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The stability of functional equation min{$f(x + y)$, $f(x - y)$} = $|f(x) - f(y)|$

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Abstract

In this paper, we prove the stability of the functional equation min $\{f(x + y), f(x - y)\} = |f(x) - f(y)|$ in the class of real, continuous functions of real variable.

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1. Introduction

In the paper [1], Simon and Volkmann examined functional equations connected with the absolute value of an additive function, that is,

$$\max\{f(x+y), f(x-y)\} = f(x) + f(y), \quad x, y \in G, \tag{1.1}$$

$$\min\{f(x+y), f(x-y)\} = |f(x) - f(y)|, \quad x, y \in G, \tag{1.2}$$

and

$$\max\{f(x+y), f(x-y)\} = f(x)f(y), \quad x, y \in G, \tag{1.3}$$

where G is an abelian group and $f : G \to \mathbb{R}$. The first two of them are satisfied by $f(x) = |a(x)|$, where a: $G \to \mathbb{R}$ is an additive function; moreover, the first one characterizes the absolute value of additive functions. The solutions of Equation (1.2) are appointed by Volkmann during the Conference on Inequalities and Applications in Noszwaj (Hungary, 2007), under the assumption that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Namely, we have

Theorem 1.1 (Jarczyk and Volkmann [2]). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying Equation* (1.2). *Then either there exists a constant $c \geq 0$ such that $f(x) = c|x|$, $x \in \mathbb{R}$, or f is periodic with period $2p$ given by $f(x) = c|x|$ with some constant $c > 0$, $x \in [-p, p]$.*

Actually, it is enough to assume continuity at a point, since this implies continuity on $\mathbb{R}$, see [2]. Moreover, some measurability assumptions force continuity. Baron in [3] showed that if G is a metrizable topological group and $f : G \to \mathbb{R}$ is Baire measurable and satisfies (1.2) then f is continuous. Kochanek and Lewicki (see [4]) proved that if G is metrizable locally compact group and $f : G \to \mathbb{R}$ is Haar measurable and satisfies (1.2), then f is continuous.

As already mentioned in [2], Kochanek noticed that every function f defined on an abelian group G which is of the form $f = g \circ a$, where $g : \mathbb{R} \to \mathbb{R}$ is a solution of (1.2) described by Theorem 1.1 and a: $G \to \mathbb{R}$ is an additive function, is a solution of Equation (1.2).

Solutions of the Equation (1.3), according to [1], with the additional assumption that G is divisible by 6, are either $f \equiv 0$ or $f = \exp(|a|)$, where a: $G \to \mathbb{R}$ is an additive function. Without this additional assumption, however, we have the following remark (see [5]).

Remark 1.1. Let $f: G \to \mathbb{R}$, where G is an abelian group. Then, f satisfies

$$\max\{f(x+y), f(x-y)\} = f(x)f(y)$$

if and only if

- $f \equiv 0$

or

- $f = \exp \circ |a|$, for some additive function a

or

- there is a subgroup G_0 of G, such that

$$x, y \notin G_0 \Rightarrow (x+y \in G_0 \ \vee x-y \in G_0), \quad x, y \in G,$$

and

$$f(x) = \begin{cases} 1, & x \in G_0; \\ -1, & x \notin G_0. \end{cases}$$

The result concerning stability of (1.1) was presented by Volkmann during the 45th ISFE in Bielsko-Biala (Poland, 2007) (for the proof see [2]) and superstability of (1.3) was proved in [6].

In this paper, we deal with the stability of Equation (1.2) in the class of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

2. Main Result

We are going to prove

Theorem 2.1. *If* $\delta \geq 0$ *and* $f: \mathbb{R} \to \mathbb{R}$ *is a continuous function satisfying*

$$|\min\{f(x+y), f(x-y)\} - |f(x) - f(y)|| \leq \delta, \quad x, y \in \mathbb{R}, \tag{2.1}$$

then either f is bounded (and in such a case is "close" to the solution $F \equiv 0$ *of* (1.2)*) or there exists a constant* $c > 0$ *such that*

$$|f(x) - c|x|| \leq 21\delta, \quad x \in \mathbb{R}, \tag{2.2}$$

that is, f is "close" to the solution $F(x) = c|x|$ *of* (1.2).

We will write $\alpha \overset{\delta}{\sim} \beta$ instead of $|\alpha - \beta| \leq \delta$ to shorten the notation.

Notice that

- if $\alpha \overset{\delta_1}{\sim} \beta \overset{\delta_2}{\sim} \gamma$ then $\alpha \overset{\delta_1+\delta_2}{\sim} \gamma$,
- if $\alpha \leq \beta \overset{\delta}{\sim} \gamma$ then $\alpha \leq \gamma + \delta$,

- if $\alpha \overset{\delta}{\sim} \beta \leq \gamma$ then $\alpha \leq \gamma + \delta$,
- if $\alpha \overset{\delta}{\sim} \beta$ then for an arbitrary γ we have $|\alpha - \gamma| \overset{\delta}{\sim} |\beta - \gamma|$.

In the following lemma, we list some properties of functions satisfying (2.1) in more general settings.

Lemma 2.1. *Let G be an abelian group, $\delta, \varepsilon \geq 0$ and let $f : G \to \mathbb{R}$ be an arbitrary function satisfying*

$$\min\{f(x+y), f(x-y)\} \overset{\delta}{\sim} |f(x) - f(y)|, \quad x, y \in G. \tag{2.3}$$

Then

(i) $f(x) \geq -\delta$, $x \in G$,
(ii) $f(0) \overset{\delta}{\sim} 0$,
(iii) $f(x) \overset{2\delta}{\sim} f(-x)$, $x \in G$,
(iv) $f(x) \overset{2\delta}{\sim} |f(x)|$, $x \in G$,
(v) *for every $x, p \in G$ it holds*

$$f(p) \overset{\varepsilon}{\sim} 0 \;\Rightarrow\; [f(x+p) \overset{3\delta+\varepsilon}{\sim} f(x) \textit{ and } f(x-p) \overset{3\delta+\varepsilon}{\sim} f(x)].$$

Proof. The first assertion follows from

$$f(x) = \min\{f(x+0), f(x-0)\} \overset{\delta}{\sim} |f(x) - f(0)| \geq 0, \quad x \in G. \tag{2.4}$$

The second one we get putting $x = 0$ in (2.4). Next, notice that, using (ii), we have

$$|f(x) - f(-x)| \overset{\delta}{\sim} \min\{f(0), f(2x)\} \leq f(0) \leq \delta, \quad x \in G,$$

which proves (iii). Moreover,

$$|f(x)| \overset{\delta}{\sim} |f(x) - f(0)| \overset{\delta}{\sim} \min\{f(x), f(x)\} = f(x), \quad x \in G,$$

so we obtain (iv). To prove (v), let us assume that $f(p) \overset{\varepsilon}{\sim} 0$ and choose an arbitrary $x \in G$. Since

$$f(x) \overset{2\delta}{\sim} |f(x)| \overset{\varepsilon}{\sim} |f(x) - f(p)| \overset{\delta}{\sim} \min\{f(x-p), f(x+p)\},$$

we have either

$$f(x) \overset{3\delta+\varepsilon}{\sim} f(x+p) \leq f(x-p) \tag{2.5}$$

or

$$f(x) \overset{3\delta+\varepsilon}{\sim} f(x-p) \leq f(x+p).$$

Let us consider the first possibility, the second can be dealt with in the analogous way. Notice that

$$f(x-p) \leq |f(x-p)| \overset{\varepsilon}{\sim} |f(x-p) - f(p)| \overset{\delta}{\sim} \min\{f(x-2p), f(x)\} \leq f(x),$$

which yields

$$f(x-p) \le f(x) + \delta + \varepsilon.$$

The last inequality together with (2.5) finishes the proof of (v). □

Proof of Theorem 2.1. First, we notice that for every x,y, $z \in \mathbb{R}$ such that $x < y < z$, we have

$$(f(x) = f(z) > f(y) + 4\delta) \Rightarrow (\exists_{p \ge y+x} f(p) \overset{\delta}{\sim} 0). \tag{2.6}$$

Indeed, assume that x, y, $z \in \mathbb{R}$, $x < y < z$ and $f(x) = f(z) > f(y) + 4\delta$. Let us choose the greatest $x' \in [x, y]$ with $f(x') = f(x)$ and the smallest $z' \in [y, z]$ with $f(z') = f(z)$. The continuity of f assures the existence of x'', $z'' \in [x', z']$, $x'' < z''$, such that $f(x'') = f(z'')$ and $z'' - x'' = y - x'$. Of course,

$$z'' + x'' = y + (x'' - x') + x'' \ge y + x.$$

Moreover, in view of (2.1), we have

$$0 = |f(z'') - f(x'')| \overset{\delta}{\sim} \min\{f(z'' - x''), f(z'' + x'')\}. \tag{2.7}$$

So, if $f(z'' - x'') \overset{\delta}{\sim} 0$, by Lemma 2.1(v), we would have

$$f(x) = f(x') \overset{4\delta}{\sim} f(x' + (z'' - x'')) = f(x' + y - x') = f(y)$$

which is impossible. Therefore, (2.7) implies $f(z'' + x'') \overset{\delta}{\sim} 0$. Now, it is enough to put $p = z'' + x''$.

Assume that f is unbounded. We will show that

$$f(x) \le f(y) + 6\delta, \quad 0 < x < y. \tag{2.8}$$

Suppose on the contrary that there exist x, $y \in \mathbb{R}$, $0 < x < y$, with $f(x) > f(y) + 6\delta$. From the unboundness of f and parts (i) and (iii) of Lemma 2.1, we infer that $\lim_{x\to\infty} f(x) = \infty$. So we can find $z_1 > y$ with $f(z_1) = f(x)$. From (2.6) (with $z = z_1$), we deduce that there exists $p_1 \ge y + x$ such that $f(p_1) \overset{\delta}{\sim} 0$. Let us suppose that we have already defined p_1, p_2, ..., p_n in such a way that $f(p_k) \overset{\delta}{\sim} 0$ and $p_k \ge y + kx$, $k = 1, 2, ..., n$. Notice that, in view of Lemma 2.1(i),

$$f(x) > f(y) + 6\delta \ge -\delta + 6\delta = \delta + 4\delta \ge f(p_n) + 4\delta$$

so we can find $z_n > p_n$ with $f(z_n) = f(x)$. By (2.6) (with $y = p_n$ and $z = z_n$), we obtain that there exists $p_{n+1} \ge p_n + x \ge y + (n + 1)x$ such that $f(p_{n+1}) \overset{\delta}{\sim} 0$. Hence, we proved that there is a sequence $(p_n)_{n\in\mathbb{N}}$ increasing to infinity such that $f(p_n) \overset{\delta}{\sim} 0$ for $n \in \mathbb{N}$. Choose $p > 0$ satisfying $f(p) \overset{\delta}{\sim} 0$ and such that

$$M := \max f([0, p]) > 7\delta. \tag{2.9}$$

Let this maximum be taken at an $x \in (0, p)$. Notice that $f(2x) \le M + 4\delta$. This is obvious if $2x \le p$, in the opposite case, if $2x > p$, it follows from Lemma 2.1 part (v) and the fact that in such a case $2x - p \in [0, p)$, more precisely,

$$f(2x) = f(2x - p + p) \overset{4\delta}{\sim} f(2x - p) \le \max f([0, p]) = M.$$

Let us now choose $y > p$ with $f(y) = 2M$. We have

$$M = |f(y) - f(x) \overset{\delta}{\sim} \min\{f(y - x), f(y + x)\},$$

whence either $f(y - x) \overset{\delta}{\sim} M$ or $f(y + x) \overset{\delta}{\sim} M$. Let us consider the first possibility, the second is analogous. From $\min\{f(y - 2x), 2M) = \min\{f(y - 2x), f(y)\} \overset{\delta}{\sim} |f(y - x) - f(x)| = |f(y - x) - M| \leq \delta$ we deduce that $f(y - 2x) \leq 2\delta$. But $2\delta \geq f(y - 2x) \geq \min\{f(y - 2x), f(y + 2x)\} \overset{\delta}{\sim} |f(y) - f(2x)| \geq M - 4\delta$ which contradicts (2.9) and, thereby, ends the proof of (2.8).

We infer that

$$f(y - x) \leq f(y + x) + 6\delta, \quad 0 < x < y,$$

whence

$$\min\{f(y - x), f(y + x)\} \overset{6\delta}{\sim} f(y - x), \quad 0 < x < y.$$

Notice that (2.8) implies

$$|f(x) - f(y)| \overset{12\delta}{\sim} f(y) - f(x), \quad 0 < x < y.$$

Thereby,

$f(y - x) \overset{6\delta}{\sim} \min\{f(y - x), f(y + x)\} \overset{\delta}{\sim} |f(x) - f(y) \overset{12\delta}{\sim} f(y) - f(x)$, for $0 < x < y$, whence $f(y - x) \overset{19\delta}{\sim} f(y) - f(x)$ for $0 < x < y$. Consequently,

$$f(x + y) \overset{19\delta}{\sim} f(x) + f(y), \quad x, y > 0.$$

Since f restricted to $(0, \infty)$ is 19δ -approximately additive, there is an additive function a: $(0, \infty) \to \mathbb{R}$ such that $f(x) \overset{19\delta}{\sim} a(x)$ (see [7]). Moreover, since f is continuous, $a(x) = cx$ for some positive c. Assertions (ii) and (iii) of Lemma 2.1 finish the proof of (2.2). □

Remark. Kochanek noticed (oral communication) that we can decrease easily 21δ appearing in (2.2) to 19δ, by repeating the consideration from the proof, which we did for positive real halfline, for the negative real halfline. We would obtain $f(x) \overset{19\delta}{\sim} cx$, for $x > 0$, $f(0) \overset{\delta}{\sim} 0$, and $f(x) \overset{19\delta}{\sim} -c'x$, for $x < 0$, where c, c' are some positive constants. But, since $f(x) \overset{21\delta}{\sim} c|x|$, $x \in \mathbb{R}$, we can deduce that $c' = c$.

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Competing interests
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