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Weak solutions of functional differential inequalities with first-order partial derivatives

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Abstract

The article deals with functional differential inequalities generated by the Cauchy problem for nonlinear first-order partial functional differential equations. The unknown function is the functional variable in equation and inequalities, and the partial derivatives appear in a classical sense. Theorems on weak solutions to functional differential inequalities are presented. Moreover, a comparison theorem gives an estimate for functions of several variables by means of functions of one variable which are solutions of ordinary differential equations or inequalities. It is shown that there are solutions of initial problems defined on the Haar pyramid.

Mathematics Subject Classification: 35R10, 35R45.

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1 Introduction

Two types of results on first-order partial differential or functional differential equations are taken into considerations in the literature. Theorems of the first type deal with initial problems which are local or global with respect to spatial variables, while the second one are concerned with initial boundary value problems. We are interested in results of the first type. More precisely, we consider initial problems which are local with respect to spatial variables. Then, the Haar pyramid is a natural domain on which solutions of differential or functional differential equations or inequalities are considered.

Hyperbolic differential inequalities corresponding to initial problems were first treated in the monographs [1]. (Chapter IX) and [2] (Chapters VII, IX). As is well known, they found applications in the theory of first-order partial differential equations, including questions such as estimates of solutions of initial problems, estimates of domains of solutions, estimates of the difference between solutions of two problems, criteria of uniqueness and continuous dependence of solution on given functions. The theory of monotone iterative methods developed in the monographs [3,4] is based on differential inequalities.

Two different types of results on differential inequalities are taken into consideration in [1,2]. The first type allows one to estimate a function of several variables by means of an another function of several variables, while the second type, the so-called comparison theorems give estimates for functions of several variables by means of functions of one variable, which are solutions of ordinary differential equations or inequalities.

There exist many generalizations of the above classical results. We list some of them below. Differential inequalities and the uniqueness of semi-classical solutions to the Cauchy problem for the weakly coupled systems were developed in [5] (Chapter VIII). Hyperbolic functional differential inequalities and suitable comparison results for initial problems are given in [6,7] (Chapter I). Infinite systems of functional differential equations and comparison results are discussed in [8,9]. Impulsive partial differential inequalities were investigated in [10]. A result on implicit functional differential inequalities can be found in [11]. Differential inequalities with unbounded delay are investigated in [12]. Functional differential inequalities with Kamke-type comparison problems can be found in [13]. Viscosity solutions of functional differential inequalities were studied in [14,15].

The aim of this article is to add a new element to the above sequence of generalizations of classical theorems on differential inequalities.

We now formulate our functional differential problem. For any metric spaces, U and V, we denote by $C(U, V)$ the class of all continuous functions from U into V. We use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Suppose that $M \in C([0,a], \mathbb{R}_+^n)$, $a > 0$, $\mathbb{R}_+ = [0, +\infty)$, is nondecreasing and $M(0) = 0_{[n]}$ where $0_{[n]} = (0, ..., 0) \in \mathbb{R}^n$. Let E be the Haar pyramid:

$$E = \{(t,x) \in \mathbb{R}^{1+n} : t \in [0,a], -b + M(t) \leq x \leq b - M(t)\}$$

where $b \in \mathbb{R}^n$ and $b > M(a)$. Write $E_0 = [-b_0, 0] \times [-b, b]$ where $b_0 \in \mathbb{R}_+$. For $(t, x) \in E$ define

$$D[t,x] = \{(\tau,s) \in \mathbb{R}^{1+n} : \tau \leq 0, (t+\tau, x+s) \in E_0 \cup E\}.$$

Then, $D[t, x] = D_0[t, x] \cup [D_\star[t, x]$ where

$$D_0[t,x] = [-b_0 - t, -t] \times [-b - x, b - x],$$
$$D_\star[t,x] = \{(\tau,s) : -t \leq \tau \leq 0, -b - x + M(\tau + t) \leq s \leq b - x - M(\tau + t)\}.$$

Write $r_0 = -b_0 - a$, $r = 2b$ and $B = [-r_0, 0] \times [-r, r]$. Then, $D[t, x] \subset B$ for $(t, x) \in E$. Given z: $E_0 \cup E \to \mathbb{R}$ and $(t, x) \in E$, define $z_{(t,x)}$: $D[t, x] \to \mathbb{R}$ by $z_{(t,x)}(\tau, s) = z(t + \tau, x + s)$, $(\tau, s) \in D[t, x]$. Then $z_{(t,x)}$ is the restriction of z to the set $(E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n)$ and this restriction is shifted to $D[t, x]$.

Put $\Omega = E \times \mathbb{R} \times C(B, \mathbb{R}) \times \mathbb{R}^n$ and suppose that $f : \Omega \to \mathbb{R}$ is a given function of the variables (t, x, p, w, q), $x = (x_1, ..., x_n)$, $q = (q_1, ..., q_n)$. Let us denote by z an unknown function of the variables (t, x). Given ψ: $E_0 \to \mathbb{R}$, we consider the functional differential equation:

$$\partial_t z(t,x) = f(t, x, z(t,x), z_{(t,x)}, \partial_x z(t,x)) \tag{1}$$

with the initial condition

$$z(t,x) = \psi(t,x) \text{ on } E_0, \tag{2}$$

where $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$. We will say that f satisfies condition (V), if for each $(t, x, p, q) \in E \times \mathbb{R} \times \mathbb{R}^n$ and for $w, \bar{w} \in C(B, \mathbb{R})$ such that $w(\tau, s) = \bar{w}(\tau, s)$ for $(\tau, s) \in D[t, x]$

then we have $f(t, x, p, w, q) = f(t, x, p, \bar{w}, q)$. It is clear that condition (V) means that the value of f at the point $(t, x, p, w, q) \in \Omega$ depends on (t, x, p, q) and on the restriction of w to the set $D[t, x]$ only.

We assume that F satisfies condition (V). Let us write

$$S_t = [-b + M(t), b - M(t)], \quad E_t = (E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n), \quad t \in [0, a],$$
$$I[x] = \{t \in [0, a] : -b + M(t) \le x \le b - M(t)\}, \quad x \in [-b, b].$$

We consider weak solutions of initial problems. A function $\tilde{z} : E_c \to \mathbb{R}$ where $0 < c \le a$, is a weak solution of (1), (2) provided

(i) $\tilde{z}$ is continuous, and $\partial_x \tilde{z}$ exists on $E \cap ([0, c] \times \mathbb{R}^n)$ and $\partial_x \tilde{z}(t, \cdot) \in C(S_t, \mathbb{R}^n)$ for $t \in [0, c]$,
(ii) for $x \in [-b, b]$, the function $\tilde{z}(\cdot, x) : I[x] \to \mathbb{R}$ is absolutely continuous,
(iii) for each $x \in [-b, b]$, the function $\tilde{z}$ satisfies equation 1 for almost all $t \in I[x] \cap [0, c]$ and condition (2) holds.

This class of solutions for nonlinear equations was introduced and widely studied in nonfunctional setting by Cinquini and Cinquini Cibrario [16,17].

The paper is organized as follows. In Sections 2 and 3 we present theorems on functional differential inequalities corresponding to (1), (2). They can be used for investigations of solutions to (1), (2). We show that the set of solutions is not empty. In Section 4 we prove that there is a weak solution to (1), (2) defined on E_c where $c \in (0, a]$ is a sufficiently small constant.

2 Functional differential inequalities

Let $\mathbb{L}([\tau, t], \mathbb{R}^n)$, $[\tau, t] \subset \mathbb{R}$, be the class of all integrable functions $\Psi: [\tau, t] \to \mathbb{R}^n$. The maximum norm in the space $C(B, \mathbb{R})$ will be denoted by $||\cdot||_B$. We will need the following assumptions on given functions.

Assumption H_0. The function $f : \Omega \to \mathbb{R}$ satisfies the condition (V) and

(1) $f(\cdot, x, p, w, q) \in \mathbb{L}(I[x], \mathbb{R})$ where $(x, p, w, q) \in [-b, b] \times \mathbb{R} \times C(B, \mathbb{R}) \times \mathbb{R}^n$ and $f(t, \cdot)$: $S_t \times \mathbb{R} \times C(B, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ is continuous for almost all $t \in [0, a]$,

(2) there exist the derivatives $(\partial_{q_1} f, \dots, \partial_{q_n} f) = \partial_q f$ and $\partial_q f(\cdot, p, x, w, q) = \mathbb{L}(I[x], \mathbb{R}^n)$ where $(x, p, w, q) \in [-b, b] \times \mathbb{R} \times C(B, \mathbb{R}) \times \mathbb{R}^n$, and the function $\partial_q f(t, \cdot)$: $S_t \times \mathbb{R} \times C(B, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous for almost all $t \in [0, a]$,

(3) there is $L \in \mathbb{L}([0, a], \mathbb{R}^n_+)$, $L = (L_1, \dots, L_n)$, such that $(|\partial_{q_1} f(P)|, \dots, |\partial_{q_n} f(P)|) \le L(t)$ where $P = (t, x, p, w, q) \in \Omega$, and

$$M(t) = \int_0^t L(\tau)\, d\tau, \quad t \in [0, a], \tag{3}$$

(4) there is $L_0 \in \mathbb{L}([0, a], \mathbb{R}_+)$ such that

$$|f(t, x, p, w, q) - f(t, x, \tilde{p}, w, q)| \le L_0(t)|p - \tilde{p}| \text{ on } \Omega, \tag{4}$$

(5) f is nondecreasing with respect to the functional variable and $\bar{z}, \tilde{z} \in C(E_0, \cup E, \mathbb{R})$ and

(i) the derivatives $\partial_x \bar{z}$, $\partial_x \tilde{z}$ exist on E and $\partial_x \bar{z}(t, \cdot)$, $\partial_x \tilde{z}(t, \cdot) \in C(S_t, \mathbb{R}^n)$ for $t \in [0, a]$,

(ii) for each $x \in [-b, b]$ the functions $\bar{z}(\cdot, x)$, $\tilde{z}(\cdot, x) : I[x] \to \mathbb{R}$ are absolutely continuous.

We start with a theorem on strong inequalities. Write

$$\mathbf{f}[z](t,x) = f(t,x,z(t,x), z_{(t,x)}, \partial_x z(t,x)).$$

Theorem 2.1. *Suppose that Assumption H_0 is satisfied and*

(1) for each $x \in [-b, b]$, the functional differential inequality

$$\partial_t \bar{z}(t,x) - \mathbf{f}[\bar{z}](t,x) < \partial_t \tilde{z}(t,x) - \mathbf{f}[\tilde{z}](t,x) \tag{5}$$

is satisfied for almost all $t \in I[x]$,
(2) $\bar{z}(t,x) \le \tilde{z}(t,x)$ for $(t,x) \in E_0$ and

$$\bar{z}(0,x) < \tilde{z}(0,x) \quad for \quad x \in [-b,b]. \tag{6}$$

Under these assumptions, we have

$$\bar{z}(t,x) < \tilde{z}(t,x) \quad on \quad E. \tag{7}$$

Proof Suppose by contradiction, that assertion (7) fails to be true. Then, the set

$$A_+ = \{t \in [0,a] : \bar{z}(t,x) \ge \tilde{z}(t,x) \text{ for some } x \in S_t\}$$

is not empty. Put $\tilde{t} = \min A_+$. From (6), we conclude that $\tilde{t} > 0$ and there is $\tilde{x} \in S_{\tilde{t}}$ such that

$$\bar{z}(t,x) < \tilde{z}(t,x) \text{ for } (t,x) \in E \cap ([0,\tilde{t}) \times \mathbb{R}^n) \tag{8}$$

and

$$\bar{z}(\tilde{t},\tilde{x}) = \tilde{z}(\tilde{t},\tilde{x}). \tag{9}$$

Write

$$A(t,x) = f(t,x,\bar{z}(t,x), \tilde{z}_{(t,x)}, \partial_x \bar{z}(t,x)) - f(t,x,\tilde{z}(t,x), \tilde{z}_{(t,x)}, \partial_x \bar{z}(t,x)),$$
$$B(t,x) = f(t,x,\tilde{z}(t,x), \tilde{z}_{(t,x)}, \partial_x \bar{z}(t,x)) - f(t,x,\tilde{z}(t,x), \tilde{z}_{(t,x)}, \partial_x \tilde{z}(t,x)),$$

where $(t,x) \in E \cap ([0,\tilde{t}] \times \mathbb{R}^n)$. It follows from (5) and (8) that for $x \in [-b, b]$ and for almost all $t \in I[x] \cap [0,\tilde{t}]$, we have

$$\partial_t(\bar{z} - \tilde{z})(t,x) < A(t,x) + B(t,x). \tag{10}$$

Set

$$Q(t,x,\xi) = (t,x,\tilde{z}(t,x), \tilde{z}_{(t,x)}, \xi \partial_x \bar{z}(t,x) + (1-\xi)\partial_x \tilde{z}(t,x)). \tag{11}$$

We conclude from the Hadamard mean value theorem that

$$B(t,x) = \sum_{j=1}^{n} \int_0^1 \partial_{q_j} f(Q(t,x,\xi))\, d\xi \; \partial_{x_j}(\bar{z} - \tilde{z})(t,x).$$

Let us denote by $g(\cdot, t, x)$ the solution of the Cauchy problem:

$$\gamma'(\tau) = -\int_0^1 \partial_q f(Q(\tau, \gamma(\tau), \xi))\, d\xi, \quad \gamma(t) = x, \tag{12}$$

where $(t, x) \in E$ and $0 \le t \le \tilde{t}$. Suppose that $[t_0, t]$ is the interval on which the solution $g(\cdot, t, x)$ is defined. Then,

$$-L(\tau) \le \frac{d}{d\tau} g(\tau, t, x) \le L(\tau) \text{ for } \tau \in [t_0, t],$$

and consequently,

$$-b + M(\tau) \le g(\tau, t, x) \le b - M(\tau), \quad \tau \in [t_0, t].$$

We conclude that $(\tau, g(\tau, t, x)) \in E$ for $\tau \in [t_0, t]$ and, consequently, the function $g(\cdot, t, x)$ is defined on $[0, t]$. It follows from (10) that

$$\frac{d}{d\tau}(\bar{z} - \tilde{z})(\tau, g(\tau, t, x)) < L_0(\tau)|(\bar{z} - \tilde{z})(\tau, g(\tau, t, x))| \text{ for almost all } \tau \in [0, t], \tag{13}$$

Where $(t, x) \in E \cap ([0, \tilde{t}] \times \mathbb{R}^n)$. We conclude from (8), (13) that

$$\int_0^t \frac{d}{d\tau}\{(\bar{z} - \tilde{z})(\tau, g(\tau, t, x)) \exp[\int_0^\tau L_0(\xi)\, d\xi]\}\, d\tau < 0.$$

This gives

$$(\bar{z} - \tilde{z})(t, x) < (\bar{z} - \tilde{z})(0, g(0, t, x)) \exp\{-\int_0^t L_0(\xi)\, d\xi\}, \quad (t, x) \in E \cap ([0, \tilde{t}\,] \times \mathbb{R}^n),$$

and consequently $\bar{z}(\tilde{t}, \tilde{x}) < \tilde{z}(\tilde{t}, \tilde{x})$ which contradicts (9). Hence, A_+ is empty and the statement (7) follows.

Now we prove that a weak initial inequality for $\bar{z}$ and $\tilde{z}$ on E_0 and weak functional differential inequalities on E imply weak inequality for $\bar{z}$ and $\tilde{z}$ on E.

Assumption $H[\sigma]$. The function $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the conditions:

(1) $\sigma(t, \cdot): \mathbb{R}_+ \to \mathbb{R}_+$ is continuous for almost all $t \in [0, a]$,

(2) $\sigma(\cdot, p): [0, a] \to \mathbb{R}_+$ is measurable for every $p \in \to \mathbb{R}_+$ and there is $m_\sigma = \mathbb{L}([0, a], \mathbb{R}_+)$ such

that $\sigma(t, p) \le m_\sigma(t)$ for $p \in \mathbb{R}_+$ and for almost all $t \in [0, a]$,

(3) the function $\tilde{\omega}(t) = 0$ for $t \in [0, a]$ is the maximal solution of the Cauchy problem:

$$\omega'(t) = L_0(t)\omega(t) + \sigma(t, \omega(t)), \quad \omega(0) = 0.$$

Theorem 2.2. *Suppose that Assumptions H_0 and $H[\sigma]$ are satisfied and*

(1) the estimate

$$f(t, x, p, \tilde{w}, q) - f(t, x, p, w, q) \le \sigma(t, ||\tilde{w} - w||_B) \tag{14}$$

holds on Ω for $w \le \tilde{w}$, *(2)* $\bar{z}(t,x) \le \tilde{z}(t,x)$ *for* $(t, x) \in E_0$, *and for each* $x \in [-b, b]$ *the functional differential inequality*

$$\partial_t \bar{z}(t,x) - \mathbf{f}[\bar{z}](t,x) \le \partial_t \tilde{z}(t,x) - \mathbf{f}[\tilde{z}](t,x) \tag{15}$$

is satisfied for almost all $t \in I[x]$.

Under these assumptions, we have

$$\bar{z}(t,x) \le \tilde{z}(t,x) \quad on \quad E. \tag{16}$$

Proof Let us denote by $\omega(\cdot, \varepsilon)$, $\varepsilon > 0$, the right-hand maximal solution of the Cauchy problem

$$\omega'(t) = L_0(t)\omega(t) + \sigma(t, \omega(t)) + \varepsilon, \quad \omega(0) = \varepsilon.$$

There is $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, the solution $\omega(\cdot, \varepsilon)$ is defined on $[0, a]$ and

$$\lim_{\varepsilon \to 0} \omega(t, \varepsilon) = 0 \text{ uniformly on } \quad [0, a].$$

Let $\tilde{z}_\varepsilon : E_0 \cup E \to \mathbb{R}$ be defined by

$$\tilde{z}_\varepsilon(t,x) = \tilde{z}(t,x) + \varepsilon \text{ on } E_0 \text{ and } \tilde{z}_\varepsilon(t,x) = \tilde{z}(t,x) + \omega(t,\varepsilon) \text{ on } E.$$

Then, we have $\bar{z}(t,x) < \tilde{z}_\varepsilon(t,x)$ on E_0. We prove that for each $x \in [-b, b]$ the functional differential inequality

$$\partial_t \bar{z}(t,x) - \mathbf{f}[\bar{z}](t,x) < \partial_t \tilde{z}_\varepsilon(t,x) - \mathbf{f}[\tilde{z}_\varepsilon](t,x) \tag{17}$$

is satisfied for almost all $t \in I[x]$. It follows from (4), (14), that

$$\begin{aligned} \partial_t \bar{z}(t,x) - \mathbf{f}[\bar{z}](t,x) &\le \partial_t \tilde{z}_\varepsilon(t,x) - \mathbf{f}[\tilde{z}_\varepsilon](t,x) - \omega'(t,\varepsilon) \\ &+ f(t,x,\tilde{z}_\varepsilon(t,x),(\tilde{z}_\varepsilon)_{(t,x)},\partial_x \tilde{z}(t,x)) - f(t,x,\tilde{z}(t,x),\tilde{z}_{(t,x)},\partial_x \tilde{z}(t,x)) \\ &\le \partial_t \tilde{z}_\varepsilon(t,x) - \mathbf{f}[\tilde{z}_\varepsilon](t,x) - \omega'(t,\varepsilon) + L_0(t)\omega(t,\varepsilon) + \sigma(t,\omega(t,\varepsilon)) \\ &= \partial_t \tilde{z}_\varepsilon(t,x) - \mathbf{f}[\tilde{z}_\varepsilon](t,x) - \varepsilon, \end{aligned}$$

which completes the proof of (17). It follows from Theorem 2.1 that $\bar{z}(t,x) < \tilde{z}(t,x) + \omega(t,\varepsilon)$ on E. From this inequality, we obtain in the limit, letting ε tend to zero, inequality (16). This completes the proof.

The results presented in Theorems 2.1 and 2.2 have the following properties. In both the theorems, we have assumed that $\bar{z}(t,x) \le \tilde{z}(t,x)$ on E_0. It follows from Theorem 2.1 that the strong inequality (6) and the strong functional differential inequality (5) for almost all $t \in I[x]$ imply the strong inequality (7). Theorem 2.2 shows that the weak initial inequality $\bar{z}(t,x) \le \tilde{z}(t,x)$ on E and the weak functional differential inequality (15) for almost all $t \in I[x]$ imply the weak inequality (16).

In the next two lemmas, we assume that $\bar{z}(t,x) \le \tilde{z}(t,x)$ on E_0 and we prove that the strong initial inequality (6) and the weak functional inequality (15) imply the strong inequality (7).

We prove also that the weak initial inequality $\bar{z}(t,x) \leq \tilde{z}(t,x)$ on E_0 and the strong functional differential inequality (5) imply the inequality $\bar{z}(t,x) < \tilde{z}(t,x)$ for $(t, x) \in E$, $0 < t \leq a$.

Lemma 2.3. *Suppose that Assumptions H_0 and $H[\sigma]$ are satisfied and*

(1) the estimate (14) holds on Ω for $w \leq \tilde{w}$,
(2) $\bar{z}(t,x) \leq \tilde{z}(t,x)$ for $(t, x) \in E_0$ and for each $x \in [-b, b]$ the functional differential inequality (5) is satisfied for almost all $t \in I[x]$.

Under these assumption, we have $\bar{z}(t,x) < \tilde{z}(t,x)$ for $(t, x) \in E$, $0 < t \leq a$.

Proof It follows from Theorem 2.2 that $\bar{z}(t,x) \leq \tilde{z}(t,x)$ for $(t, x) \in E$. Suppose that there is $(\tilde{t},\tilde{x}) \in E,\ \ 0 < \tilde{t} \leq a$, such that $\bar{z}(\tilde{t},\tilde{x}) = \tilde{z}(\tilde{t},\tilde{x})$. By repeating the argument used in the proof of Theorem 2.1, we obtain

$$(\bar{z}-\tilde{z})(\tilde{t},\tilde{x}) < (\bar{z}-\tilde{z})(0, g(0,\tilde{t},\tilde{x})\ \exp[-\int_0^{\tilde{t}} L_0(\xi)\, d\xi],$$

where $g(\cdot, t, x)$ is the solution to (12). Then, $\bar{z}(\tilde{t},\tilde{x}) < \tilde{z}(\tilde{t},\tilde{x})$, which completes the proof of the lemma.

Lemma 2.4. *Suppose that Assumption H_0 and $H[\sigma]$ are satisfied and*

(1) the estimate (14) holds on Ω for $w \leq \tilde{w}$,
(2) $\bar{z}(t,x) \leq \tilde{z}(t,x)$ for $(t, x) \in E_0$ and $\bar{z}(0,x) < \tilde{z}(0,x)$ for $x \in [-b, b]$,
(3) for each $x \in [-b, b]$ the functional differential inequality (15) is satisfied for almost all $t \in I[x]$.

Under these assumption, we have

$$\bar{z}(t,x) < \tilde{z}(t,x) \ \ on \ \ E. \tag{18}$$

Proof Let

$$0 < p_0 < \min\{\tilde{z}(0,x) - \bar{z}(0,x) : \ \ x \in [-b,b]\,\}.$$

For $\delta > 0$, we denote by $\omega(\cdot, \delta)$ the solution of the Cauchy problem

$$\omega'(t) = -L_0(t)\omega(t) - \delta, \ \ \omega(0) = p_0. \tag{19}$$

There is $\delta_0 > 0$ such that for $0 < \delta \leq \delta_0$, we have

$$\omega(t,\delta) > 0 \ \ \text{for} \ \ t \in [0,a]. \tag{20}$$

Let us denote by $\tilde{\omega} : E_0 \to \mathbb{R}$ a continuous function such that $\bar{z}(t,x) \leq \tilde{\omega}(t,x) \leq \tilde{z}(t,x)$ on E_0 and $\tilde{\omega}(0,x) = \bar{z}(0,x) + p_0$ for $x \in [-b, b]$. Suppose that $z^\star : E_0 \cup E \to \mathbb{R}$ is defined by

$$z^\star(t,x) = \tilde{\omega}(t,x) \ \ \text{on} \ \ E_0, \ \ z^\star(t,x) = \bar{z}(t,x) + \omega(t,\delta) \ \ \text{on} \ \ E,$$

where $0 < \delta \leq \delta_0$. We prove that

$$z^\star(t,x) < \tilde{z}(t,x) \ \ \text{on} \ \ E. \tag{21}$$

Note that $z^\star(t,x) \le \tilde{z}(t,x)$ on E_0 and $z^\star(0,x) < \tilde{z}(0,x)$ for $x \in [-b, b]$. We prove that for each $x \in [-b, b]$, the functional differential inequality

$$\partial_t z^\star(t,x) - \mathbf{f}[z^\star](t,x) < \partial_t \tilde{z}(t,x) - \mathbf{f}[\tilde{z}](t,x) \tag{22}$$

is satisfied for almost all $t \in I[x]$. By Assumption H_0 and (19), we have

$$\begin{aligned}\partial_t z^\star(t,x) - \mathbf{f}[z^\star](t,x) &= \partial_t \bar{z}(t,x) - \mathbf{f}[\bar{z}](t,x) + \omega'(t,\delta)\\ +f(t,x,\bar{z}(t,x),\bar{z}_{(t,x)},\partial_x\bar{z}(t,x)) &- f(t,x,z^\star(t,x),(z^\star)_{(t,x)},\partial_x\bar{z}(t,x))\\ &\le \partial_t \tilde{z}(t,x) - \mathbf{f}[\tilde{z}](t,x) + L_0(t)\omega(t,\delta) + \omega'(t,\delta)\\ &= \partial_t \tilde{z}(t,x) - \mathbf{f}[\tilde{z}](t,x) - \delta,\end{aligned}$$

which completes the proof of (22). We get from Theorem 2.1 that (21 holds. Inequalities (20), (21), imply (18), which completes the proof of the lemma.

Remark 2.5. *The results presented in Section 2 can be extended on functional differential inequalities corresponding to the system:*

$$\partial_t z_i(t,x) = f_i(t,x,z(t,x),z_{(t,x)},\partial_x z_i(t,x)), \quad i = 1,\ldots,k,$$

where $z = (z_1, ..., z_k)$ *and* $f = (f_1, ..., f_k)$: $E \times \mathbb{R}^k \times C(B, \mathbb{R}^k) \times \mathbb{R}^n \to \mathbb{R}^n$ *is a given function of the variables* (t, x, p, w, q), $p = (p_1, ..., p_k)$, $w = (w_1, ..., w_k)$, *Some quasi-monotone assumptions on the function f with respect to p are needed in this case.*

3 Comparison theorem

For $z \in C(E_0 \cup E, \mathbb{R})$, we put

$$||z||_{(t,\mathbb{R})} = \max\{|z(\tau,s)| : (\tau,s) \in E_t\}, \quad 0 \le t \le a.$$

Assumption $H_\star$. The functions Δ: $E \times C(B, \mathbb{R}) \to \mathbb{R}^n$, $\Delta = (\Delta_1, ..., \Delta_n)$, and ϱ: $[0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the conditions:

(1) Δ satisfies condition (V) and $\Delta(\cdot,x,w) \in \mathbb{L}(I[x],\mathbb{R}^n)$ where $(x, w) \in [-b, b] \times C(B, \mathbb{R})$ and $\Delta(t, \cdot)$: $S_t \times C(B, \mathbb{R}) \to \mathbb{R}^n$ is continuous for almost all $t \in [0, a]$,

(2) there is $L \in \mathbb{L}([0,a],\mathbb{R}^n_+)$, $L = (L_1, ..., L_n)$, such that

$$(|\Delta_1(t,x,w)|,\ldots,|\Delta_n(t,x,w)|) \le L(t) \text{ on } E \times C(B,\mathbb{R})$$

and $M : [0,a] \to \mathbb{R}^n_+$ is given by (3),

(3) $\varrho(\cdot, p)$: $[0, a] \to \mathbb{R}_+$ is measurable for $p \in \mathbb{R}_+$ and $\varrho(t, \cdot)$: $\mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing for almost all $t \in [0, a]$, and there is $m_\varrho \in \mathbb{L}([0,a],\mathbb{R}_+)$ such that $\varrho(t, p) \le m\varrho(t)$ for $p \in \mathbb{R}_+$ and for almost all $t \in [0, a]$,

(4) $z^\star$: $E_0 \cup E \to \mathbb{R}$ is continuous and

(i) the derivatives $(\partial_{x_1}z^\star,\ldots,\partial_{x_n}z^\star) = \partial_x z^\star$ exist on E and $\partial_x z^\star(t, \cdot) \in C(S_t, \mathbb{R}^n)$ for $t \in [0, a]$,

(ii) for each $x \in [-b, b]$ the function $z^\star(\cdot, x)$: $I[x] \to \mathbb{R}$ is absolutely continuous.

Theorem 3.1. *Suppose that Assumption* $H_\star$ *is satisfied and*

(1) for each $x \in [-b, b]$ *the functional differential inequality*

$$|\partial_t z^\star(t,x) + \sum_{i=1}^{n} \Delta_i(t,x,(z^\star)_{(t,x)})\,\partial_{x_i} z^\star(t,x)| \le \varrho(t,||z^\star||_{(t,\mathbb{R})}) \tag{23}$$

is satisfied for almost all $t \in I[x]$,

(2) the number $\eta \in \mathbb{R}_+$ *is defined by the relation:* $|z^\star(t, x)| \leq \eta$ *for* $(t, x) \in E_0$.

Under these assumptions we have

$$||z^\star||_{(t,\mathbb{R})} \leq \omega(t, \eta), \quad t \in [0, a], \tag{24}$$

where $\omega(\cdot, \eta)$ *is the maximal solution of the Cauchy problem*

$$\omega'(t) = \varrho(t, \omega(t)), \quad \omega(0) = \eta. \tag{25}$$

Proof Let us denote by $g[z^\star](\cdot, t, x)$ the solution of the Cauchy problem

$$y'(\tau) = \Delta(\tau, y(\tau), (z^\star)_{(\tau,y(\tau))}), \quad y(t) = x,$$

where $(t, x) \in E$. It follows from condition 1) of Assumption $H_\star$ that $g[z^\star](\cdot, t, x)$ is defined on $[0, t]$. We conclude from (23) that for each $x \in [-b, b]$, the differential inequality

$$|\frac{d}{d\tau} z^\star(\tau, g[z^\star](\tau, t, x))| \leq \varrho(\tau, ||z^\star||_{(\tau,\mathbb{R})})$$

is satisfied for almost all $\tau \in [0, t]$. This gives the integral inequality

$$||z^\star||_{(t,\mathbb{R})} \leq \eta + \int_0^t \varrho(\tau, ||z^\star||_{(\tau,\mathbb{R})})\, d\tau, \quad t \in [0, a].$$

The function $\omega(\cdot, \eta)$ satisfies the integral equation corresponding to the above inequality. From condition 3) of Assumption $H_\star$ we obtain (24), which completes the proof.

We give an estimate of the difference between two solutions of equation 1.

Theorem 3.2. *Suppose that the function* $f : \Omega \to \mathbb{R}$ *satisfies condition (V) and*

(1) conditions (1)-(3) of Assumption H_0 *hold,*

(2) there is $\varrho : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that condition (3) of Assumption* $H_\star$ *is satisfied and*

$$f(t, x, p, w, q) - f(t, x, \tilde{p}, \tilde{w}, q)| \leq \varrho(t, \max\{|p - \tilde{p}, \; ||w - \tilde{w}||_B) \;\; on \;\; \Omega, \tag{26}$$

(3) the functions $\bar{z}, \tilde{z} : E_0 \to \mathbb{R}_+$*are weak solutions to (1) and* $\eta \in \mathbb{R}_+$ *is defined by the relation:* $|\bar{z}(t, x) - \tilde{z}(t, x)| \leq \eta$*for* $(t, x) \in E_0$.

Under these assumptions, we have

$$||\bar{z} - \tilde{z}||_{(t,\mathbb{R})} \leq \omega(t, \eta) \;\; for \;\; t \in [0, a], \tag{27}$$

where $\omega(\cdot, \eta)$ *is the maximal solution to (25).*

Proof Let us write

$$\tilde{A}(t,x) = f(t,x,\bar{z}(t,x),\bar{z}_{(t,x)},\partial_x\bar{z}(t,x)) - f(t,x,\tilde{z}(t,x),\tilde{z}_{(t,x)},\partial_x\bar{z}(t,x)),$$
$$\tilde{B}(t,x) = F(t,x,\tilde{z}(t,x),\tilde{z}_{(t,x)},\partial_x\bar{z}(t,x)) - F(t,x,\tilde{z}(t,x),\tilde{z}_{(t,x)},\partial_x\tilde{z}(t,x)).$$

Then, for each $x \in [-b, b]$ and for almost all $t \in I[x]$, we have

$$\partial_t(\bar{z}-\tilde{z})(t,x) = \tilde{A}(t,x) + \tilde{B}(t,x).$$

Set $z^\star = \bar{z} - \tilde{z}$. It follows from the Hadamard mean value theorem that

$$\tilde{B}(t,x) = \sum_{i=1}^{n}\int_0^t \partial_{q_i} f(Q(t,x,\xi))\,d\xi\,\partial_{x_i} z^\star(t,x)$$

where $D(t, x\xi)$ is given by (11). We conclude from (26 that

$$|\tilde{A}(t,x)| \le \varrho(t, ||z^\star||_{(t,\mathbb{R})}), \quad (t,x) \in E.$$

Thus, we see that for each $x \in [-b, b]$ the functional differential inequality

$$|\partial_t z^\star(t,x) - \sum_{i=1}^{n}\int_0^1 \partial_{q_i} f(Q(t,x,\xi))\,d\xi\,\partial_{x_i} z^\star(t,x)| \le \varrho(t, ||z^\star||_{(t,\mathbb{R})})$$

is satisfied for almost all $\in I[x]$. From Theorem 3.1 we obtain (27), which completes the proof.

The next lemma on the uniqueness of weak solutions is a consequence of Theorem 3.2.

Lemma 3.3. *Suppose that the function* $f : \Omega \to \mathbb{R}$ *satisfies condition* (V) *and*

(1) assumptions (1), (2) of Theorem 3.2 hold,
(2) the function $\tilde{\omega}(t)$*for* $t \in [0, a]$ *is the maximal solution to (25) with* $\eta = 0$.

Then, problem (1), (2) admits one weak solution at the most.

Proof From (27) we deduce that for $\eta = 0$ we have $\bar{z} = \tilde{z}$ on E and the lemma follows.

4 Existence of solutions of initial problems

Put $\Xi = E \times C(B, \mathbb{R}) \times \mathbb{R}^n$ and suppose that $F : \Xi \to \mathbb{R}$ is a given function of the variables (t, x, w, q). Given $\psi : E_0 \to \mathbb{R}$, we consider the functional differential equation:

$$\partial_t z(t,x) = F(t,x,z_{(t,x)},\partial_x z(t,x)) \tag{28}$$

with the initial condition

$$z(t,x) = \psi(t,x) \text{ on } E_0. \tag{29}$$

We assume that F satisfies condition (V) and we consider weak solutions to (28), (29).

Let us denote by $M_{n\times n}$ the class of all $n \times n$ matrices with real elements. For $x \in \mathbb{R}^n$, $W \in M_{n\times n}$, where $x = (x_1, ..., x_n)$, $W = [w_{ij}]_{i,j=1,...,n}$, we put

$$||x|| = \sum_{i=1}^{n}|x_i|, \quad ||W||_{n\times n} = \max\{\sum_{j=1}^{n}|w_{ij}| : 1 \le i \le n\}.$$

If $W \in M_{n \times n}$, then W^T denotes the transpose matrix. Suppose that $v \in C(E_0 \cup R, \mathbb{R}^n)$, $U \in C(E_0 \cup R, M_{n \times n})$. The following seminorms will be needed in our considerations:

$$||v||_{(t,\mathbb{R}^n)} = \max\{||v(\tau,s)|| : (\tau,s) \in E_t\},$$
$$||U||_{(t,M_{n\times n})} = \max\{||U(\tau,s)||_{n\times n} : (\tau,s) \in E_t\},$$

where $t \in [0, a]$. The scalar product in $\mathbb{R}^n$ will be denoted by "$\circ$". We will use the symbol $CL(B, \mathbb{R})$ to denote the class of all linear and continuous operators defined on $C(B, \mathbb{R})$ and taking values in $\mathbb{R}$. The norm in the space $CL(B, \mathbb{R})$ generated by the maximum norm in $C(B, \mathbb{R})$ will be denoted by $||\cdot||_\star$. The maximum norms in $C(E_0, \mathbb{R})$ and $C(E_0, \mathbb{R}^n)$ will be denoted by $||\cdot||_{(E_0,\mathbb{R})}$ and $||\cdot||_{(E_0,\mathbb{R}^n)}$, respectively.

Assumption $H_0[F]$. The function $F : \Xi \to \mathbb{R}$ satisfies the condition (V) and

(1) $F(\cdot, x, w, q) \in \mathbb{L}(I[x], \mathbb{R})$ where $(x, w, q) \in [-b, b] \times C(B, \mathbb{R}) \times \mathbb{R}^n$ and $F(t, \cdot)$: $S_t \times C(B, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ is continuous for almost all $t \in [0, a]$,

(2) there is $\alpha \in \mathbb{L}([0, a], \mathbb{R}_+)$ such that

$$|F(t, x, \theta, 0_{[n]})| \le \alpha(t) \quad \text{on} \ \ E,$$

where $\theta \in C(B, \mathbb{R})$ is given by $\theta(\tau, s) = 0$ on B,

(3) for $P = (t, x, w, q) \in \Xi$ there exist the derivatives

$$\partial_x F(P) = (\partial_{x_1} F(P), \ldots, \partial_{x_n} F(P)), \quad \partial_w F(P),$$
$$\partial_q F(P) = (\partial_{q_1} F(P), \ldots, \partial_{q_n} F(P))$$

and $\partial_x F(\cdot, x, w, q)$, $\partial_x F(\cdot, x, w, q) \in \mathbb{L}(I[x], \mathbb{R}^n)$ and $\partial_w F(\cdot, x, w, q)\tilde{w} \in \mathbb{L}(I[x], \mathbb{R})$ where $(x, w, q) \in [-b, b] \times C(B, \mathbb{R}) \times \mathbb{R}^n$, $\tilde{w} \in C(B, \mathbb{R})$,

(4) the functions

$$\partial_x F(t, \cdot),\ \partial_q F(t, \cdot) : S_t \times C(B, \mathbb{R}) \times \mathbb{R}^n \to n\mathbb{R}^n,$$
$$\partial_w F(t, \cdot) : S_t \times C(B, \mathbb{R}) \times \mathbb{R}^n \to CL(B, R)$$

are continuous for almost all $t \in [0, a]$ and there are $L \in \mathbb{L}([0, a], \mathbb{R}^n_+) L \in \mathbb{L}([0, a], \mathbb{R}^n_+)$, $L = (L_1, \ldots, L_n)$, such that

$$||\partial_x F(t, x, w, q)||, \quad ||\partial_w F(t, x, w, q)||_\star \le \beta(t),$$

and

$$(|\partial_{q_1} F(t, x, w, q)|, \ldots, |\partial_{q_n} F(t, x, w, q)|) \le L(t),$$

where $(t, x, w, q) \in \Xi$, and $M : [0, a] \to \mathbb{R}^n_+$ is given by (3).

Now we define some function spaces. Given $\bar{c} = (c_0, c_1, c_2) \in \mathbb{R}^3_+$, we denote by $\mathbb{X}$ the set of all $\psi \in C(E_0, \mathbb{R})$ such that

(i) the derivatives $(\partial_{x_1}\psi, \ldots, \partial_{x_n}\psi) = \partial_x\psi$ exist on E_0 and $\partial_x\psi \in C(E_0, \mathbb{R}^n)$,

(ii) the estimates

$$|\psi(t, x)| \le c_0, \quad ||\partial_x\psi(t, x)|| \le c_1, \quad ||\partial_x\psi(t, x) - \partial_x\psi(t, \bar{x})|| \le c_2||x - \bar{x}||$$

are satisfied on E_0.

Let $\psi \in \mathbb{X}$ be given and $0 < c \le a$. We denote by $C_{\psi.c}$ the class of all $z \in C(E_c, \mathbb{R})$ such that $z(t, x) = \psi(t, x)$ on E_0. For the above ψ and c we denote by $C_{\partial\psi.c}$ the class of all $v \in C(Ec, \mathbb{R}^n)$ such that $v(t, x) = \partial_x \psi(t, x)$ on E_0.

Suppose that Assumption $H_0[F]$ is satisfied and $\psi \in \mathbb{X}$, $z \in C_{\psi.c}$, $u \in C_{\partial\psi.c}$ where $0 < c \le a$. We consider the Cauchy problem

$$\gamma'(\tau) = -\partial_q F(\tau, \gamma(\tau), z_{(\tau,\gamma(\tau))}, u(\tau, \gamma(\tau))), \quad \gamma(t) = x, \tag{30}$$

where $(t, x) \in E$ and $0 \le t \le c$. Let us denote by $g[z, u](\cdot, t, x)$ the solution of (30). The function $g[z, u](\cdot, t, x)$ is the bicharacteristic of (28) corresponding to (z, u).

For $u \in C_{\partial\psi.c}$, $u = (u_1, ..., u_n)$, and $P \in \Xi$, $(t, x) \in E \cap ([0, c] \times \mathbb{R}^n)$, we write

$$\partial_w F(P) \star u_{(t,x)} = (\partial_w F(P)(u_1)_{(t,x)}, \ldots, \partial_w F(P)(u_n)_{(t,x)}).$$

Set

$$P[z, u](\tau, t, x) = (\tau, g[z, u](\tau, t, x), z_{(\tau, g[z,u](\tau,t,x))}, u(\tau, g[z, u](\tau, t, x))).$$

Let $\mathbb{F}[z, u]$ be defined by

$$\mathbb{F}[z, u](t, x) = \psi(t, x) \ \text{ on } \ E_0 \tag{31}$$

and

$$\begin{aligned} \mathbb{F}[z, u](t, x) = \psi(0, g[z, u](0, t, x)) + \int_0^t F(P[z, u](\tau, t, x))\, d\tau \\ - \int_0^t \partial_q F(P[z, u](\tau, t, x)) \circ u(\tau, g[z, u](\tau, t, x))\, d\tau \ \text{ on } \ E \cap ([0, c] \times \mathbb{R}^n). \end{aligned} \tag{32}$$

Set $\mathbb{G}[z, u] = (\mathbb{G}_1[z, u], \ldots \mathbb{G}_n[z, u])$ where

$$\mathbb{G}[z, u](t, x) = \partial_x \psi(t, x) \ \text{ on } \ E_0 \tag{33}$$

and

$$\begin{aligned} \mathbb{G}[z, u](t, x) = \partial_x \psi(0, g[z, u](0, t, x)) + \int_0^t \partial_x F(P[z, u](\tau, t, x))\, d\tau \\ + \int_0^t \partial_w F(P[z, u](\tau, t, x)) \star u_{(\tau, g[z,u](\tau,t,x))}\, d\tau \ \text{ on } \ E \cap ([0, c] \times \mathbb{R}^n). \end{aligned} \tag{34}$$

We consider the system of integral functional equations

$$z = \mathbb{F}[z, u], \quad u = \mathbb{G}[z, u]. \tag{35}$$

System (35) is obtained in the following way. We first introduce an additional unknown function $u = \partial_x z$ in (28). Then, we consider the linearization of (28) with respect to the last variable, and we obtain the equation

$$\partial_t z(t, x) = F(t, x, z_{(t,x)}, u(t, x)) + \partial_q F(t, x, z_{(t,x)}, u(t, x)) \circ (\partial_x z(t, x) - u(t, x)). \tag{36}$$

By virtue of (28) we get the following differential equation for the unknown function u ::

$$\begin{aligned} \partial_t u(t, x) = \partial_x F(t, x, z_{(t,x)}, u(t, x)) + \partial_w F(t, x, z_{(t,x)}, u(t, x)) \star (\partial_x z)_{(t,x)} \\ + \partial_q F(t, x, z_{(t,x)}, u(t, x))[\partial_x u(t, x)]^T. \end{aligned} \tag{37}$$

Finally, we put $u = \partial_x z$ in (37). If we consider (36) and (37) along the bicharacteristic $g[z, u](\cdot, t, x)$, then we obtain

$$\frac{d}{d\tau}z(\tau, g[z,u](\tau,t,x)) = F(P[z,u](\tau,t,x)) - \partial_q F(P[z,u](\tau,t,x)) \circ u(\tau, g[z,u](\tau,t,x)) \quad (38)$$

and

$$\frac{d}{d\tau}u(\tau, g[z,u](\tau,t,x)) = \partial_x F(P[z,u](\tau,t,x)) + \partial_w F(P[z,u](\tau,t,x)) \star u_{(\tau,g[z,u](\tau,t,x))}. \quad (39)$$

By integrating of (38) and (39) on $[0, t]$ with respect to τ, we get (35).

We prove that there is a solution $(\bar{z}, \bar{u})$ to (35) defined on E_c where $c \in (0, a]$ is sufficiently a small constant, and $\partial_x \bar{z} = \bar{u}$ and $\bar{z}$ are weak solutions to (28), (29). We first give estimates of solutions to (35).

Lemma 4.1. *Suppose that Assumption $H_0[F]$ is satisfied and*

(1) $\psi \in \mathbb{X}$ and $0 < c \leq a$.
(2) the functions $\tilde{z} : E_c \to \mathbb{R}$, $\tilde{u} : E_c \to \mathbb{R}^n$ are continuous and they satisfy (35).

Then

$$\|\tilde{z}\|_{(t,\mathbb{R})} \leq \tilde{\zeta}(t), \quad \|\tilde{u}\|_{(t,\mathbb{R}^n)} \leq \tilde{\chi}(t) \quad \textit{for} \quad t \in [0, c],$$

where

$$\tilde{\zeta}(t) = c_0 \exp\left\{\int_0^t \beta(\tau)\,d\tau\right\} + \int_0^t \tilde{\gamma}(\xi)\,\exp\left\{\int_\xi^t \beta(\tau)\,d\tau\right\} d\xi,$$
$$\tilde{\gamma}(\xi) = \alpha(\xi) + \tilde{\chi}(\xi)\,[\beta(\xi) + ||L(\xi)||],$$
$$\tilde{\chi}(t) = (c_1 + 1)\,\exp\left\{\int_0^t \beta(\tau)\,d\tau\right\} - 1.$$

Proof. Write

$$\bar{\zeta}(t) = ||\tilde{z}||_{(t,\mathbb{R})}, \quad \bar{\chi}(t) = ||\tilde{u}||_{(t,\mathbb{R}^n)}, \quad t \in [0, c].$$

It follows from Assumption $H_0[F]$ and from (31) - (34) that $(\bar{\zeta}, \bar{\chi})$ satisfy the integral inequalities

$$\bar{\zeta}(t) \leq c_0 + \int_0^t \alpha(\tau)\,d\tau + \int_0^t \beta(\tau)[\bar{\zeta}(\tau) + \bar{\chi}(\tau)]\,d\tau + \int_0^t ||L(\tau)||\,\bar{\chi}(\tau)\,d\tau,$$
$$\bar{\chi}(t) \leq c_1 + \int_0^t \beta(\tau)\,d\tau + \int_0^t \beta(\tau)\bar{\chi}(\tau)\,d\tau,$$

where $t \in [0, c]$. The functions $(\bar{\zeta}, \bar{\chi})$ satisfy integral equations corresponding to the above inequalities. This proves the lemma.

Suppose that $\zeta, \chi : [0, c] \to \mathbb{R}_+$ are continuous and they satisfy the integral inequalities

$$\zeta(t) \geq c_0 + \int_0^t \alpha(\tau)\,d\tau + \int_0^t \beta(\tau)[\zeta(\tau) + \chi(\tau)]\,d\tau + \int_0^t ||L(\tau)||\chi(\tau)\,d\tau,$$
$$\chi(t) \geq c_1 + \int_0^t \beta(\tau)\,d\tau + \int_0^t \beta(\tau)\,\chi(\tau)\,d\tau,$$

where $t \in [0, c]$. It is clear that $(\bar{\zeta}, \bar{\chi})$ satisfy the above conditions.

Given $d, h \in \mathbb{R}_+$, $d \geq c_1$, $h \geq c_2$ and $0 < c \leq a$. Suppose that $\psi \in \mathbb{X}$. We denote by $C_{\psi.c}[\zeta, d]$ the class of all $z \in C_{\psi.c}$ such that

$$||z||_{(t,\mathbb{R})} \leq \zeta(t) \text{ for } t \in [0,c],$$
$$|z(t,x) - z(t,\bar{x})| \leq d||x - \bar{x}|| \text{ on } E \cap ([0,c] \times \mathbb{R}^n).$$

For the above ψ , we denote by $C_{\partial\psi c}[\chi, h]$ the class of all $v \in C_{\partial\psi_i,c}$ satisfying the conditions

$$||v||_{(t,\mathbb{R}^n)} \leq \chi(t) \text{ for } t \in [0,c],$$
$$||v(t,x) - v(t,\bar{x})|| \leq h||x - \bar{x}|| \text{ on } E \cap ([0,c] \times \mathbb{R}^n).$$

Write $A = \zeta(a)$, $C = \chi(a)$ and $\Xi[A, C] = E \times K_{C(B,\mathbb{R})}[A] \times K_{\mathbb{R}^n}[C]$ where

$$K_{C(B,\mathbb{R})}[A] = \{w \in C(B,\mathbb{R}) : ||w||_B \leq A\}, \quad K_{\mathbb{R}^n}[C] = \{q \in \mathbb{R}^n : ||q|| \leq C\}.$$

Assumption $H[F]$. The function $F : \Xi \to \mathbb{R}$ satisfies Assumption $H_0[F]$, and there is $\gamma \in \mathbb{L}([0,a], \mathbb{R}_+)$ such that the terms

$$||\partial_x F(t,x,w,q) - \partial_x F(t,\bar{x},\bar{w},\bar{q})||, \quad ||\partial_w F(t,x,w,q) - \partial_w F(t,\bar{x},\bar{w},\bar{q})||_\star,$$
$$||\partial_q F(t,x,w,q) - \partial_q F(t,\bar{x},\bar{w},\bar{q})||$$

are bounded from above on $\Xi[A,C]$ by $\gamma(t)[||x - \bar{x}|| + ||w - \bar{w}||_B + ||q - \bar{q}||]$.

Remark 4.2. *It is important that we have assumed the Lipschitz condition for $\partial_x F$, $\partial_w F$, $\partial_q F$ for $w, \bar{w}$satisfying the condition: $||w||_B, ||\bar{w}||_B \leq A$.*

There are differential integral equations and differential equations with deviated variables such that Assumption $H[F]$ is satisfied and the functions $\partial_x F$, $\partial_w F$, and $\partial_q F$ do not satisfy the Lipschitz condition with respect to the functional variable on Ξ.

It is clear that there are functional differential equations which satisfy Assumptions $H[F]$ and they do not satisfy the assumptions of the existence theorem presented in [18].

Lemma 4.3. *Suppose that Assumption $H[F]$, $H[\phi]$ are satisfied and*

$$\psi, \tilde{\psi} \in \mathbb{X}, \quad z \in C_{\psi.c}[\zeta, d], \quad \tilde{z} \in C_{\tilde{\psi},c}[\zeta, d], \quad u \in C_{\partial\psi.c}[\chi, h], \quad \tilde{u} \in C_{\partial\tilde{\psi}.c}[\chi, h]$$

where $0 < c \leq a$.

Then the bicharacteristics $g[z, u](\cdot, t, x)$ and $g[\tilde{z}, \tilde{u}](\cdot, t, x)$exist on intervals $[0, \delta[z, u](t, x)]$ and $[0, \delta[\tilde{z}, \tilde{u}](t, x)]$such that for $\tau = \delta[z, u](t, x)$, $\tilde{\tau} = \delta[\tilde{z}, \tilde{u}](t, x)$, we have $(\tau, g[z, u](\tau, t, x)) \in \partial E_c$, $(\tilde{\tau}, g[z, u](\tilde{\tau}, t, x)) \in \partial E_c$, where ∂E_c is the boundary of E_c.

The solution of (30) is unique, and we have the estimates

$$||g[z,u](\tau,t,x) - g[z,u](\tau,t,\bar{x})|| \leq ||x - \bar{x}|| \exp\{\bar{C}\int_\tau^t \gamma(\xi)\,d\xi\} \tag{40}$$

and

$$||g[z,u](\tau,t,x) - g[\tilde{z},\tilde{u}](\tau,t,x)||$$
$$\leq |\int_\tau^t \gamma(\xi)\,d\xi[||z - \tilde{z}||_{(\xi,\mathbb{R})} + ||u - \tilde{u}||_{(\xi,\mathbb{R}^n)}]\,d\xi| \exp\{\bar{C}\int_\tau^t \gamma(\xi)\,d\xi\}, \tag{41}$$

where $\bar{C} = 1 + d + h$, (t,x), $(t,\bar{x}) \in E \cap ([0,c] \times \mathbb{R}^n)$, $\tau \in [0, c]$.

Proof The existence and uniqueness of the solution to (30) follows from classical theorems on Carathéodory solutions of ordinary differential equations. We conclude from

Assumption $H[F]$ that the integral inequalities

$$\begin{aligned}&||g[z,u](\tau,t,x)-g[z,u](\tau,t,\bar{x})||\\ &\leq ||x-\bar{x}||+\bar{C}\left|\int_\tau^t \gamma(\xi)||g[z,u](\xi,t,x)-g[z,u](\xi,t,\bar{x})||\ d\xi\right|\end{aligned}$$

and

$$\begin{aligned}\|g[z,u](\tau,t,x)-g[\tilde{z},\tilde{u}](\tau,t,x)\| \leq &\left|\int_\tau^t \gamma(\xi)[||z-\tilde{z}||_{(\xi,\mathbb{R})}+||u-\tilde{u}||_{(\xi,\mathbb{R}^n)}]\,d\xi\right|\\ &+\bar{C}\left|\int_\tau^t \gamma(\xi)||g[z,u](\xi,t,x)-g[\tilde{z},\tilde{u}](\xi,t,x)||d\xi\right|,\end{aligned}$$

are satisfied. Then, we obtain (40) and (41) from the Gronwall inequality.

Write

$$\Lambda(t)=\exp\left\{\int_0^t \gamma(\tau)\,d\tau\right\}\left[c_1+(1+d)\int_0^t \beta(\tau)\,d\tau+C\bar{C}\int_0^t \gamma(\tau)\,d\tau+2h\int_0^t ||L(\tau)||\,d\tau\right],$$
$$\Gamma(t)=\exp\left\{\int_0^t \gamma(\tau)\,d\tau\right\}\left[c_2+\bar{C}(1+C)\int_0^t \gamma(\tau)\,d\tau+h\int_0^t \beta(\tau)\,d\tau\right].$$

Assumption $H[c]$. The constants $c \in (0, a]$, d, $h > 0$ satisfy the relations: $\Lambda(c) \leq d$, $\Gamma(c) \leq h$.

Remark 4.4. *If we assume that*

$$\exp\left\{\int_0^a \gamma(\tau)\,d\tau\right\}c_1 < d \quad \textit{and} \quad \exp\left\{\int_0^a \gamma(\tau)\,d\tau\right\}c_2 < h,$$

then there is $c \in (0, a]$ *such that* $\Lambda(c) \leq d$ *and* $\Gamma(c) \leq h$.

Theorem 4.5. *Suppose that Assumptions* $H[F]$, $H[c]$ *are satisfied and* $\psi \in \mathbb{X}$. *Then there is a solution* $\bar{z}: E_c \to \mathbb{R}$*of (28), (29).*

If $\tilde{\psi} \in \mathbb{X}$*and* $\tilde{z}: E_c \to \mathbb{R}$*is a solution to (28) with the initial condition*

$$z(t,x)=\tilde{\psi}(t,x) \quad \text{on} \quad E_0,$$

then there is $C_\star \in \mathbb{R}_+$ *such that for* $t \in [0, c]$, *we have.*

$$||\bar{z}-\tilde{z}||_{(t,\mathbb{R})}+||\partial_x\bar{z}-\partial_x\tilde{z}||_{(t,\mathbb{R}^n)} \leq C_\star[||\psi-\tilde{\psi}||_{(E_0,\mathbb{R})}+||\partial_x\psi-\partial_x\tilde{\psi}||_{(E_0,\mathbb{R}^n)}]. \tag{42}$$

Proof The proof will be divided into four steps

I. We define the sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$, where

$$z^{(m)}: E_c \to \mathbb{R}, \quad u^{(m)}: E_c \to \mathbb{R}^n, \quad u^{(m)}=(u_1^{(m)},\ldots u_n^{(m)}),$$

In the following way. We put first

$$z^{(0)}(t,x)=\psi(t,x) \ \text{ on } \ E_0,\ z^{(0)}(t,x)=\psi(0,x) \ \text{ on } \ E\cap([0,c]\times\mathbb{R}^n),$$
$$u^{(0)}(t,x)=\partial_x\psi(t,x) \ \text{ on } \ E_0,\ u^{(0)}(t,x)=\partial_x\psi(0,x) \ \text{ on } \ E\cap([0,c]\times\mathbb{R}^n).$$

If $z^{(m)}$: $E_c \to \mathbb{R}$, $u^{(m)}: E_c \to \mathbb{R}^n$ are already defined then $u^{(m+1)}$ is a solution of the equation

$$v=\mathbb{G}^{(m)}[v] \tag{43}$$

where

$$\mathbb{G}^{(m)}[v](t,x) = \partial_x\psi(t,x) \text{ on } E_0 \tag{44}$$

and

$$\begin{aligned}\mathbb{G}^{(m)}[v](t,x) = \partial_x\psi(0,g[z^{(m)},v](0,t,x)) + \int_0^t \partial_x F(P[z^{(m)},v](\tau,t,x))\,d\tau \\ + \int_0^t \partial_w F(P[z^{(m)},v](\tau,t,x)) \star \left(u^{(m)}\right)_{(\tau,g[z^{(m)},v](\tau,t,x))} d\tau.\end{aligned} \tag{45}$$

The function $z^{(m+1)}$ is given by

$$z^{(m+1)}(t,x) = \mathbb{F}[z^{(m)},u^{(m+1)}](t,x) \text{ on } E_c. \tag{46}$$

We prove that

(I_m) the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are defined on E_c and for $m \geq 0$, we have

$$z^{(m)} \in C_{\psi.c}[\zeta,d] \text{ and } u^{(m)} \in C_{\partial\psi.c}[\chi,h].$$

(II_m) there exist the sequences $\{\partial_x z^{(m)}\}$ and for $m \geq 0$ we have

$$\partial_x z^{(m)}(t,x) = u^{(m)}(t,x) \text{ on } E_c.$$

We prove (I_m), (II_m) by induction. It is easily seen that conditions (I_0), (II_0) are satisfied. Suppose that (I_m) and (II_m) hold for a given $m \geq 0$. We first prove that there is $u^{(m+1)} : E_c \to \mathbb{R}^n$, and $u^{(m+1)} \in C_{\partial\psi.c}[\chi, h]$. We claim that

$$\mathbb{G}^{(m)} : C_{\partial\psi.c}[\chi,h] \to C_{\partial\psi.c}[\chi,h]. \tag{47}$$

Indeed, it follows from Assumption $H[F]$ and from (44), (45) that for $v \in C_{\partial\psi.c}$ we have

$$||\mathbb{G}^{(m)}[v](t,x)|| \leq c_1 + \int_0^t \beta(\tau)\,d\tau + \int_0^t \beta(\tau)\,\chi(\tau)\,d\tau, \quad (t,x) \in E \cap ([0,c] \times \mathbb{R}^n),$$

and consequently

$$||\mathbb{G}^{(m)}||_{(t,\mathbb{R}^n)} \leq \chi(t) \text{ for } t \in [0,c].$$

It follows easily that

$$||\mathbb{G}^{(m)}[v](t,x) - \mathbb{G}^{(m)}[v](t,\bar{x})|| \leq \Gamma(c)||x-\bar{x}|| \text{ on } E \cap ([0,c] \times \mathbb{R}^n).$$

From the above estimates and from (44), we deduce (47).

There is $K \in \mathbb{L}([0,c],\mathbb{R}_+)$ such that for v, $\tilde{v} \in C_{\partial\psi.c}(\chi,h)$, we have

$$||\mathbb{G}^{(m)}[v](t,x) - \mathbb{G}^{(m)}[\tilde{v}](t,x)|| \leq \int_0^t K(\tau)\left\|v-\tilde{v}\right\|_{(\tau,\,\mathbb{R}^n)} d\tau \text{ on } E \cap ([0,c] \times \mathbb{R}^n).$$

For the above v, $\tilde{v}$ we put

$$[|v-\tilde{v}|] = \max\{||v-\tilde{v}||_{(t,\mathbb{R}^n)} \exp[-2\int_0^t K(\tau)\,d\tau] : t \in [0,c]\}.$$

Then, we have

$$||\mathbb{G}^{(m)}[v](t,x) - \mathbb{G}^{(m)}[\tilde{v}](t,x)|| \leq \frac{1}{2}[|v - \tilde{v}|] \exp[2\int_0^t K(\tau)\,d\tau], \quad (t,x) \in E \cap ([0,c] \times \mathbb{R}^n),$$

and consequently

$$[|\mathbb{G}^{(m)}[v] - \mathbb{G}^{(m)}[\tilde{v}]|] \leq \frac{1}{2}[|v - \tilde{v}|].$$

If follows from the Banach fixed point theorem that there is $u^{(m+1)} \in C_{\partial\psi.c}[\chi, h]$ and it is unique.

Then $u^{(m+1)}$ is defined E_c. It follows from Assumption $H[F]$ and from (I_m) that

$$||z^{(m+1)}||_{(t,\mathbb{R})} \leq \zeta(t) \text{ for } t \in [0,c],$$

and

$$|z^{(m+1)}(t,x) - z^{(m+1)}(t,\bar{x})| \leq \Lambda(c)||x - \bar{x}|| \text{ on } E \cap ([0,c] \times \mathbb{R}^n).$$

We conclude from the above estimates that $z^{(m+1)} \in C_{\partial\psi.c}[\zeta, d]$ which completes the proof of (II_{m+1}).

Put

$$U^{(m+1)}(t,x,\bar{x}) = z^{(m+1)}(t,\bar{x}) - z^{(m+1)}(t,x) - u^{(m+1)}(t,x) \circ (\bar{x} - x).$$

It follows easily that there is $C^\star \in \mathbb{R}_+$ such that

$$|U^{(m+1)}(t,x,\bar{x})| \leq C^\star \|x - \bar{x}\|^2, (t,x), (t,\bar{x}) \in E \cap ([0,c] \times \mathbb{R}^n). \tag{48}$$

We conclude from (48) that there exist the derivatives $\partial_x z^{(m+1)}$ and

$$\partial_x z^{(m+1)}(t,x) = u^{(m+1)}(t,x) \text{ on } E \cap ([0,c] \times \mathbb{R}^n.$$

This proves (II_{m+1}).

II. We prove that the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are uniformly convergent on E_c.

It follows from (43)-(46) that there are $K_0, K_1 \in \mathbb{L}([0,c], \mathbb{R}_+)$

$$\begin{aligned} &||z^{(m+1)} - z^{(m)}||_{(t,\mathbb{R})} \\ &\leq \int_0^t K_0(\tau)[||z^{(m)} - z^{(m-1)}||_{(\tau,\mathbb{R})} + ||u^{(m+1)} - u^{(m)}||_{(\tau,\mathbb{R}^n)}]\,d\tau \end{aligned} \tag{49}$$

and

$$\begin{aligned} &||u^{(m+1)} - u^{(m)}||_{(t,\mathbb{R}^n)} \\ &\leq \int_0^t K_1(\tau)[||z^{(m)} - z^{(m-1)}||_{(\tau,\mathbb{R})} + ||u^{(m)} - u^{(m-1)}||_{(\tau,\mathbb{R}^n)} + ||u^{(m+1)} - u^{(m)}||_{(\tau,\mathbb{R}^n)}]\,d\tau, \end{aligned} \tag{50}$$

where $t \in [0, c]$. From (50) and from the Gronwall inequality, we deduce that there is $K_2 \in \mathbb{L}([0,c], \mathbb{R}_+)$ such that

$$\begin{aligned} &||u^{(m+1)} - u^{(m)}||_{(t,\mathbb{R}^n)} \\ &\leq \int_0^t K_2(\tau)[||z^{(m)} - z^{(m-1)}||_{(\tau,\mathbb{R})} + ||u^{(m)} - u^{(m-1)}||_{(\tau,\mathbb{R}^n)}]\,d\tau. \end{aligned} \tag{51}$$

Write

$$V^{(m)}(t) = ||z^{(m)} - z^{(m-1)}||_{(t,\mathbb{R})} + ||u^{(m)} - u^{(m-1)}||_{(t,\mathbb{R}^n)}, \quad t \in [0,c], \quad m \geq 1.$$

It follows from (49), (51) that there is $\tilde{K} \in \mathbb{L}([0,c],\mathbb{R}_+)$ such that

$$V^{(m+1)}(t) \leq \int_0^t \tilde{K}(\tau)\, V^{(m)}(\tau)\, d\tau, \quad t \in [0,c]. \tag{52}$$

Set

$$[|V^{(m)}|] = \max\{V^{(m)}(t)\ \exp[-2\int_0^t \tilde{K}(\tau\, d\tau] : t \in [0,c]\}.$$

We conclude from (52) that

$$V^{(m+1)}(t) \leq [|V^{(m)}|]\ \exp[2\int_0^t \tilde{K}(\tau)\, d\tau] \leq \frac{1}{2}[|V^{(m)}|]\ \exp[2\int_0^t \tilde{K}(\tau)\, d\tau], \quad t \in [0,c],$$

and consequently

$$[|V^{(m+1)}|] \leq \frac{1}{2}[|V^{(m)}|], \quad m \geq 1.$$

There is $C_1 \in \mathbb{R}_+$ such that $[|V^{(1)}|] \leq C_1$. Then,

$$\lim_{m\to\infty}[|V^{(m)}|] = 0$$

and there are

$$\bar{z} \in C(E_c,\mathbb{R}),\ \bar{u} \in C(E_c,\mathbb{R}^n), \quad \bar{u} = (\bar{u}_1,\dots,\bar{u}_n)$$

such that

$$\bar{z}(t,x) = \lim_{m\to\infty} z^{(m)}(t,x), \quad \bar{u}(t,x) = \lim_{m\to\infty} u^{(m)}(t,x) \ \text{ uniformly on } \ E \cap ([0,c]\times\mathbb{R}^n).$$

It follows from (II_m) that there exist the derivatives $\partial_x\bar{z} = (\partial_{x_1}\bar{z},\dots,\partial_{x_n}\bar{z})$, and

$$\partial_x\bar{z}(t,x) = \lim_{m\to\infty} u^{(m)}(t,x) \ \text{ uniformly on } \ E \cap ([0,c]\times\mathbb{R}^n).$$

III. We prove that $\bar{z}$ is a solution to (28), (29). We conclude from (46) that the functions $\bar{z}$, $\partial_z\bar{z}$ satisfy the relations

$$\begin{aligned} \bar{z}(t,x) = \psi(0,g[\bar{z},\partial_x\bar{z}](0,t,x)) + \int_0^t F(P[\bar{z},\partial_z\bar{z}](\tau,t,x))\, d\tau \\ -\int_0^t \partial_q F(P[\bar{z},\partial_x\bar{z}](\tau,t,x)) \circ \partial_x\bar{z}(\tau, g[\bar{z},\partial_x\bar{z}](\tau,t,x))\, d\tau \ \text{ on } \ E \cap ([0,c]\times\mathbb{R}^n). \end{aligned} \tag{53}$$

For a given $(t, x) \in E\cap([0, c] \times \mathbb{R}^n)$ set $y = g[\bar{z},\partial_x\bar{z}](0,t,x)$. Then $g[\bar{z},\partial_x\bar{z}](\tau,t,x) = g[\bar{z},\partial_x\bar{z}](\tau,0,y)$ for $\tau \in [0,\delta[\bar{z},\partial_x\bar{z}](t,x)\,]$. Then relations (53) imply

$$\begin{aligned} \bar{z}(t,g[\bar{z},\partial_x\bar{z}](t,0,y)) = \psi(0,y) + \int_0^t F(P[\bar{z},\partial_x\bar{z}](\tau,0,y))\, d\tau \\ -\int_0^t \partial_q F(P[\bar{z},\partial_x\bar{z}](\tau,0,y)) \circ \partial_x\bar{z}(\tau, g[\bar{z},\partial_x\bar{z}](\tau,0,y))\, d\tau. \end{aligned} \tag{54}$$

The relations $y = g[\bar{z}, \partial_x\bar{z}](0, t, x))$ and $x = g[\bar{z}, \partial_x\bar{z}](\tau, 0, y))$ are equivalent. By differentiating (54) with respect to τ and by putting again $x = g[\bar{z}, \partial_x\bar{z}](\tau, 0, y))$, we find that $\bar{z}$ is a weak solution to (28). Since $\bar{z} \in C_{\psi.c}[\zeta, d]$, it follows that initial condition (29) is satisfied.

IV. Now we prove (42). It follows from (31) - (35) and from Assumption $H[F]$ that there are $\tilde{\alpha}, \tilde{\beta} \in \mathbb{L}([0, c], \mathbb{R}_+)$ such that

$$|\bar{z}(t,x) - \tilde{z}(t,x)| \leq ||\psi - \tilde{\psi}||_{(E_0,\mathbb{R})} + \int_0^t \tilde{\alpha}(\xi)[||\bar{z} - \tilde{z}||_{(\xi,\mathbb{R})} + ||\partial_x\bar{z} - \partial_x\tilde{z}||_{(\xi,\mathbb{R}^n)}]\, d\xi$$

and

$$\left\|\partial_x\bar{z}(t,x) - \partial_x\tilde{z}(t,x)\right\| \leq \left\|\partial_x\psi - \partial_x\tilde{\psi}\right\|_{(E_0,\mathbb{R}^n)} + \int_0^t \tilde{\beta}(\xi)[\|\bar{z} - \tilde{z}\|_{(\xi,\mathbb{R})} + \|\partial_x\bar{z} - \partial_x\tilde{z}\|_{(\xi,\mathbb{R}^n)}]\, d\xi.$$

Hence, there is $\tilde{\gamma} \in \mathbb{L}([0, c], \mathbb{R}_+)$ such that the integral inequality

$$\begin{aligned}\|\bar{z} - \tilde{z}\|_{(t,\mathbb{R})} + \|\partial_x\bar{z} - \partial_x\tilde{z}\|_{(t,\mathbb{R}^n)} &\leq \left\|\psi - \tilde{\psi}\right\|_{(E_0,\mathbb{R})} + \left\|\partial_x\psi - \partial_x\tilde{\psi}\right\|_{(E_0,\mathbb{R}^n)} \\ &+ \int_{\tilde{a}}^t \tilde{\gamma}(\xi)[\|\bar{z} - \tilde{z}\|_{(\xi,\mathbb{R})} + \|\partial_x\bar{z} - \partial_x\tilde{z}\|_{(\xi,\mathbb{R}^n)}]\, d\xi, \quad t \in [0, c],\end{aligned}$$

is satisfied, We conclude from the Gronwall inequality that estimate (42) is satisfied with

$$C_\star = \int_0^c \tilde{\gamma}(\xi)\, d\xi.$$

This completes the proof of the theorem.

Remark 4.6. *It is easy to see that differential integral equations and equations with deviated variables are particular cases of (28).*

Suppose that f: $\Omega \to \mathbb{R}$ *is a given function. Let F*: $\Xi \to \mathbb{R}$ *be defined by*

$$F(t, x, w, q) = f(t, x, w(0, 0_{[n]}), w, q).$$

Then, equation 1 is equivalent to (28). It follows that existence results for (1), (2) can be obtained from Theorem 4.5.

Competing interests
The author declares that they have no competing interests.

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References

1. Lakshmikantham, V, Leela, S: Differential and Integral Inequalities. Acadamic Press, New York (1969)
2. Szarski, J: Differential Inequalities. Polish Scientific Publishers, Warsaw (1967)
3. Ladde, GS, Lakshmikantham, V, Vatsala, A: Monotone Iterative Techniques for Nonlinear Differential Equations. Pitmann Advanced Publishing Program, Boston (1985)
4. Lakshimkantham, V, Vatsala, AS: Generalized Quasilinearization for Nonlinear Problems. Kluwer Acadamic Publication, Dordrecht (1995)
5. Van, TD, Tsuji, M, Thai Son, ND: The Characteristics method and Its Generalizations for First Order Nonlinear Partial Differential equations. Chapmann and Hall/CRC, Boca Raton, FL (2000)
6. Brandi, P, Marcelli, C: Haar Inequality in hereditary setting and applications. Rend Sem Math Univ Padova. **96**, 177–194 (1966)
7. Kamont, Z: Hyperbolic Functional Differential Inequalities. Kluwer Acadamic Publishers, Dordrecht (1999)
8. Kamont, Z: Infinite systems of hyperbolic functional differential inequalities. Nonlinear Anal TMA. **51**, 1429–1445 (2002). doi:10.1016/S0362-546X(01)00907-5
9. Szarski, J: Comparison theorems for infinite systems of differential functional equations and strongly coupled infinite systems of first order partial differential equations. Rocky Mt J Math. **10**, 237–246 (1980)

10. Bainov, D, Kamont, Z, Minchev, E: On first order impulsive partial differential inequalities. Appl Math Comput. **61**, 207–230 (1994). doi:10.1016/0096-3003(94)90048-5
11. Byszewski, L: Finite systems of strong nonlinear differential functional degenerate implicit inequalities with first order partial derivatives. Univ Iagell Acta Math. **29**, 77–84 (1992)
12. Kamont, Z, Kozieł, S: Functional differential inequalities with unbounded delay. Ann Polon Math. **88**, 19–37 (2006). doi:10.4064/ap88-1-2
13. Augustynowicz, A, Kamont, Z: On Kamke's functions in uniqueness theorems for first order partial differential functional equations. Nonlinear Anal TMA. **14**, 837–850 (1990). doi:10.1016/0362-546X(90)90024-B
14. Topolski, K: On the uniqueness of viscosity solutions for first order partial differential functional equations. Ann Plon Math. **59**, 65–75 (1994)
15. Topolski, K: Classical methods for viscosity solutions of differential-functional inequalities. Nonlinear World. **4**, 1–18 (1997)
16. Cinquini, S: Sopra i sistemi iperbolici di equazion a derivate parzaili (nonlinear) in piú variabili indipendenti. Ann Mat pura ed appl. **120**, 201–214 (1979). doi:10.1007/BF02411944
17. Cinquini Cibrario, M: Sopra una class di sistemi de equazioni nonlineari a derivate parziali in piú variabili indipendenti. Ann mat pura ed appl. **140**, 223–253 (1985). doi:10.1007/BF01776851
18. Puźniakowska-Gałuch, E: On the local Cauchy problem for first order partial differential functional equations. Ann Polon Math. **98**, 39–61 (2010). doi:10.4064/ap98-1-3

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