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A geometrical constant and normal normal structure in Banach Spaces

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Abstract

Recently, we introduced a new coefficient as a generalization of the modulus of smoothness and Pythagorean modulus such as $J_{X,\,p}(t)$. In this paper, We can compute the constant $J_{X,\,p}(1)$ under the absolute normalized norms on $\mathbb{R}^2$ by means of their corresponding continuous convex functions on [0, 1]. Moreover, some sufficient conditions which imply uniform normal structure are presented.

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1. Introduction and preliminaries

We assume that X and X^* stand for a Banach space and its dual space, respectively. By S_X and B_X we denote the unit sphere and the unit ball of a Banach space X, respectively. Let C be a non-empty bounded closed convex subset of a Banach space X. A mapping $T : C \to C$ is said to be non-expansive provided the inequality

$$\|Tx - Ty\| \le \|x - y\|$$

holds for every $x, y \in C$. A Banach space X is said to have the fixed point property if every non-expansive mapping $T : C \to C$ has a fixed point, where C is a non-empty bounded closed convex subset of a Banach space X.

Recall that a Banach space X is called uniformly non-square if there exists $\delta > 0$ such that $||x + y||/2 \le 1 - \delta$ or $||x - y||/2 \le 1 - \delta$ whenever $x, y \in S_X$. A bounded convex subset K of a Banach space X is said to have normal structure if for every convex subset H of K that contains more than one point, there exists a point $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < \sup\{\|x - y\| : x, y \in H\}.$$

A Banach space X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any closed bounded convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| : y \in K\} < c\,\sup\{\|x - y\| : x, y \in K\}.$$

It was proved by Kirk that every reflexive Banach space with normal structure has the fixed point property.

There are several constants defined on Banach spaces such as the James [1] and von Neumann-Jordan constants [2]. It has been shown that these constants are very useful in geometric theory of Banach spaces, which enable us to classify several important concept of Banach spaces such as uniformly non-squareness and uniform normal structure [3-8]. On the other hand, calculation of the constant for some concrete spaces is also of some interest [2,5,6,9].

Recently, we introduced a new coefficient as a generalization of the modulus of smoothness and Pythagorean modulus such as $J_{X,\,p}(t)$.

Definition 1.1. Let $x \in S_X$, $y \in S_X$. For any $t > 0$, $1 \le p < \infty$ we set

$$J_{X,\,p}(t) = \sup\left\{\left(\frac{||x+ty||^p + ||x-ty||^p}{2}\right)^{\frac{1}{p}}\right\}.$$

Some basic properties of this new coefficient are investigated in [6]. In particular, we compute the new coefficient in the Banach spaces l_r, L_r, $l_{1,\,\infty}$ and give rough estimates of the constant in some concrete Banach spaces. In fact, the constant $J_{X,\,p}(1)$ is also important from the below Corollary in [6].

Corollary 1.2. If $J_{X,\,p}(1) < 2^{1-\frac{1}{p}}(1+\omega(X)^p)^{\frac{1}{p}}$. Then $R(X) < 2$, where $R(X)$ and $\omega(X)$ stand for García-Falset constant and the coefficient of weak orthogonality, respectively (see [10,11]). It is well known that a reflexive Banach space X with $R(X) < 2$ enjoys the fixed property (see [10]).

In this paper, we compute the constant $J_{X,\,p}(1)$ under the absolute normalized norms on $\mathbb{R}^2$, and give exact values of the constant $J_{X,\,p}(1)$ in some concrete Banach spaces. Moreover, some sufficient conditions which imply uniform normal structure are presented.

Recall that a norm on $\mathbb{R}^2$ is called absolute if $||(z, w)|| = ||(|z|, |w|)||$ for all $z, w \in \mathbb{R}$ and normalized if $||(1,0)|| = ||(0,1)||$. Let N_α denote the family of all absolute normalized norms on $\mathbb{R}^2$, and let Ψ denote the family of all continuous convex functions on $[0, 1]$ such that $\psi(1) = \psi(0) = 1$ and $\max\{1 - s, s\} \le \psi(s) \le 1 (0 \le s \le 1)$. It has been shown that N_α and Ψ are a one-to-one correspondence in view of the following proposition in [12].

Proposition 1.3. If $||\cdot|| \in N_\alpha$, then $\psi(s) = ||(1 - s, s)|| \in \Psi$. On the other hand, if $\psi(s) \in \Psi$, defined a norm $||\cdot||_\psi$ as

$$\|(z,\omega)\|_\psi := \begin{cases} (|z|+|\omega|)\psi\left(\dfrac{|\omega|}{|z|+|\omega|}\right), & (z,\omega) \neq (0,0), \\ 0, & (z,\omega) = (0,0). \end{cases}$$

then the norm $||\cdot||_\psi \in N_\alpha$.

A simple example of absolute normalized norm is usual $l_r (1 \le r \le \infty)$ norm. From Proposition 1.3, one can easily get the corresponding function of the l_r norm:

$$\psi_r(s) = \begin{cases} \{(1-s)^r + s^r\}^{1/r}, & 1 \le r < \infty, \\ \max\{1-s, s\}, & r = \infty. \end{cases}$$

Also, the above correspondence enable us to get many non-l_r norms on $\mathbb{R}^2$. One of the properties of these norms is stated in the following result.

Proposition 1.4. Let ψ, $\phi \in \Psi$ and $\phi \leq \psi$. Put $M = \max_{0\leq s\leq 1} \frac{\psi(s)}{\varphi(s)}$, then

$$\|\cdot\|_\varphi \leq \|\cdot\|_\psi \leq M\|\cdot\|_\varphi.$$

The Cesàro sequence space was defined by Shue [13] in 1970. It is very useful in the theory of matrix operators and others. Let l be the space of real sequences.

For $1 < p < \infty$, the Cesàro sequence space ces_p is defined by

$$ces_p = \left\{ x \in l : \|x\| = \left\|(x(i))\right\| = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right)^p\right)^{1/p} < \infty \right\}$$

The geometry of Cesàro sequence spaces have been extensively studied in [14-16]. Let us restrict ourselves to the two-dimensional Cesàro sequence space $\mathrm{ces}_p^{(2)}$ which is just $\mathbb{R}^2$ equipped with the norm defined by

$$\left\|(x, y)\right\| = \left(|x|^p + \left(\frac{|x| + |y|}{2}\right)^p\right)^{1/p}$$

2. Geometrical constant $J_{X,\,p}(1)$ and absolute normalized norm

In this section, we give a simple method to determine and estimate the constant $J_{X,\,p}(1)$ of absolute normalized norms on $\mathbb{R}^2$. For a norm $||\cdot||$ on $\mathbb{R}^2$, we write $J_{X,\,p}(1)(||\cdot||)$ for $J_{X,\,p}(1)(\mathbb{R}^2, ||\cdot||)$. The following is a direct result of Proposition 2.4 in [6].

Proposition 2.1. Let X be a non-trivial Banach space. Then

$$J_{X,\,p}(t) = \sup\left\{\left(\frac{||x+ty||^p + ||x-ty||^p}{2\max(||x||^p, ||y||^p)}\right)^{\frac{1}{p}} x, y \in X, ||x|| + ||y|| \neq 0\right\}.$$

Proposition 2.2. Let X be the space l_r or $L_r[0, 1]$ with $\dim X \geq 2$ (see [6])

(1) Let $1 < r \leq 2$ and $1/r + 1/r' = 1$. Then for all $t > 0$

if $1 < p < r'$ then $J_{X,\,p}(t) = (1 + t^r)^{\frac{1}{r}}$.

if $r' \leq p < \infty$ then $J_{X,\,p}(t) \leq (1 + Kt^r)^{\frac{1}{r}}$, for some $K \geq 1$.

(2) Let $2 \leq r < \infty$, $1 \leq p < \infty$ and $h = \max\{r, p\}$. Then

$$J_{X,\,p}(t) = \left(\frac{(1+t)^h + |1-t|^h}{2}\right)^{\frac{1}{h}} \text{ for all } t > 0.$$

Proposition 2.3. Let $\phi \in \Psi$ and $\psi(s) = \phi(1 - s)$. Then

$$J_{X,\,p}(t)(||\cdot||_\varphi) = J_{X,\,p}(t)(||\cdot||_\psi)$$

Proof. For any $x = (a, b) \in \mathbb{R}^2$ and $a \neq 0$, $b \neq 0$, put $\tilde{x} = (b, a)$. Then

$$||x||_\varphi = (|a| + |b|)\varphi\left(\frac{|b|}{|a| + |b|}\right) = (|b| + |a|)\psi\left(\frac{|a|}{|a| + |b|}\right) = ||\tilde{x}||_\psi.$$

Consequently, we have

$$\begin{aligned} J_{X,p}(t)(\|\cdot\|_\varphi) &= \sup\left\{\left(\frac{\|x+ty\|^p+\|x-ty\|^p}{2\max(\|x\|^p,\|y\|^p)}\right)^{\frac{1}{p}} x,y\in X, \|x\|+\|y\| \neq 0\right\} \\ &= \sup\left\{\left(\frac{\|\tilde{x}+t\tilde{y}\|^p+\|\tilde{x}-t\tilde{y}\|^p}{2\max(\|\tilde{x}\|^p,\|\tilde{y}\|^p)}\right)^{\frac{1}{p}} \tilde{x},\tilde{y}\in X, \|\tilde{x}\|+\|\tilde{y}\| \neq 0\right\} \\ &= J_{X,p}(t)(\|\cdot\|_\psi). \end{aligned}$$

We now consider the constant $J_{X,\ p}(1)$ of a class of absolute normalized norms on $\mathbb{R}^2$. Now let us put

$$M_1 = \max_{0\le s\le 1}\frac{\psi_r(s)}{\psi(s)} \text{ and } M_2 = \max_{0\le s\le 1}\frac{\psi(s)}{\psi_r(s)}.$$

Theorem 2.4. Let $\psi \in \Psi$ and $\psi \le \psi_r$ $(2 \le r < \infty)$. If the function $\frac{\psi_r(s)}{\psi(s)}$ attains its maximum at $s = 1/2$ and $r \ge p$, then

$$J_{X,p}(1)(\|\cdot\|_\psi) = \frac{1}{\psi(1/2)}.$$

Proof. By Proposition 1.4, we have $\|\cdot\|_\psi \le \|\cdot\|_r \le M_1\|\cdot\|_\psi$. Let $x, y \in X$, $(x, y) \neq (0, 0)$, where $X = \mathbb{R}^2$. Then

$$\begin{aligned} \|x+ty\|_\psi^p+\|x-ty\|_\psi^p &\le \|\boldsymbol{x}+ty\|_r^p+\|x-ty\|_r^p \\ &\le 2\boldsymbol{J}_{X,p}^p(t)(\|\cdot\|_r)\max\{\|x\|_r^p,\|y\|_r^p\} \\ &\le 2\boldsymbol{J}_{X,p}^p(t)(\|\cdot\|_r)M_1^p\max\{\|x\|_\psi^p,\|y\|_\psi^p\} \end{aligned}$$

from the definition of $J_{X,\ p}(t)$, implies that

$$J_{X,p}(t)(\|\cdot\|_\psi) \le J_{X,p}(t)(\|\cdot\|_r)M_1.$$

Note that $r \ge p$ and the function $\frac{\psi_r(s)}{\psi(s)}$ attains its maximum at $s = 1/2$, i.e.,$M_1 = \frac{\psi_r(1/2)}{\psi(1/2)}$. From Proposition 2.2, implies that

$$J_{X,p}(1)(\|\cdot\|_\psi) \le J_{X,p}(1)(\|\cdot\|_r)M_1 = \frac{1}{\psi(1/2)}. \tag{1}$$

On the other hand, let us put $x = (a, a)$, $y = (a, -a)$, where $a = \frac{1}{2\psi(1/2)}$. Hence $\|x\|_\psi = \|y\|_\psi = 1$, and

$$\left(\frac{\|x+y\|_\psi^p+\|x-y\|_\psi^p}{2}\right)^{\frac{1}{p}} = 2a = \frac{1}{\psi(1/2)}. \tag{2}$$

From (1) and (2), we have

$$J_{X,p}(1)(\|\cdot\|_\psi) = \frac{1}{\psi(1/2)}.$$

Theorem 2.5. Let $\psi \in \Psi$ and $\psi \ge \psi_r$ $(1 \le r \le 2)$. If the function $\frac{\psi(s)}{\psi_r(s)}$ attains its maximum at $s = 1/2$ and $1 \le p < r'$, then

$$J_{X,\,p}(1)(\|\cdot\|_\psi) = 2\psi(1/2).$$

Proof. By Proposition 1.4, we have $\|\cdot\|_r \le \|\cdot\|_\psi \le M_2\|\cdot\|_r$. Let $x, y \in X$, $(x, y) \neq (0, 0)$, where $X = \mathbb{R}^2$. Then

$$\begin{aligned} \|x+ty\|_\psi^p + \|x-ty\|_\psi^p &\le M_2^p(\|x+ty\|_r^p + \|x-ty\|_r^p) \\ &\le 2J_{X,\,p}^p(t)(\|\cdot\|_r)M_2^p \max\{\|x\|_r^p, \|y\|_r^p\} \\ &\le 2J_{X,\,p}^p(t)(\|\cdot\|_r)M_2^p \max\{\|x\|_\psi^p, \|y\|_\psi^p\}. \end{aligned}$$

From the definition of $J_{X,\,p}(t)$, it implies that

$$J_{X,\,p}(t)(\|\cdot\|_\psi) \le J_{X,\,p}(t)(\|\cdot\|_r)M_2$$

note that $1 \le p < r'$ and the function $\frac{\psi(s)}{\psi_r(s)}$ attains its maximum at $s = 1/2$, i.e.,$M_2 = \frac{\psi(1/2)}{\psi_r(1/2)}$. From Proposition 2.2, it implies that

$$J_{X,\,p}(1)(\|\cdot\|_\psi) \le J_{X,\,p}(1)(\|\cdot\|_r)M_2 = 2\psi(1/2). \tag{3}$$

On the other hand, let us put $x = (1, 0)$, $y = (0, 1)$. Then $\|x\|_\psi = \|y\|_\psi = 1$, and

$$\left(\frac{\|x+y\|_\psi^p + \|x-y\|_\psi^p}{2}\right)^{\frac{1}{p}} = 2\psi(1/2). \tag{4}$$

From (3) and (4), we have

$$J_{X,\,p}(1)(\|\cdot\|_\psi) = 2\psi(1/2).$$

Lemma 2.6 (see [6]). Let $\|\cdot\|$ and $|.|$ be two equivalent norms on a Banach space. If $a|.| \le \|\cdot\| \le b|.|$ $(b \ge a > 0)$, then

$$\frac{a}{b}J_{X,\,p}(t)(|.|) \le J_{X,\,p}(t)(\|\cdot\|) \le \frac{b}{a}J_{X,\,p}(t)(|.|).$$

Example 2.7. Let $X = \mathbb{R}^2$ with the norm

$$\|x\| = \max\{\|x\|_2, \lambda\|x\|_1\} \ (1/\sqrt{2} \le \lambda \le 1).$$

Then

$$J_{X,\,p}(1)(\|\cdot\|) = 2\lambda.\ (1 \le p < 2)$$

Proof. It is very easy to check that $\|x\| = \max\{\|x\|_2, \lambda\|x\|_1\} \in \mathbb{N}_\alpha$ and its corresponding function is

$$\psi(s) = \|(1-s, s)\| = \max\{\psi_2(s), \lambda\} \ge \psi_2(s).$$

Therefore,

$$\frac{\psi(s)}{\psi_2(s)} = \max\left\{1, \frac{\lambda}{\psi_2(s)}\right\}.$$

Since $\psi_2(s)$ attains minimum at $s = 1/2$ and hence $\frac{\psi(s)}{\psi_2(s)}$ attains maximum at $s = 1/2$. Therefore, from Theorem 2.5, we have

$$J_{X,p}(1)(||\cdot||) = 2\psi(1/2) = 2\lambda.$$

Example 2.8. Let $X = \mathbb{R}^2$ with the norm

$$||x|| = \max\{||x||_2, \lambda||x||_\infty\} \ (1 \leq \lambda \leq \sqrt{2}).$$

Then

$$J_{X,p}(1)(||\cdot||) = \sqrt{2}\lambda. \ (1 \leq p \leq 2)$$

Proof. It is obvious to check that the norm $||x|| = \max\{||x||_2, \lambda||x||_\infty\}$ is absolute, but not normalized, since $||(1, 0)|| = ||(0, 1)|| = \lambda$. Let us put

$$|.| = \frac{||\cdot||}{\lambda} = \max\left\{\frac{||\cdot||_2}{\lambda}, ||\cdot||_\infty\right\}.$$

Then $|.| \in \mathbb{N}_\alpha$ and its corresponding function is

$$\psi(s) = ||(1-s, s)|| = \max\left\{\frac{\psi_2(s)}{\lambda}, \psi_\infty(s)\right\} \leq \psi_2(s).$$

Then

$$\frac{\psi_2(s)}{\psi(s)} = \min\left\{\lambda, \frac{\psi_2(s)}{\psi_\infty(s)}\right\}.$$

Consider the increasing continuous function $g(s) = \frac{\psi_2(s)}{\psi(s)} (0 \leq s \leq 1/2)$. Because $g(0) = 1$ and $g(1/2) = \sqrt{2}$, there exists a unique $0 \leq a \leq 1$ such that $g(a) = \lambda$. In fact $g(s)$ is symmetric with respect to $s = 1/2$. Then we have

$$g(s) = \begin{cases} \frac{\psi_2(s)}{\psi(s)}, & s \in [0, a] \cup [1-a, a]; \\ \lambda, & s \in [a, 1-a] \end{cases}$$

Obviously, $g(s)$ attains its maximum at $s = 1/2$. Hence, from Theorem 2.4 and Lemma 2.6, we have

$$J_{X,p}(1)(||\cdot||) = J_{X,p}(1)(|.|) = \frac{1}{\psi(1/2)} = \sqrt{2}\lambda.$$

Example 2.9. Let $X = \mathbb{R}^2$ with the norm

$$||x|| = (||x||_2^2 + \lambda||x||_\infty^2) \ (\lambda \geq 0).$$

Then

$$J_{X,p}(1)(||\cdot||) = 2\sqrt{\frac{1+\lambda}{\lambda+2}} \ (1 \leq p \leq 2).$$

Proof. It is obvious to check that the norm $||x|| = (||x||_2^2 + \lambda||x||_\infty^2)$ is absolute, but not normalized, since $||(1, 0)|| = ||(0, 1)|| = (1 + \lambda)^{1/2}$. Let us put

$$|.| = \frac{||\cdot||}{\sqrt{1+\lambda}}.$$

Therefore, $|.| \in \mathbb{N}_\alpha$ and its corresponding function is

$$\psi(s) = ||(1-s,s)|| = \begin{cases} [(1-s)^2 + s^2/(1+\lambda)]^{1/2}, & s \in [0,1/2], \\ [s^2 + (1-s)^2/(1+\lambda)]^{1/2}, & s \in [1/2,1]. \end{cases}$$

Obvious $\psi(s) \le \psi_2(s)$. Since $\lambda \ge 0$, $\frac{\psi_2(s)}{\psi(s)}$ is symmetric with respect to $s = 1/2$, it suffices to consider $\frac{\psi_2(s)}{\psi(s)}$ for $s \in [0, 1/2]$. Note that, for any $s \in [0, 1/2]$, put $g(s) = \frac{\psi_2(s)^2}{\psi(s)^2}$. Taking derivative of the function $g(s)$, we have

$$g'(s) = \frac{2\lambda}{1+\lambda} \times \frac{s(1-s)}{[(1-s)^2 + s^2/(1+\lambda)]^2}.$$

We always have $g'(s) \ge 0$ for $0 \le s \le 1/2$. This implies that the function $g(s)$ is increased for $0 \le s \le 1/2$. Therefore, the function $\frac{\psi_2(s)}{\psi(s)}$ attains its maximum at $s = 1/2$. By Theorem 2.4 and Lemma 2.6, we have

$$J_{X,p}(1)(||\cdot||) = J_{X,p}(1)(|.|) = \frac{1}{\psi(1/2)} = 2\sqrt{\frac{1+\lambda}{\lambda+2}}.$$

Example 2.10. (Lorentz sequence spaces). Let $\omega_1 \ge \omega_2 > 0$, $2 \le r < \infty$. Two-dimensional Lorentz sequence space, i.e. $\mathbb{R}^2$ with the norm

$$||(z,\omega)||_{\omega,r} = (\omega_1|x_1^*|^r + \omega_2|x_2^*|^r)^{1/r},$$

where (x_1^*, x_2^*) is the rearrangement of $(|z|, |\omega|)$ satisfying $x_1^* \ge x_2^*$, then

$$J_{X,p}(1)(||(z,\omega)||_{\omega,r}) = 2\left(\frac{\omega_1}{\omega_1+\omega_2}\right)^{\frac{1}{r}} \ (1 \le p \le r)$$

Proof. It is obvious that $|.| = (||(z,\omega)||_{\omega,r})/\omega_1^{1/q} \in \mathbb{N}_\alpha$, and the corresponding convex function is given by

$$\psi(s) = \begin{cases} [(1-s)^r + (\omega_2/\omega_1)s^r]^{1/r}, & s \in [0,1/2], \\ [s^r + (\omega_2/\omega_1)(1-s)^r]^{1/r}, & s \in [1/2,1]. \end{cases}$$

Obviously $\psi(s) \le \psi_r(s)$ and $\Phi(s) = \frac{\psi_r(s)}{\psi(s)}$. It suffices to consider $\Phi(s)$ for $s \in [0, 1/2]$ since $\Phi(s)$ is symmetric with respect to $s = 1/2$. Note that for $s \in [0, 1/2]$

$$\Phi^r(s) = \frac{\psi_r^r(s)}{\psi^r(s)} = \frac{(1-s)^r + s^r}{(1-s)^r + (\omega_2/\omega_1)s^r} = \frac{u(s)}{v(s)}.$$

Some elementary computation shows that $u(s) - v(s) = (1-(\omega_2/\omega_1))s^r$ attains its maximum and $v(s)$ attains its minimum at $s = 1/2$. Hence,

$$\Phi^r(s) = \frac{u(s) - v(s)}{v(s)} + 1$$

attains its maximum at $s = 1/2$ and so does $\Phi(s)$. Then by Theorem 2.4 and Lemma 2.6, we have

$$J_{X,p}(1)(\|(z,\omega)\|_{\omega,r}) = J_{X,p}(1)(|.|) = 2\left(\frac{\omega_1}{\omega_1+\omega_2}\right)^{\frac{1}{r}}.$$

Example 2.11. Let X be two-dimensional Cesàro space $ces_2^{(2)}$, then

$$J_{X,p}(1)(ces_2^{(2)}) = \sqrt{2+\frac{2\sqrt{5}}{5}}.\ (1 \le p < 2).$$

Proof. We first define

$$|x,y| = \left\|\left(\frac{2x}{\sqrt{5}}, 2y\right)\right\|_{ces_2^{(2)}}$$

for $(x, y) \in \mathbb{R}^2$. It follows that $ces_2^{(2)}$ is isometrically isomorphic to $(\mathbb{R}^2, |.|)$ and $|.|$ is an absolute and normalized norm, and the corresponding convex function is given by

$$\psi(s) = \left[\frac{4(1-s)^2}{5} + \left(\frac{1-s}{\sqrt{5}}+s\right)^2\right]^{\frac{1}{2}}$$

Indeed, $T: ces_2^{(2)} \to (\mathbb{R}^2, |.|)$ defined by $T(x,y) = \left(\frac{x}{\sqrt{5}}, 2y\right)$ is an isometric isomorphism. We prove that $\psi(s) \ge \psi_2(s)$. Note that

$$\left(\frac{1-s}{\sqrt{5}}+s\right)^2 \ge \left(\frac{1-s}{\sqrt{5}}\right)^2 + s^2.$$

Consequently,

$$\psi(s) \ge ((1-s)^2+s^2)^{1/2} = \psi_2(s).$$

Some elementary computation shows that $\frac{\psi(s)}{\psi_2(s)}$ attains its maximum at $s = 1/2$. Therefore, from Theorem 2.5, we have

$$J_{X,p}(1)(ces_2^{(2)}) = 2\psi(1/2) = \sqrt{2+\frac{2\sqrt{5}}{5}}.$$

3. Constant and uniform normal structure

First, we recall some basic facts about ultrapowers. Let $l_\infty(X)$ denote the subspace of the product space $\Pi_{n\in\mathbb{N}}X$ equipped with the norm $\|(x_n)\| := \sup_{n\in\mathbb{N}}\|x_n\| < \infty$. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and let

$$N_{\mathcal{U}} = \left\{(x_n) \in l_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\right\}.$$

The ultrapower of X, denoted by $\tilde{X}$, is the quotient space $l_\infty(X)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $\tilde{x}_n$ to denote the elements of the ultrapower. Note that if $\mathcal{U}$ is non-trivial, then X can be embedded into $\tilde{X}$ isometrically. We also note that if X is super-reflexive, that is $\tilde{X}^* = (\tilde{X})^*$, then X has uniform normal structure if and only if $\tilde{X}$ has normal structure (see [17]).

Theorem 3.1. Let X be a Banach space with

$$J_{X,p}(t) < \frac{\sqrt{4+t^2}+t}{2}$$

for some $t \in (0, 1]$. Then X has uniform normal structure.

Proof. Observe that X is uniform non-square (see [6]) and then X is super-reflexive, it is enough to show that X has normal structure. Suppose that X lacks normal structure, then by Saejung [18, Lemma 2], there exist $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \in S_{\tilde{X}}$ and $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3 \in S_{\widetilde{X^*}}$ satisfying:

(1) $\|\tilde{x}_i - \tilde{x}_j\| = 1$ and $\tilde{f}_i(\tilde{x}_j) = 0$ for all $i \neq j$.

(2) $\tilde{f}_i(\tilde{x}_i) = 1$ for $i = 1, 2, 3$.

(3) $\|\tilde{x}_3 - (\tilde{x}_2 + \tilde{x}_1)\| \geq \|\tilde{x}_2 + \tilde{x}_1\|$.

Let $h(t) = (2 - t + \sqrt{4+t^2})/2$ and consider three possible cases.

First, if $\|\tilde{x}_1 + \tilde{x}_2\| \leq h(t)$. In this case, let us put $\tilde{x} = \tilde{x}_1 - \tilde{x}_2$ and $\tilde{y} = (\tilde{x}_1 + \tilde{x}_2)/h(t)$. It follows that $\tilde{x}, \tilde{y} \in B_{\tilde{X}}$, and

$$\begin{aligned} \|\tilde{x} + t\tilde{y}\| &= \|(1 + (t/h(t)))\tilde{x}_1 - (1 - (t/h(t)))\tilde{x}_2\| \\ &\geq (1 + (t/h(t)))\tilde{f}_1(\tilde{x}_1) - (1 - (t/h(t)))\tilde{f}_1(\tilde{x}_2) \\ &= 1 + (t/h(t)), \end{aligned}$$

$$\begin{aligned} \|\tilde{x} - t\tilde{y}\| &= \|(1 + (t/h(t)))\tilde{x}_2 - (1 - (t/h(t)))\tilde{x}_1\| \\ &\geq (1 + (t/h(t)))\tilde{f}_2(\tilde{x}_2) - (1 - (t/h(t)))\tilde{f}_2(\tilde{x}_1) \\ &= 1 + (t/h(t)). \end{aligned}$$

Secondly, if $\|\tilde{x}_1 + \tilde{x}_2\| \geq h(t)$ and $\|\tilde{x}_3 + \tilde{x}_2 - \tilde{x}_1\| \leq h(t)$. In this case, let us put $\tilde{x} = \tilde{x}_2 - \tilde{x}_3$ and $\tilde{y} = (\tilde{x}_3 + \tilde{x}_2 - \tilde{x}_1)/h(t)$. It follows that $\tilde{x}, \tilde{y} \in B_{\tilde{X}}$, and

$$\begin{aligned} \|\tilde{x} + t\tilde{y}\| &= \|(1 + (t/h(t)))\tilde{x}_2 - (1 - (t/h(t)))\tilde{x}_3 - (t/h(t))\tilde{x}_1\| \\ &\geq (1 + (t/h(t)))\tilde{f}_2(\tilde{x}_2) - (1 - (t/h(t)))\tilde{f}_2(\tilde{x}_3) - (t/h(t))\tilde{f}_2(\tilde{x}_1) \\ &= 1 + (t/h(t)), \end{aligned}$$

$$\begin{aligned} \|\tilde{x} - t\tilde{y}\| &= \|(1 + (t/h(t)))\tilde{x}_3 - (1 - (t/h(t)))\tilde{x}_2 - (t/h(t))\tilde{x}_1\| \\ &\geq (1 + (t/h(t)))\tilde{f}_3(\tilde{x}_3) - (1 - (t/h(t)))\tilde{f}_3(\tilde{x}_2) - (t/h(t))\tilde{f}_3(\tilde{x}_1) \\ &= 1 + (t/h(t)). \end{aligned}$$

Thirdly, $\|\tilde{x}_1 + \tilde{x}_2\| \geq h(t)$ and $\|\tilde{x}_3 + \tilde{x}_2 - \tilde{x}_1\| \geq h(t)$. In this case, let us put $\tilde{x} = \tilde{x}_3 - \tilde{x}_1$ and $\tilde{y} = \tilde{x}_2$. It follows that $\tilde{x}, \tilde{y} \in S_{\tilde{X}}$, and

$$\begin{aligned} \|\tilde{x} + t\tilde{y}\| &= \|\tilde{x}_3 + t\tilde{x}_2 - \tilde{x}_1\| \\ &\geq \|\tilde{x}_3 + \tilde{x}_2 - \tilde{x}_1\| - (1 - t) \\ &\geq h(t) + t - 1, \end{aligned}$$

$$\begin{aligned} \|\tilde{x} - t\tilde{y}\| &= \|\tilde{x}_3 - (t\tilde{x}_2 + \tilde{x}_1)\| \\ &\geq \|\tilde{x}_3 - (\tilde{x}_2 + \tilde{x}_1)\| - (1 - t) \\ &\geq h(t) + t - 1. \end{aligned}$$

Then, by definition of $J_{X,\,p}(t)$ and the fact $J_{X,\,p}(t) = J_{\tilde{X},\,p}(t)$,

$$\begin{aligned} J_{X,\,p}(t) &\geq \max\{1 + (t/h(t)), h(t) + t - 1\} \\ &= \frac{\sqrt{4 + t^2} + t}{2}. \end{aligned}$$

This is a contradiction and thus the proof is complete.

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Authors' contributions
ZZF designed and performed all the steps of proof in this research and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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