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On the stability of pexider functional equation in non-archimedean spaces

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Abstract

In this paper, the Hyers-Ulam stability of the Pexider functional equation

$$f_1(x+y) + f_2(x+\sigma(y)) = f_3(x) + f_4(y)$$

in a non-Archimedean space is investigated, where σ is an involution in the domain of the given mapping *f*.

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1.Introduction

The stability problem for functional equations first was planed in 1940 by Ulam [1]:

Let G_1 be group and G_2 be a metric group with the metric $d(\cdot,\cdot)$. Does, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any mapping $f : G_1 \to G_2$ which satisfies $d(f(xy), f(x)f(y)) \leq \delta$ for all $x, y \in G_1$, there exists a homomorphism $h : G_1 \to G_2$ so that, for any $x \in G_1$, we have $d(f(x), h(x)) \leq \varepsilon$?

In 1941, Hyers [2] answered to the Ulam's question when G_1 and G_2 are Banach spaces. Subsequently, the result of Hyers was generalized by Aoki [3] for additive mappings and Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [4] has provided a lot of influences in the development of the Hyers-Ulam-Rassias stability of functional equations (for more details, see [5] where a discussion on definitions of the Hyers-Ulam stability is provided by Moszner, also [6-12]).

In this paper, we give a modification of the approach of Belaid et al. [13] in non-Archimedean spaces. Recently, Ciepliński [14] studied and proved stability of multi-additive mappings in non-Archimedean normed spaces, also see [15-22].

Definition 1.1. The function $|\cdot| : K \to \mathbb{R}$ is called a *non-Archimedean valuation* or *absolute value* over the field K if it satisfies following conditions: for any $a, b \in K$,

(1) $|a| \geq 0$;
(2) $|a| = 0$ if and only if $a = 0$;
(3) $|ab| = |a|\,|b|$
(4) $|a + b| \leq \max\{|a|, |b|\}$;

(5) there exists a member $a_0 \in K$ such that $|a_0| \neq 0, 1$.

A field K with a non-Archimedean valuation is called a *non-Archimedean field*.

Corollary 1.2. *$|-1| = |1| = 1$ and so, for any $a \in K$, we have $|-a| = |a|$. Also, if $|a| < |b|$ for any $a, b \in K$, then $|a + b| = |b|$.*

In a non-Archimedean field, the triangle inequality is satisfied and so a metric is defined. But an interesting inequality changes the usual *Archimedean* sense of the absolute value. For any $n \in \mathbb{N}$, we have $|n \cdot 1| \leq \mathbb{R}$. Thus, for any $a \in K$, $n \in \mathbb{N}$ and nonzero divisor $k \in \mathbb{Z}$ of n, the following inequalities hold:

$$|na| \leqslant |ka| \leqslant |a| \leqslant \left|\frac{a}{k}\right| \leqslant \left|\frac{a}{n}\right|. \qquad (1.1)$$

Definition 1.3. Let V be a vector space over a non-Archimedean field K. A *non-Archimedean norm* over V is a function $|| \cdot || : V \to \mathrm{R}$ satisfying the following conditions: for any $\alpha \in K$ and $u, v \in V$,

(1) $||u|| = 0$ if and only if $u = 0$;
(2) $||\alpha u|| = |\alpha|\ ||u||$;
(3) $||u + v|| \leq \max\{||u||, ||v||\}$.

Since $0 = ||0|| = ||v - v|| \leq \max\{||v||, ||-v||\} = ||v||$ for any $v \in V$, we have $||v|| \geq 0$. Any vector space V with a non-Archimedean norm $|| \cdot || : V \to \mathbb{R}$ is called a *non-Archimedean space*. If the metric $d(u, v) = ||u - v||$ is induced by a non-Archimedean norm $|| \cdot || : V \to \mathbb{R}$ on a vector space V which is complete, then $(V, || \cdot ||)$ is called a *complete non-Archimedean space*.

Proposition 1.4. ([23]) *A sequence $\{x_n\}_{n=1}^{\infty}$in a non-Archimedean space is a Cauchy sequence if and only if the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$converges to zero.*

Since any non-Archimedean norm satisfies the triangle inequality, any non-Archimedean norm is a continuous function from its domain to real numbers.

Proposition 1.5. *Let V be a normed space and E be a non-Archimedean space. Let $f : V \to E$ be a function, continuous at $0 \in V$ such that, for any $\times \in V$, $f(2x) = 2f(x)$ (for example, additive functions). Then, $f = 0$.*

Proof. Since $f(0) = 0$, for any $\varepsilon > 0$, there exists $\delta > 0$ that, for any $x \in V$ with $||x|| \leq \delta$,

$$||f(x) - f(0)|| = ||f(x)|| \leqslant \varepsilon$$

and, for any $x \in V$, there exists $n \in \mathbb{N}$ that $\left\|\frac{x}{2^n}\right\| \leqslant \delta$ and hence

$$||f(x)|| = \left\|2^n f\left(\frac{x}{2^n}\right)\right\| \leqslant \left\|f\left(\frac{x}{2^n}\right)\right\| \leqslant \varepsilon.$$

Since this inequality holds for all $\varepsilon > 0$, it follows that, for any $x \in V$, $f(x) = 0$. This completes the proof.

The preceding fact is a special case of a general result for non-Archimedean spaces, that is, *every continuous function from a connected space to a non-Archimedean space is constant*. This is a consequence of *totally disconnectedness* of every non-Archimedean space (see [23]).

2. Stability of quadratic and Cauchy functional equations

Throughout this section, we assume that V_1 is a normed space and V_2 is a complete non-Archimedean space. Let $\sigma : V_1 \to V_1$ be a continuous involution (i.e., $\sigma(x + y) = \sigma(x) + \sigma(y)$ and $\sigma(\sigma(x)) = x$) and $\phi : V_1 \times V_1 \to \mathbb{R}$ be a function with

$$\lim_{(x,y)\to(0,0)} \varphi(x,y) = 0 \tag{2.1}$$

and define a function $\varphi : V_1 \times V_1 \to \mathbb{R}$ by

$$\phi(x,y) = \sup_{n\in\mathbb{N}} \left\{ \varphi\left(\frac{x-\sigma(x)}{2}, \frac{y+\sigma(y)}{2}\right), \varphi\left(\frac{x+\sigma(x)}{2^n}, \frac{y+\sigma(y)}{2^n}\right), \varphi\left(\frac{x-\sigma(x)}{2^n}, \frac{y-\sigma(y)}{2^n}\right) \right\}, \tag{2.2}$$

which easily implies

$$\lim_{(x,y)\to(0,0)} \phi(x,y) = 0. \tag{2.3}$$

Theorem 2.1. *Suppose that* ϕ *satisfies the condition* 2.1 *and let* φ *is defined by Equation* 2.2. *If* $f : V_1 \to V_2$ *satisfies the inequality*

$$\left\| \frac{1}{2}f(x+y) + \frac{1}{2}f(x+\sigma(y)) - f(x) - f(y) \right\| \leqslant \varphi(x,y) \tag{2.4}$$

for all $x, y \in V_1$, *then there exists a unique solution* $q : V_1 \to V_2$ *of the functional equation*

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y) \tag{2.5}$$

such that

$$\|f(x) - q(x)\| \leqslant \phi(x,x) \tag{2.6}$$

for all $x \in V_1$.

Proof. Replacing x and y in Equation 2.4 with $\frac{x-\sigma(x)}{2}$ and $\frac{x+\sigma(x)}{2}$, respectively, we obtain

$$\left\| f(x) - f\left(\frac{x+\sigma(x)}{2}\right) - f\left(\frac{x-\sigma(x)}{2}\right) \right\| \leqslant \varphi\left(\frac{x-\sigma(x)}{2}, \frac{x+\sigma(x)}{2}\right). \tag{2.7}$$

Replacing x and y in Equation 2.4 with $\frac{x+\sigma(x)}{2}$ and $\frac{x-\sigma(x)}{2}$, respectively, we obtain

$$\left\| \frac{f(x)+f(\sigma(x))}{2} - f\left(\frac{x+\sigma(x)}{2}\right) - f\left(\frac{x-\sigma(x)}{2}\right) \right\| \leqslant \varphi\left(\frac{x+\sigma(x)}{2}, \frac{x-\sigma(x)}{2}\right). \tag{2.8}$$

Also, replacing both of x, y in Equation 2.4 with $\frac{x+\sigma(x)}{2}$, we get

$$\left\| f(x+\sigma(x)) - 2f\left(\frac{x+\sigma(x)}{2}\right) \right\| \leqslant \varphi\left(\frac{x+\sigma(x)}{2}, \frac{x+\sigma(x)}{2}\right)$$

and so, for any $n \in \mathbb{N}$, we get

$$\left\| f\left(\frac{x+\sigma(x)}{2^n}\right) - 2f\left(\frac{x+\sigma(x)}{2^{n+1}}\right) \right\| \leqslant \varphi\left(\frac{x+\sigma(x)}{2^{n+1}}, \frac{x+\sigma(x)}{2^{n+1}}\right). \tag{2.9}$$

Similarly, replacing both of x, y in Equation 2.4 with $\frac{x-\sigma(x)}{2}$, we get

$$\begin{aligned}\left\|f\left(x-\sigma(x)\right)+f(0)-4f\left(\frac{x-\sigma(x)}{2}\right)\right\| &\leqslant \left\|\frac{1}{2}f(x-\sigma(x))+\frac{1}{2}f(0)-2f\left(\frac{x-\sigma(x)}{2}\right)\right\| \\ &\leqslant \varphi\left(\frac{x-\sigma(x)}{2},\frac{x-\sigma(x)}{2}\right).\end{aligned} \tag{2.10}$$

Replacing x in Equation 2.7 with $\frac{x+\sigma(x)}{2}$, we obtain

$$\|f(0)\| \leqslant \varphi\left(0,\frac{x+\sigma(x)}{2}\right)$$

for all $x \in V_1$ and so, by assumption Equation 2.1,

$$\lim_{n\to\infty}\varphi\left(0,\frac{x+\sigma(x)}{2^n}\right)=0.$$

Thus, $f(0) = 0$ and the inequality Equation 2.10 reduces to

$$\left\|f(x-\sigma(x))-4f\left(\frac{x-\sigma(x)}{2}\right)\right\| \leqslant \varphi\left(\frac{x-\sigma(x)}{2},\frac{x-\sigma(x)}{2}\right)$$

and so,

$$\left\|f\left(\frac{x-\sigma(x)}{2^n}\right)-4f\left(\frac{x-\sigma(x)}{2^{n+1}}\right)\right\| \leqslant \varphi\left(\frac{x-\sigma(x)}{2^{n+1}},\frac{x-\sigma(x)}{2^{n+1}}\right). \tag{2.11}$$

For any $n \in \mathbb{N}$, define

$$q_n(x)=2^{n-1}f\left(\frac{x+\sigma(x)}{2^n}\right)+2^{2n-2}f\left(\frac{x-\sigma(x)}{2^n}\right)$$

and

$$\phi_n(x,y)=\max_{1\leqslant i\leqslant n}\left\{\varphi\left(\frac{x-\sigma(x)}{2},\frac{y+\sigma(y)}{2}\right),\varphi\left(\frac{x+\sigma(x)}{2^i},\frac{y+\sigma(y)}{2^i}\right),\varphi\left(\frac{x-\sigma(x)}{2^i},\frac{y-\sigma(y)}{2^i}\right)\right\}.$$

Then,

$$\phi_n(x,y) \leqslant \phi(x,y) \tag{2.12}$$

for all x, $y \in V_1$.

From Equations (2.9) and (2.11), we get

$$\begin{aligned}\|q_n(x)-q_{n+1}(x)\| &\leqslant \max\left\{\left\|2^{n-1}f\left(\frac{x+\sigma(x)}{2^n}\right)-2^nf\left(\frac{x+\sigma(x)}{2^{n+1}}\right)\right\|,\right.\\ &\qquad\left.\left\|2^{2n-2}f\left(\frac{x-\sigma(x)}{2^n}\right)-2^{2n}f\left(\frac{x-\sigma(x)}{2^{n+1}}\right)\right\|\right\}\\ &\leqslant \max\left\{\left\|f\left(\frac{x+\sigma(x)}{2^n}\right)-2f\left(\frac{x+\sigma(x)}{2^{n+1}}\right)\right\|,\right.\\ &\qquad\left.\left\|f\left(\frac{x-\sigma(x)}{2^n}\right)-4f\left(\frac{x-\sigma(x)}{2^{n+1}}\right)\right\|\right\}\\ &\leqslant \max\left\{\varphi\left(\frac{x+\sigma(x)}{2^{n+1}},\frac{x+\sigma(x)}{2^{n+1}}\right),\varphi\left(\frac{x-\sigma(x)}{2^{n+1}},\frac{x-\sigma(x)}{2^{n+1}}\right)\right\}\end{aligned}$$

and so Proposition 1.4 and the hypothesis Equation 2.1 imply that $\{q_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence. Since V_2 is complete, the sequence $\{q_n(x)\}_{n=1}^{\infty}$ converges to a point of V_2 which defines a mapping $q : V_1 \to V_2$.

Now, we prove

$$\|f(x) - q_n(x)\| \leqslant \phi(x, x) \tag{2.13}$$

for all $n \in \mathbb{N}$. Since Equation 2.7 implies

$$\|f(x) - q_1(x)\| \leqslant \varphi\left(\frac{x - \sigma(x)}{2}, \frac{x + \sigma(x)}{2}\right) \leqslant \phi_1(x, x).$$

Assume that $||f(x) - q_n(x)|| \leq \varphi_n(x, x)$ holds for some $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}\|f(x) - q_{n+1}(x)\| &\leqslant \max\{\|f(x) - q_n(x)\|, \|q_n(x) - q_{n+1}(x)\|\}\\ &\leqslant \max\left\{\phi_n(x, x), \varphi\left(\frac{x + \sigma(x)}{2^{n+1}}, \frac{y + \sigma(y)}{2^{n+1}}\right), \varphi\left(\frac{x - \sigma(x)}{2^{n+1}}, \frac{y - \sigma(y)}{2^{n+1}}\right)\right\}\\ &= \phi_{n+1}(x, x).\end{aligned}$$

Therefore, by induction on n, Equation 2.13 follows from Equation 2.12. Taking the limit of both sides of Equation 2.13, we prove that q satisfies Equation 2.6.

For any $n \in \mathbb{N}$ and $x, y \in V_1$, we have

$$\begin{aligned}&\|q_n(x + y) + q_n(x + \sigma(y)) - 2q_n(x) - 2q_n(y)\|\\ &\leqslant \max\left\{\left\|f\left(\frac{x + y + \sigma(x + y)}{2^n}\right) + f\left(\frac{x + \sigma(y) + \sigma(x) + y}{2^n}\right) - 2f\left(\frac{x + \sigma(x)}{2^n}\right) - 2f\left(\frac{y + \sigma(y)}{2^n}\right)\right\|,\right.\\ &\qquad\left.\left\|f\left(\frac{x + y - \sigma(x + y)}{2^n}\right) + f\left(\frac{x + \sigma(y) - \sigma(x) - y}{2^n}\right) - 2f\left(\frac{x - \sigma(x)}{2^n}\right) - 2f\left(\frac{y - \sigma(y)}{2^n}\right)\right\|\right\}\\ &\leqslant \max\left\{\varphi\left(\frac{x + \sigma(x)}{2^n}, \frac{y + \sigma(y)}{2^n}\right), \varphi\left(\frac{x - \sigma(x)}{2^n}, \frac{y - \sigma(y)}{2^n}\right)\right\}\end{aligned}$$

and so, by the continuity of non-Archimedean norm and taking the limit of both sides of the above inequality, we get

$$\|q(x + y) + q(x + \sigma(y)) - 2q(x) - 2q(y)\| = 0.$$

Thus, q is a solution of the Equation 2.5 which satisfies Equation 2.6.

Then, by replacing x, y with $\frac{x + \sigma(x)}{2}$ in Equation 2.5, we obtain the following identities: for any solution $g : V_1 \to V_2$ of the Equation (2.5),

$$g(x + \sigma(x)) = 2g\left(\frac{x + \sigma(x)}{2}\right), \quad g(x - \sigma(x)) = 4g\left(\frac{x - \sigma(x)}{2}\right)$$

and

$$g(x) = g\left(\frac{x + \sigma(x)}{2}\right) + g\left(\frac{x - \sigma(x)}{2}\right). \tag{2.14}$$

By induction on n, one can show that

$$g(x + \sigma(x)) = 2^n g\left(\frac{x + \sigma(x)}{2^n}\right) \tag{2.15}$$

and

$$g\left(x-\sigma(x)\right)=4^{n}g\left(\frac{x-\sigma(x)}{2^{n}}\right) \tag{2.16}$$

for all $n \in \mathbb{N}$.

Now, suppose that $q' : V_1 \to V_2$ is another solution of 2.5 that satisfies the Equation 2.6. It follows from Equations 2.14 to 2.16 that

$$\begin{aligned}
&\|q(x)-q'(x)\|\\
&\leqslant \max\left\{\left\|2^{n-1}q(\frac{x+\sigma(x)}{2^{n}})-2^{n-1}q'\left(\frac{x+\sigma(x)}{2^{n}}\right)\right\|,\right.\\
&\qquad\left.\left\|2^{2n-2}q\left(\frac{x-\sigma(x)}{2^{n}}\right)-2^{2n-2}q'\left(\frac{x-\sigma(x)}{2^{n}}\right)\right\|\right\}\\
&\leqslant \max\left\{\left\|q\left(\frac{x+\sigma(x)}{2^{n}}\right)-q'\left(\frac{x+\sigma(x)}{2^{n}}\right)\right\|,\left\|q\left(\frac{x-\sigma(x)}{2^{n}}\right)-q'\left(\frac{x-\sigma(x)}{2^{n}}\right)\right\|\right\}\\
&\leqslant \max\left\{\left\|f\left(\frac{x+\sigma(x)}{2^{n}}\right)-q\left(\frac{x+\sigma(x)}{2^{n}}\right)\right\|,\left\|f\left(\frac{x+\sigma(x)}{2^{n}}\right)-q'\left(\frac{x+\sigma(x)}{2^{n}}\right)\right\|,\right.\\
&\qquad\left.\left\|f\left(\frac{x-\sigma(x)}{2^{n}}\right)-q\left(\frac{x-\sigma(x)}{2^{n}}\right)\right\|,\left\|f\left(\frac{x-\sigma(x)}{2^{n}}\right)-q'\left(\frac{x-\sigma(x)}{2^{n}}\right)\right\|\right\}\\
&\leqslant \max\left\{\phi\left(\frac{x+\sigma(x)}{2^{n}},\frac{x+\sigma(x)}{2^{n}}\right),\phi\left(\frac{x-\sigma(x)}{2^{n}},\frac{x-\sigma(x)}{2^{n}}\right)\right\}.
\end{aligned}$$

Therefore, since

$$\lim_{n\to\infty}\phi\left(\frac{x+\sigma(x)}{2^{n}},\frac{x+\sigma(x)}{2^{n}}\right)=\lim_{n\to\infty}\phi\left(\frac{x-\sigma(x)}{2^{n}},\frac{x-\sigma(x)}{2^{n}}\right)=0,$$

we have $q(x) = q'(x)$ for all $x \in V_1$. This completes the proof.

In the proof of the next theorem, we need a result concerning the Cauchy functional equation

$$f(x+y)=f(x)+f(y), \tag{2.17}$$

which has been established in [20].

Theorem 2.2. ([20]) *Suppose that* $\phi(x, y)$ *satisfies the condition* 2.1 *and, for a mapping* $f : V_1 \to V_2$,

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varphi(x,y) \tag{2.18}$$

for all $x, y \in V_1$. *Then, there exists a unique solution* $q : V_1 \to V_2$ *of the Equation* 2.17 *such that*

$$\|f(x)-q(x)\| \leqslant \psi(x,x) \tag{2.19}$$

for all $x \in V_1$, *where*

$$\psi(x,y)=\sup_{n\in\mathbb{N}}\varphi\left(\frac{x}{2^{n}},\frac{y}{2^{n}}\right)$$

for all $x, y \in V_1$

3. Stability of the Pexider functional equation

In this section, we assume that V_1 is a normed space and V_2 is a complete non-Archimedean space. For any mapping $f : V_1 \to V_2$, we define two mappings F^e and F^o as follows:

$$F^e(x) = \frac{F(x) + F\left(\sigma(x)\right)}{2}, \quad F^o(x) = \frac{F(x) - F\left(\sigma(x)\right)}{2}$$

and also define $F(x) = f(x) - f(0)$. Then, we have obviously

$$\begin{gathered} F(0) = F^e(0) = F^o(0) = 0, \quad F^e(x + \sigma(x)) = F(x + \sigma(x)), \quad F^o(x + \sigma(x)) = 0 \\ F^o(\sigma(x)) = -F^o(x), \quad F^e(\sigma(x)) = F^e(x). \end{gathered} \tag{3.1}$$

Theorem 3.1. *Let $\sigma : V_1 \to V_1$ be a continuous involution and the mappings $f_i : V_1 \to V_2$ for $i = 1, 2, 3, 4$ and $\delta > 0$, satisfy*

$$\left\| f_1(x + y) + f_2(x + \sigma(y)) - f_3(x) - f_4(y) \right\| \leqslant \delta \tag{3.2}$$

for all $x, y \in V_1$, then there exists a unique solution $q : V_1 \to V_2$ of the Equation 2.5 and a mapping $v : V_1 \to V_2$ which satisfies

$$v(x + y) = v(x + \sigma(y))$$

for all $x, y \in V_1$ and exists two additive mappings $\mathbb{A}_1, \mathbb{A}_2 : V_1 \to V_2$ such that $\mathbb{A}_i \circ \sigma = -\mathbb{A}_i$ for $i = 1, 2$ and, for all $x \in V_1$,

$$\left\| 2f_1(x) - \mathbb{A}_1(x) - \mathbb{A}_2(x) - v(x) - q(x) - 2f_1(0) \right\| \leqslant \frac{1}{|2|}\delta, \tag{3.3}$$

$$\left\| 2f_2(x) - \mathbb{A}_1(x) + \mathbb{A}_2(x) + v(x) - q(x) - 2f_2(0) \right\| \leqslant \frac{1}{|2|}\delta, \tag{3.4}$$

$$\left\| f_3(x) - \mathbb{A}_2(x) - q(x) - f_3(0) \right\| \leqslant \frac{1}{|2|}\delta, \tag{3.5}$$

$$\left\| f_4(x) - \mathbb{A}_1(x) - q(x) - f_4(0) \right\| \leqslant \frac{1}{|2|}\delta. \tag{3.6}$$

Proof. It follows from (3.2) that

$$\begin{aligned} &\left\| F_1(x + y) + F_2(x + \sigma(y)) - F_3(x) - F_4(y) \right\| \\ &\leqslant \max\left\{ \left\| f_1(x + y) + f_2(x + \sigma(y)) - f_3(x) - f_4(y) \right\|, \right. \\ &\qquad \left. \left\| f_1(0) + f_2(0) - f_3(0) - f_4(0) \right\| \right\} \\ &\leqslant \max\{\delta, \delta\} \\ &= \delta \end{aligned}$$

and so, for all $x, y \in V_1$,

$$\begin{aligned} &\left\| 2F_1^e(x + y) + 2F_2^e(x + \sigma(y)) - 2F_3^e(x) - 2F_4^e(y) \right\| \\ &\leqslant \max\left\{ \left\| F_1(x + y) + F_2(x + \sigma(y)) - F_3(x) - F_4(y) \right\|, \right. \\ &\qquad \left. \left\| F_1(\sigma(x) + \sigma(y)) + F_2(\sigma(x) + \sigma(\sigma(y))) - F_3(\sigma(x)) - F_4(\sigma(y)) \right\| \right\} \\ &\leqslant \delta. \end{aligned}$$

then,

$$\left\| F_1^e(x + y) + F_2^e(x + \sigma(y)) - F_3^e(x) - F_4^e(y) \right\| \leqslant \frac{1}{|2|}\delta. \tag{3.7}$$

Similarly, we have

$$\left\|F_1^o(x+y)+F_2^o(x+\sigma(y))-F_3^o(x)-F_4^o(y)\right\| \leqslant \frac{1}{|2|}\delta \tag{3.8}$$

for all $x, y \in V_1$.

Now, first by putting $y = 0$ in Equation 3.7 and applying Equation 3.2 and second by putting $x = 0$ in Equation 3.7 and applying Equation 3.2 once again, we obtain

$$\left\|F_1^e(x)+F_2^e(x)-F_3^e(x)\right\| \leqslant \frac{1}{|2|}\delta, \tag{3.9}$$

$$\left\|F_1^e(y)+F_2^e(y)-F_4^e(y)\right\| \leqslant \frac{1}{|2|}\delta, \tag{3.10}$$

for all $x, y \in V_1$ and so these inequalities with Equation 3.7 imply

$$\begin{aligned} &\left\|F_1^e(x+y)+F_2^e(x+\sigma(y))-(F_1^e+F_2^e)(x)-(F_1^e+F_2^e)(y)\right\| \\ &\leqslant \max\left\{\left\|F_1^e(x+y)+F_2^e(x+\sigma(y))-F_3^e(x)-F_4^e(y)\right\|,\right. \\ &\qquad \left.\left\|F_1^e(x)+F_2^e(x)-F_3^e(x)\right\|, \left\|F_1^e(y)+F_2^e(y)-F_4^e(y)\right\|\right\} \\ &\leqslant \frac{1}{|2|}\delta. \end{aligned} \tag{3.11}$$

Replacing y with $\sigma(y)$ in Equation 3.11, we get

$$\begin{aligned} &\left\|F_1^e(x+\sigma(y))+F_2^e(x+y)-(F_1^e+F_2^e)(x)-(F_1^e+F_2^e)(\sigma(y))\right\| \\ &\leqslant \frac{1}{|2|}\delta. \end{aligned} \tag{3.12}$$

It follows from Equations 3.1, 3.11 and 3.12 that

$$\begin{aligned} &\left\|(F_1^e+F_2^e)(x+y)+(F_1^e+F_2^e)(x+\sigma(y))-2(F_1^e+F_2^e)(x)-2(F_1^e+F_2^e)(y)\right\| \\ &\leqslant \frac{1}{|2|}\delta. \end{aligned}$$

By Theorem 2.1 of [24], there exists a unique solution $q : V_1 \to V_2$ of the functional Equation 2.5 such that

$$\left\|(F_1^e+F_2^e)(x)-q(x)\right\| \leqslant \frac{1}{|2|}\delta \tag{3.13}$$

for all $x \in V_1$.

As a result of the inequalities Equations 3.11 and 3.12, we have

$$\left\|(F_1^e-F_2^e)(x+y)-(F_1^e-F_2^e)(x+\sigma(y))\right\| \leqslant \frac{1}{|2|}\delta. \tag{3.14}$$

It is easily seen that the mapping $v : V_1 \to V_2$ defined by

$$v(x) = (F_1^e-F_2^e)\left(\frac{x+\sigma(x)}{2}\right)$$

is a solution of the functional equation

$$v(x+y) = v(x+\sigma(y))$$

for all $x, y \in V_1$.

Replacing both of x, y in Equation 3.14 with $\frac{x}{2}$, We get

$$\left\|(F_1^e - F_2^e)(x) - v(x)\right\| \leqslant \frac{1}{|2|}\delta \tag{3.15}$$

for all $x \in V_1$. Now, Equations 3.13 and 3.15 imply

$$\begin{aligned}\left\|2F_1^e(x) - q(x) - v(x)\right\| &\leqslant \left\|(F_1^e + F_2^e)(x) - q(x) + (F_1^e - F_2^e)(x) - v(x)\right\| \\ &\leqslant \max\left\{\left\|(F_1^e + F_2^e)(x) - q(x)\right\|, \left\|(F_1^e - F_2^e)(x) - v(x)\right\|\right\} \\ &\leqslant \frac{1}{|2|}\delta\end{aligned} \tag{3.16}$$

and

$$\left\|2F_2^e(x) - q(x) + v(x)\right\| \leqslant \frac{1}{|2|}\delta. \tag{3.17}$$

Similarly, it follows from the inequalities Equations 3.7, 3.10 and 3.13 that

$$\left\|F_3^e(x) - q(x)\right\| \leqslant \frac{1}{|2|}\delta, \tag{3.18}$$

$$\left\|F_4^e(x) - q(x)\right\| \leqslant \frac{1}{|2|}\delta. \tag{3.19}$$

Since Equation 3.8 implies

$$\left\|F_3^o(x) - F_1^o(x) - F_2^o(x)\right\| \leqslant \frac{1}{|2|}\delta, \tag{3.20}$$

$$\left\|F_4^o(y) - F_1^o(y) - F_2^o(y)\right\| \leqslant \frac{1}{|2|}\delta \tag{3.21}$$

for all $x, y \in V_1$, we have

$$\left\|2F_1^o(x) - F_3^o(x) - F_4^o(x)\right\| \leqslant \frac{1}{|2|}\delta, \tag{3.22}$$

$$\left\|2F_2^o(x) - F_3^o(x) + F_4^o(x)\right\| \leqslant \frac{1}{|2|}\delta \tag{3.23}$$

for all $x \in V_1$. Now, from Equations 3.8 and 3.20, we obtain

$$\begin{aligned}&\left\|F_3^o(x+y) + F_3^o(x+\sigma(y)) - 2F_3^o(x)\right\| \\ &\leqslant \max\left\{\left\|F_3^o(x+y) - F_1^o(x+y) - F_2^o(x+y)\right\|,\right. \\ &\qquad \left\|F_3^o(x+\sigma(y)) - F_1^o(x+\sigma(y)) - F_2^o(x+\sigma(y))\right\|, \\ &\qquad \left\|F_1^o(x+y) + F_2^o(x+\sigma(y)) - F_3^o(x) - F_4^o(y)\right\|, \\ &\qquad \left.\left\|F_1^o(x+\sigma(y)) + F_2^o(x+y) - F_3^o(x) - F_4^o(\sigma(y))\right\|\right\} \\ &\leqslant \frac{1}{|2|}\delta\end{aligned} \tag{3.24}$$

and so, by interchanging role of x, y in the preceding inequality,

$$\begin{aligned} &\left\| F_3^o(y+x) + F_3^o(y+\sigma(x)) - 2F_3^o(y) \right\| \\ &\leqslant \frac{1}{|2|}\delta \end{aligned} \tag{3.25}$$

for all $x, y \in V_1$. Since $y + \sigma(x) = \sigma(x + \sigma(y))$, it follows from Equations 3.1, 3.24 and 3.25 that

$$\left\| 2F_3^o(x+y) - 2F_3^o(x) - 2F_3^o(y) \right\| \leqslant \frac{1}{|2|}\delta. \tag{3.26}$$

By Theorem 2.2, there exists a unique additive mapping $\mathbb{A}_1 : V_1 \to V_2$ such that

$$\left\| F_3^o(x) - \mathbb{A}_1(x) \right\| \leqslant \frac{1}{|2|}\delta. \tag{3.27}$$

Since

$$\left\| \mathbb{A}_1(x) + \mathbb{A}_1(\sigma(x)) \right\| \leqslant \frac{1}{|2|}\delta,$$

for all $x \in V_1$, we deduce $\mathbb{A}_1(\sigma(x)) = -\mathbb{A}_1(x)$ for all $x \in V_1$.

By a similar deduction, Equations 3.8 and 3.21 imply that there exists a unique additive mapping $\mathbb{A}_2 : V_1 \to V_2$ such that

$$\left\| F_4^o(x) - \mathbb{A}_2(x) \right\| \leqslant \frac{1}{|2|}\delta. \tag{3.28}$$

Moreover, we have $\mathbb{A}_2(\sigma(x)) = -\mathbb{A}_2(x)$ for all $x \in V_1$. Thus, by Equations 3.16, 3.22, 3.27 and 3.28, we obtain

$$\begin{aligned} &\left\| 2F_1(x) - q(x) - v(x) - \mathbb{A}_1(x) - \mathbb{A}_2(x) \right\| \\ &\leqslant \max\left\{ \left\| 2F_1^e(x) - q(x) - v(x) \right\|, \left\| 2F_1^o(x) - F_3^o(x) - F_4^o(x) \right\|, \right. \\ &\qquad \left. \left\| F_3^o(x) - \mathbb{A}_1(x) \right\|, \left\| F_4^o(x) - \mathbb{A}_2(x) \right\| \right\} \\ &\leqslant \frac{1}{|2|}\delta. \end{aligned} \tag{3.29}$$

This proves Equation 3.3. Similarly, one can prove Equations 3.4 to 3.6.

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Authors' contributions

All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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