 x)^{\frac{1}{2}}$, $i = 1, 2,..., l$. Thus, a number of new symmetric dual pairs and duality results can be established.

Acknowledgements
This study was supported by the Education Committee Project Research Foundation of Chongqing (No.KJ110624), the Doctoral Foundation of Chongqing Normal University (No.10XLB015) and Chongqing Key Lab of Operations Research and System Engineering.

Authors' contributions
All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 21 January 2011 Accepted: 21 July 2011 Published: 21 July 2011

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doi:10.1186/1029-242X-2011-23
Cite this article as: Li and Gao: **Non-differentiable multiobjective mixed symmetric duality under generalized convexity.** *Journal of Inequalities and Applications* 2011 **2011**:23.