

RESEARCH

Open Access

# Some exponential inequalities for acceptable random variables and complete convergence

Aiting Shen<sup>1</sup>, Shuhe Hu<sup>1</sup>, Andrei Volodin<sup>2\*</sup> and Xuejun Wang<sup>1</sup>

\* Correspondence: volodin@math.uregina.ca

<sup>2</sup>Department of Mathematics and Statistics, University of Regina, Regina Saskatchewan S4S 0A2, Canada

Full list of author information is available at the end of the article

## Abstract

Some exponential inequalities for a sequence of acceptable random variables are obtained, such as Bernstein-type inequality, Hoeffding-type inequality. The Bernstein-type inequality for acceptable random variables generalizes and improves the corresponding results presented by Yang for NA random variables and Wang et al. for NOD random variables. Using the exponential inequalities, we further study the complete convergence for acceptable random variables.

**MSC(2000):** 60E15, 60F15.

**Keywords:** acceptable random variables, exponential inequality, complete convergence

## 1 Introduction

Let  $\{X_n, n \geq 1\}$  be a sequence of random variables defined on a fixed probability space  $(\Omega, \mathcal{F}, P)$ . The exponential inequality for the partial sums  $\sum_{i=1}^n (X_i - EX_i)$  plays an important role in various proofs of limit theorems. In particular, it provides a measure of convergence rate for the strong law of large numbers. There exist several versions available in the literature for independent random variables with assumptions of uniform boundedness or some, quite relaxed, control on their moments. If the independent case is classical in the literature, the treatment of dependent variables is more recent.

First, we will recall the definitions of some dependence structure.

**Definition 1.1.** A finite collection of random variables  $X_1, X_2, \dots, X_n$  is said to be negatively associated (NA) if for every pair of disjoint subsets  $A_1, A_2$  of  $\{1, 2, \dots, n\}$ ,

$$\text{Cov}\{f(X_i : i \in A_1), g(X_j : j \in A_2)\} \leq 0, \quad (1.1)$$

whenever  $f$  and  $g$  are coordinatewise nondecreasing (or coordinatewise nonincreasing) such that this covariance exists. An infinite sequence of random variables  $\{X_n, n \geq 1\}$  is NA if every finite subcollection is NA.

**Definition 1.2.** A finite collection of random variables  $X_1, X_2, \dots, X_n$  is said to be negatively upper orthant dependent (NUOD) if for all real numbers  $x_1, x_2, \dots, x_n$ ,

$$P(X_i > x_i, i = 1, 2, \dots, n) \leq \prod_{i=1}^n P(X_i > x_i), \quad (1.2)$$

and negatively lower orthant dependent (NLOD) if for all real numbers  $x_1, x_2, \dots, x_n$

$$P(X_i \leq x_i, i = 1, 2, \dots, n) \leq \prod_{i=1}^n P(X_i \leq x_i). \tag{1.3}$$

A finite collection of random variables  $X_1, X_2, \dots, X_n$  is said to be negatively orthant dependent (NOD) if they are both NUOD and NLOD. An infinite sequence  $\{X_n, n \geq 1\}$  is said to be NOD if every finite subcollection is NOD.

The concept of NA random variables was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [2]. Joag-Dev and Proschan [2] pointed out that a number of well-known multivariate distributions possesses the negative association property, such as multinomial, convolution of unlike multinomial, multivariate hypergeometric, Dirichlet, permutation distribution, negatively correlated normal distribution, random sampling without replacement, and joint distribution of ranks. The notion of NOD random variables was introduced by Lehmann [3] and developed in Joag-Dev and Proschan [2]. Obviously, independent random variables are NOD. Joag-Dev and Proschan [2] pointed out that NA random variables are NOD, but neither NUOD nor NLOD implies NA. They also presented an example in which  $X = (X_1, X_2, X_3, X_4)$  possesses NOD, but does not possess NA. Hence, we can see that NOD is weaker than NA.

Recently, Giuliano et al. [4] introduced the following notion of acceptability.

**Definition 1.3.** We say that a finite collection of random variables  $X_1, X_2, \dots, X_n$  is acceptable if for any real  $\lambda$ ,

$$E \exp \left( \lambda \sum_{i=1}^n X_i \right) \leq \prod_{i=1}^n E \exp(\lambda X_i). \tag{1.4}$$

An infinite sequence of random variables  $\{X_n, n \geq 1\}$  is acceptable if every finite subcollection is acceptable.

Since it is required that the inequality (1.4) holds for all  $\lambda$ , Sung et al. [5] weakened the condition on  $\lambda$  and gave the following definition of acceptability.

**Definition 1.4.** We say that a finite collection of random variables  $X_1, X_2, \dots, X_n$  is acceptable if there exists  $\delta > 0$  such that for any real  $\lambda \in (-\delta, \delta)$ ,

$$E \exp \left( \lambda \sum_{i=1}^n X_i \right) \leq \prod_{i=1}^n E \exp(\lambda X_i). \tag{1.5}$$

An infinite sequence of random variables  $\{X_n, n \geq 1\}$  is acceptable if every finite subcollection is acceptable.

First, we point out that Definition 1.3 of acceptability will be used in the current article. As is mentioned in Giuliano et al. [4], a sequence of NOD random variables with a finite Laplace transform or finite moment generating function near zero (and hence a sequence of NA random variables with finite Laplace transform, too) provides us an example of acceptable random variables. For example, Xing et al. [6] consider a strictly stationary NA sequence of random variables. According to the sentence above, a sequence of strictly stationary and NA random variables is acceptable.

Another interesting example of a sequence  $\{Z_n, n \geq 1\}$  of acceptable random variables can be constructed in the following way. Feller [[7], Problem III.1] (cf. also

Romano and Siegel [[8], Section 4.30]) provides an example of two random variables  $X$  and  $Y$  such that the density of their sum is the convolution of their densities, yet they are not independent. It is easy to see that  $X$  and  $Y$  are not negatively dependent either. Since they are bounded, their Laplace transforms  $E \exp(\lambda X)$  and  $E \exp(\lambda Y)$  are finite for any  $\lambda$ . Next, since the density of their sum is the convolution of their densities, we have

$$E \exp(\lambda(X + Y)) = E \exp(\lambda X)E \exp(\lambda Y).$$

The announced sequence of acceptable random variables  $\{Z_n, n \geq 1\}$  can be now constructed in the following way. Let  $(X_k, Y_k)$  be independent copies of the random vector  $(X, Y)$ ,  $k \geq 1$ . For any  $n \geq 1$ , set  $Z_n = X_k$  if  $n = 2k + 1$  and  $Z_n = Y_k$  if  $n = 2k$ . Hence, the model of acceptable random variables that we consider in this article (Definition 1.3) is more general than models considered in the previous literature. Studying the limiting behavior of acceptable random variables is of interest.

Recently, Sung et al. [5] established an exponential inequality for a random variable with the finite Laplace transform. Using this inequality, they obtained an exponential inequality for identically distributed acceptable random variables which have the finite Laplace transforms. The main purpose of the article is to establish some exponential inequalities for acceptable random variables under very mild conditions. Furthermore, we will study the complete convergence for acceptable random variables using the exponential inequalities.

Throughout the article, let  $\{X_n, n \geq 1\}$  be a sequence of acceptable random variables and denote  $S_n = \sum_{i=1}^n X_i$  for each  $n \geq 1$ .

**Remark 1.1.** If  $\{X_n, n \geq 1\}$  is a sequence of acceptable random variables, then  $\{-X_n, n \geq 1\}$  is still a sequence of acceptable random variables. Furthermore, we have for each  $n \geq 1$ ,

$$\begin{aligned} E \exp \left( \lambda \sum_{i=1}^n (X_i - EX_i) \right) &= \exp \left( -\lambda \sum_{i=1}^n EX_i \right) E \exp \left( \lambda \sum_{i=1}^n X_i \right) \\ &\leq \left[ \prod_{i=1}^n \exp(-\lambda EX_i) \right] \left[ \prod_{i=1}^n E \exp(\lambda X_i) \right] \\ &= \prod_{i=1}^n E \exp(\lambda(X_i - EX_i)). \end{aligned}$$

Hence,  $\{X_n - EX_n, n \geq 1\}$  is also a sequence of acceptable random variables.

The following lemma is useful.

**Lemma 1.1.** *If  $X$  is a random variable such that  $a \leq X \leq b$ , where  $a$  and  $b$  are finite real numbers, then for any real number  $h$ ,*

$$Ee^{hX} \leq \frac{b - EX}{b - a} e^{ha} + \frac{EX - a}{b - a} e^{hb}. \tag{1.6}$$

*Proof.* Since the exponential function  $\exp(hX)$  is convex, its graph is bounded above on the interval  $a \leq X \leq b$  by the straight line which connects its ordinates at  $X = a$  and  $X = b$ . Thus

$$e^{hX} \leq \frac{e^{hb} - e^{ha}}{b - a}(X - a) + e^{ha} = \frac{b - X}{b - a}e^{ha} + \frac{X - a}{b - a}e^{hb},$$

which implies (1.6).

The rest of the article is organized as follows. In Section 2, we will present some exponential inequalities for a sequence of acceptable random variables, such as Bernstein-type inequality, Hoeffding-type inequality. The Bernstein-type inequality for acceptable random variables generalizes and improves the corresponding results of Yang [9] for NA random variables and Wang et al. [10] for NOD random variables. In Section 3, we will study the complete convergence for acceptable random variables using the exponential inequalities established in Section 2.

## 2 Exponential inequalities for acceptable random variables

In this section, we will present some exponential inequalities for acceptable random variables, such as Bernstein-type inequality and Hoeffding-type inequality.

**Theorem 2.1.** *Let  $\{X_n, n \geq 1\}$  be a sequence of acceptable random variables with  $EX_i = 0$  and  $EX_i^2 = \sigma_i^2 < \infty$  for each  $i \geq 1$ . Denote  $B_n^2 = \sum_{i=1}^n \sigma_i^2$  for each  $n \geq 1$ . If there exists a positive number  $c$  such that  $|X_i| \leq cB_n$  for each  $1 \leq i \leq n, n \geq 1$ , then for any  $\varepsilon > 0$ ,*

$$P(S_n/B_n \geq \varepsilon) \leq \begin{cases} \exp\left[-\frac{\varepsilon^2}{2}\left(1 - \frac{\varepsilon c}{2}\right)\right] & \text{if } \varepsilon c \leq 1, \\ \exp\left(-\frac{\varepsilon}{4c}\right) & \text{if } \varepsilon c \geq 1. \end{cases} \quad (2.1)$$

**Proof.** For fixed  $n \geq 1$ , take  $t > 0$  such that  $tcB_n \leq 1$ . It is easily seen that

$$|EX_i^k| \leq (cB_n)^{k-2}EX_i^2, \quad k \geq 2.$$

Hence,

$$\begin{aligned} Ee^{tX_i} &= 1 + \sum_{k=2}^{\infty} \frac{t^k}{k!} EX_i^k \leq 1 + \frac{t^2}{2} EX_i^2 \left(1 + \frac{t}{3} cB_n + \frac{t^2}{12} c^2 B_n^2 + \dots\right) \\ &\leq 1 + \frac{t^2}{2} EX_i^2 \left(1 + \frac{t}{2} cB_n\right) \leq \exp\left[\frac{t^2}{2} EX_i^2 \left(1 + \frac{t}{2} cB_n\right)\right]. \end{aligned}$$

By Definition 1.3 and the inequality above, we have

$$Ee^{tS_n} = E\left(\prod_{i=1}^n e^{tX_i}\right) \leq \prod_{i=1}^n Ee^{tX_i} \leq \exp\left[\frac{t^2}{2} B_n^2 \left(1 + \frac{t}{2} cB_n\right)\right],$$

which implies that

$$P(S_n/B_n \geq \varepsilon) \leq \exp\left[-t\varepsilon B_n + \frac{t^2}{2} B_n^2 \left(1 + \frac{t}{2} cB_n\right)\right]. \quad (2.2)$$

We take  $t = \frac{\varepsilon}{B_n}$  when  $\varepsilon c \leq 1$ , and take  $t = \frac{1}{cB_n}$  when  $\varepsilon c > 1$ . Thus, the desired result (2.1) can be obtained immediately from (2.2).

**Theorem 2.2.** *Let  $\{X_n, n \geq 1\}$  be a sequence of acceptable random variables with  $EX_i = 0$  and  $|X_i| \leq b$  for each  $i \geq 1$ , where  $b$  is a positive constant. Denote  $\sigma_i^2 = EX_i^2$  and  $B_n^2 = \sum_{i=1}^n \sigma_i^2$  for each  $n \geq 1$ . Then, for any  $\varepsilon > 0$ ,*

$$P(S_n \geq \varepsilon) \leq \exp \left\{ -\frac{\varepsilon^2}{2B_n^2 + \frac{2}{3}b\varepsilon} \right\} \quad (2.3)$$

and

$$P(|S_n| \geq \varepsilon) \leq 2 \exp \left\{ -\frac{\varepsilon^2}{2B_n^2 + \frac{2}{3}b\varepsilon} \right\}. \quad (2.4)$$

**Proof.** For any  $t > 0$ , by Taylor's expansion,  $EX_i = 0$  and the inequality  $1 + x \leq e^x$ , we can get that for  $i = 1, 2, \dots, n$ ,

$$\begin{aligned} E \exp\{tX_i\} &= 1 + \sum_{j=2}^{\infty} \frac{E(tX_i)^j}{j!} \leq 1 + \sum_{j=2}^{\infty} \frac{t^j E|X_i|^j}{j!} \\ &= 1 + \frac{t^2 \sigma_i^2}{2} \sum_{j=2}^{\infty} \frac{t^{j-2} E|X_i|^j}{\frac{1}{2} \sigma_i^2 j!} \\ &= 1 + \frac{t^2 \sigma_i^2}{2} F_i(t) \leq \exp \left\{ \frac{t^2 \sigma_i^2}{2} F_i(t) \right\}, \end{aligned} \quad (2.5)$$

where

$$F_i(t) = \sum_{j=2}^{\infty} \frac{t^{j-2} E|X_i|^j}{\frac{1}{2} \sigma_i^2 j!}, \quad i = 1, 2, \dots, n.$$

Denote  $C = b/3$  and  $M_n = \frac{b\varepsilon}{3B_n^2} + 1$ . Choosing  $t > 0$  such that  $tC < 1$  and

$$tC \leq \frac{M_n - 1}{M_n} = \frac{C\varepsilon}{C\varepsilon + B_n^2}.$$

It is easy to check that for  $i = 1, 2, \dots, n$  and  $j \geq 2$ ,

$$E|X_i|^j \leq \sigma_i^2 b^{j-2} \leq \frac{1}{2} \sigma_i^2 C^{j-2} j!,$$

which implies that for  $i = 1, 2, \dots, n$ ,

$$F_i(t) = \sum_{j=2}^{\infty} \frac{t^{j-2} E|X_i|^j}{\frac{1}{2} \sigma_i^2 j!} \leq \sum_{j=2}^{\infty} (tC)^{j-2} = (1 - tC)^{-1} \leq M_n. \quad (2.6)$$

By Markov's inequality, Definition 1.3, (2.5) and (2.6), we can get

$$P(S_n \geq \varepsilon) \leq e^{-t\varepsilon} E \exp\{tS_n\} \leq e^{-t\varepsilon} \prod_{i=1}^n E \exp\{tX_i\} \leq \exp \left\{ -t\varepsilon + \frac{t^2 B_n^2}{2} M_n \right\}. \quad (2.7)$$

Taking  $t = \frac{\varepsilon}{B_n^2 M_n} = \frac{\varepsilon}{C\varepsilon + B_n^2}$ . It is easily seen that  $tC < 1$  and  $tC = \frac{C\varepsilon}{C\varepsilon + B_n^2}$ . Substituting  $t = \frac{\varepsilon}{B_n^2 M_n}$  into the right-hand side of (2.7), we can obtain (2.3) immediately. By (2.3), we have

$$P(S_n \leq -\varepsilon) = P(-S_n \geq \varepsilon) \leq \exp \left\{ -\frac{\varepsilon^2}{2B_n^2 + \frac{2}{3}b\varepsilon} \right\}, \quad (2.8)$$

since  $\{-X_n, n \geq 1\}$  is still a sequence of acceptable random variables. The desired result (2.4) follows from (2.3) and (2.8) immediately.  $\square$

**Remark 2.1.** By Theorem 2.2, we can get that for any  $t > 0$ ,

$$P(|S_n| \geq nt) \leq 2 \exp \left\{ -\frac{n^2 t^2}{2B_n^2 + \frac{2}{3} bnt} \right\}$$

and

$$P(|S_n| \geq B_n t) \leq 2 \exp \left\{ -\frac{t^2}{2 + \frac{2}{3} \cdot \frac{bt}{B_n}} \right\}.$$

It is well known that the upper bound of  $P(|S_n| \geq nt)$  is also  $2 \exp \left\{ -\frac{n^2 t^2}{2B_n^2 + \frac{2}{3} bnt} \right\}$ . So Theorem 2.3 extends corresponding results for independent random variables without necessarily adding any extra conditions. In addition, it is easy to check that

$$\exp \left\{ -\frac{\varepsilon^2}{2B_n^2 + \frac{2}{3} b\varepsilon} \right\} < \exp \left\{ -\frac{\varepsilon^2}{2(2B_n^2 + b\varepsilon)} \right\},$$

which implies that our Theorem 2.2 generalizes and improves the corresponding results of Yang [9, Lemma 3.5] for NA random variables and Wang et al. [10, Theorem 2.3] for NOD random variables.

In the following, we will provide the Hoeffding-type inequality for acceptable random variables.

**Theorem 2.3.** *Let  $\{X_n, n \geq 1\}$  be a sequence of acceptable random variables. If there exist two sequences of real numbers  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  such that  $a_i \leq X_i \leq b_i$  for each  $i \geq 1$ , then for any  $\varepsilon > 0$  and  $n \geq 1$ ,*

$$P(S_n - ES_n \geq \varepsilon) \leq \exp \left\{ -\frac{2\varepsilon^2}{\sum_{i=1}^n (b_i - a_i)^2} \right\}, \tag{2.9}$$

$$P(S_n - ES_n \leq -\varepsilon) \leq \exp \left\{ -\frac{2\varepsilon^2}{\sum_{i=1}^n (b_i - a_i)^2} \right\}, \tag{2.10}$$

and

$$P(|S_n - ES_n| \geq \varepsilon) \leq 2 \exp \left\{ -\frac{2\varepsilon^2}{\sum_{i=1}^n (b_i - a_i)^2} \right\}. \tag{2.11}$$

**Proof.** For any  $h > 0$ , by Markov's inequality, we can see that

$$P(S_n - ES_n \geq \varepsilon) \leq Ee^{h(S_n - ES_n - \varepsilon)}. \tag{2.12}$$

It follows from Remark 1.1 that

$$Ee^{h(S_n - ES_n - \varepsilon)} = e^{-h\varepsilon} E \left( \prod_{i=1}^n e^{h(X_i - EX_i)} \right) \leq e^{-h\varepsilon} \prod_{i=1}^n Ee^{h(X_i - EX_i)}. \quad (2.13)$$

Denote  $EX_i = \mu_i$  for each  $i \geq 1$ . By  $a_i \leq X_i \leq b_i$  and Lemma 1.1, we have

$$Ee^{h(X_i - EX_i)} \leq e^{-h\mu_i} \left( \frac{b_i - \mu_i}{b_i - a_i} e^{ha_i} + \frac{\mu_i - a_i}{b_i - a_i} e^{hb_i} \right) \doteq e^{L(h_i)}, \quad (2.14)$$

where

$$L(h_i) = -h_i p_i + \ln(1 - p_i + p_i e^{h_i}), \quad h_i = h(b_i - a_i), \quad p_i = \frac{\mu_i - a_i}{b_i - a_i}.$$

The first two derivatives of  $L(h_i)$  with respect to  $h_i$  are

$$L'(h_i) = -p_i + \frac{p_i}{(1 - p_i)e^{-h_i} + p_i}, \quad L''(h_i) = \frac{p_i(1 - p_i)e^{-h_i}}{[(1 - p_i)e^{-h_i} + p_i]^2}. \quad (2.15)$$

The last ratio is of the form  $u(1 - u)$ , where  $0 < u < 1$ . Hence,

$$L''(h_i) = \frac{(1 - p_i)e^{-h_i}}{(1 - p_i)e^{-h_i} + p_i} \left( 1 - \frac{(1 - p_i)e^{-h_i}}{(1 - p_i)e^{-h_i} + p_i} \right) \leq \frac{1}{4}. \quad (2.16)$$

Therefore, by Taylor's expansion and (2.16), we can get

$$L(h_i) \leq L(0) + L'(0)h_i + \frac{1}{8}h_i^2 = \frac{1}{8}h_i^2 = \frac{1}{8}h^2(b_i - a_i)^2. \quad (2.17)$$

By (2.12), (2.13), and (2.17), we have

$$P(S_n - ES_n \geq \varepsilon) \leq \exp \left\{ -h\varepsilon + \frac{1}{8}h^2 \sum_{i=1}^n (b_i - a_i)^2 \right\}. \quad (2.18)$$

It is easily seen that the right-hand side of (2.18) has its minimum at  $h = \frac{4\varepsilon}{\sum_{i=1}^n (b_i - a_i)^2}$ . Inserting this value in (2.18), we can obtain (2.9) immediately. Since  $\{X_n, n \geq 1\}$  is a sequence of acceptable random variables, (2.9) implies (2.10). Therefore, (2.11) follows from (2.9) and (2.10) immediately. This completes the proof of the theorem.

### 3 Complete convergence for acceptable random variables

In this section, we will present some complete convergence for a sequence of acceptable random variables. The concept of complete convergence was introduced by Hsu and Robbins [11] as follows. A sequence of random variables  $\{U_n, n \geq 1\}$  is said to converge completely to a constant  $C$  if  $\sum_{n=1}^{\infty} P(|U_n - C| > \varepsilon) < \infty$  for all  $\varepsilon > 0$ . In view of the Borel-Cantelli lemma, this implies that  $U_n \rightarrow C$  almost surely (a.s.). The converse is true if the  $\{U_n, n \geq 1\}$  are independent. Hsu and Robbins [11] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdős [12] proved the converse. The result of Hsu-Robbins-Erdős is a

fundamental theorem in probability theory and has been generalized and extended in several directions by many authors.

Define the space of sequences

$$\mathcal{H} = \left\{ \{b_n\} : \sum_{n=1}^{\infty} h^{b_n} < \infty \text{ for every } 0 < h < 1 \right\}.$$

The following results are based on the space of sequences  $\mathcal{H}$ .

**Theorem 3.1.** *Let  $\{X_n, n \geq 1\}$  be a sequence of acceptable random variables with  $EX_i = 0$  and  $|X_i| \leq b$  for each  $i \geq 1$ , where  $b$  is a positive constant. Assume that  $\sum_{i=1}^n EX_i^2 = O(b_n)$  for some  $\{b_n\} \in \mathcal{H}$ . Then,*

$$b_n^{-1}S_n \rightarrow 0 \text{ completely as } n \rightarrow \infty. \tag{3.1}$$

**Proof.** For any  $\varepsilon > 0$ , it follows from Theorem 2.2 that

$$\begin{aligned} \sum_{n=1}^{\infty} P(|S_n| \geq b_n \varepsilon) &\leq 2 \sum_{n=1}^{\infty} \exp \left\{ -\frac{b_n^2 \varepsilon^2}{2 \sum_{i=1}^n EX_i^2 + \frac{2}{3} b b_n \varepsilon} \right\} \\ &\leq 2 \sum_{n=1}^{\infty} \exp\{-C b_n\} < \infty, \end{aligned}$$

which implies (3.1). Here,  $C$  is a positive number not depending on  $n$ .  $\square$

**Theorem 3.2.** *Let  $\{X_n, n \geq 1\}$  be a sequence of acceptable random variables with  $|X_i| \leq c < \infty$  for each  $i \geq 1$ , where  $c$  is a positive constant. Then, for every  $\{b_n\} \in \mathcal{H}$ ,*

$$(b_n n)^{-1/2} (S_n - ES_n) \rightarrow 0 \text{ completely as } n \rightarrow \infty. \tag{3.2}$$

**Proof.** For any  $\varepsilon > 0$ , it follows from Theorem 2.3 that

$$\sum_{n=1}^{\infty} P(|S_n - ES_n| \geq (b_n n)^{1/2} \varepsilon) \leq 2 \sum_{n=1}^{\infty} \left[ \exp\left(-\frac{\varepsilon^2}{2c^2}\right) \right]^{b_n} < \infty,$$

which implies (3.2).  $\square$

**Theorem 3.3.** *Let  $\{X_n, n \geq 1\}$  be a sequence of acceptable random variables with  $EX_i = 0$  and  $EX_i^2 = \sigma_i^2 < \infty$  for each  $i \geq 1$ . Denote  $B_n^2 = \sum_{i=1}^n \sigma_i^2$  for each  $n \geq 1$ . For fixed  $n \geq 1$ , there exists a positive number  $H$  such that*

$$|EX_i^m| \leq \frac{m!}{2} \sigma_i^2 H^{m-2}, \quad i = 1, 2, \dots, n \tag{3.3}$$

for any positive integer  $m \geq 2$ . Then,

$$b_n^{-1}S_n \rightarrow 0 \text{ completely as } n \rightarrow \infty, \tag{3.4}$$

provided that  $\{b_n^2/B_n^2\} \in \mathcal{H}$  and  $\{b_n\} \in \mathcal{H}$ .

**Proof.** By (3.3), we can see that

$$Ee^{tX_i} = 1 + \frac{t^2}{2} \sigma_i^2 + \frac{t^3}{6} EX_i^3 + \dots \leq 1 + \frac{t^2}{2} \sigma_i^2 (1 + H|t| + H^2 t^2 + \dots)$$



for  $i = 1, 2, \dots, n, n \geq 1$ . When  $|t| \leq \frac{1}{2H}$ , it follows that

$$Ee^{tX_i} \leq 1 + \frac{t^2\sigma_i^2}{2} \cdot \frac{1}{1-H|t|} \leq 1 + t^2\sigma_i^2 \leq e^{t^2\sigma_i^2}, \quad i = 1, 2, \dots, n. \quad (3.5)$$

Therefore, by Markov's inequality, Definition 1.3 and (3.5), we can get that for any  $x \geq 0$  and  $|t| \leq \frac{1}{2H}$ ,

$$\begin{aligned} P\left(\left|\sum_{i=1}^n X_i\right| \geq x\right) &= P\left(\sum_{i=1}^n X_i \geq x\right) + P\left(\sum_{i=1}^n (-X_i) \geq x\right) \\ &\leq e^{-|t|x} E \exp\left\{|t| \sum_{i=1}^n X_i\right\} + e^{-|t|x} E \exp\left\{|t| \sum_{i=1}^n (-X_i)\right\} \\ &\leq e^{-tx} E \exp\left\{|t| \sum_{i=1}^n X_i\right\} + e^{-tx} E \exp\left\{|t| \sum_{i=1}^n (-X_i)\right\} \\ &= e^{-tx} E \exp\left\{t \sum_{i=1}^n X_i\right\} + e^{-tx} E \exp\left\{t \sum_{i=1}^n (-X_i)\right\} \\ &\leq e^{-tx} \left(\prod_{i=1}^n Ee^{tX_i} + \prod_{i=1}^n Ee^{-tX_i}\right) \\ &\leq 2 \exp\{-tx + t^2B_n^2\}. \end{aligned}$$

Hence,

$$P\left(\left|\sum_{i=1}^n X_i\right| \geq x\right) \leq 2 \min_{|t| \leq \frac{1}{2H}} \exp\{-tx + t^2B_n^2\}.$$

If  $0 \leq x \leq \frac{B_n^2}{H}$ , then

$$\min_{|t| \leq \frac{1}{2H}} \exp\{-tx + t^2B_n^2\} = \exp\left\{-\frac{x}{2B_n^2}x + \frac{x^2}{4B_n^4}B_n^2\right\} = \exp\left\{-\frac{x^2}{4B_n^2}\right\};$$

if  $x \geq \frac{B_n^2}{H}$ , then

$$\min_{|t| \leq \frac{1}{2H}} \exp\{-tx + t^2B_n^2\} = \exp\left\{-\frac{1}{2H}x + \frac{1}{4H^2}B_n^2\right\} \leq \exp\left\{-\frac{x}{4H}\right\}.$$

From the statements above, we can get that

$$P\left(\left|\sum_{i=1}^n X_i\right| \geq x\right) \leq \begin{cases} 2e^{-\frac{x^2}{4B_n^2}}, & 0 \leq x \leq \frac{B_n^2}{H}, \\ 2e^{-\frac{x}{4H}}, & x \geq \frac{B_n^2}{H}, \end{cases}$$

which implies that for any  $x \geq 0$ ,

$$P\left(\left|\sum_{i=1}^n X_i\right| \geq x\right) \leq 2 \exp\left\{-\frac{x^2}{4B_n^2}\right\} + 2 \exp\left\{-\frac{x}{4H}\right\}.$$

Therefore, the assumptions of  $\{b_n\}$  yield that

$$\sum_{n=1}^{\infty} P\left(\left|\frac{1}{b_n} \sum_{i=1}^n X_i\right| \geq \varepsilon\right) \leq 2 \sum_{n=1}^{\infty} \exp\left\{-\frac{b_n^2 \varepsilon^2}{4B_n^2}\right\} + 2 \sum_{n=1}^{\infty} \exp\left\{-\frac{b_n \varepsilon}{4H}\right\} < \infty.$$

This completes the proof of the theorem.  $\square$

#### Acknowledgements

The authors are most grateful to the editor and anonymous referee for the careful reading of the manuscript and valuable suggestions which helped in significantly improving an earlier version of this article. The study was supported by the National Natural Science Foundation of China (11171001, 71071002, 11126176) and the Academic Innovation Team of Anhui University (KJTD001B).

#### Author details

<sup>1</sup>School of Mathematical Science, Anhui University, Hefei 230039, China <sup>2</sup>Department of Mathematics and Statistics, University of Regina, Regina Saskatchewan S4S 0A2, Canada

#### Authors' contributions

Some exponential inequalities for a sequence of acceptable random variables are obtained, such as Bernstein-type inequality, Hoeffding-type inequality. The complete convergence is further studied by using the exponential inequalities. All authors read and approved the final manuscript.

#### Competing interests

The authors declare that they have no competing interests.

Received: 6 July 2011 Accepted: 22 December 2011 Published: 22 December 2011

#### References

1. Alam, K, Saxena, KML: Positive dependence in multivariate distributions. *Commun Stat Theory Methods*. **10**, 1183–1196 (1981). doi:10.1080/03610928108828102
2. Joag-Dev, K, Proschan, F: Negative association of random variables with applications. *Ann Stat*. **11**(1), 286–295 (1983). doi:10.1214/aos/1176346079
3. Lehmann, E: Some concepts of dependence. *Ann Math Stat*. **37**, 1137–1153 (1966). doi:10.1214/aoms/1177699260
4. Giuliano, AR, Kozachenko, Y, Volodin, A: Convergence of series of dependent  $\varphi$ -subGaussian random variables. *J Math Anal Appl*. **338**, 1188–1203 (2008). doi:10.1016/j.jmaa.2007.05.073
5. Sung, SH, Srisuradetchai, P, Volodin, A: A note on the exponential inequality for a class of dependent random variables. *J Korean Stat Soc*. **40**(1), 109–114 (2011). doi:10.1016/j.jkss.2010.08.002
6. Xing, G, Yang, S, Liu, A, Wang, X: A remark on the exponential inequality for negatively associated random variables. *J Korean Stat Soc*. **38**, 53–57 (2009). doi:10.1016/j.jkss.2008.06.005
7. Feller, W: *An Introduction to Probability Theory and its Applications*. Wiley, New York, **2II** (1971)
8. Romano, JP, Siegel, AF: *The Wadsworth & Brooks/Cole Statistics/Probability Series, Counterexamples in Probability and Statistics*. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey (1986)
9. Yang, SC: Uniformly asymptotic normality of the regression weighted estimator for negatively associated samples. *Stat Probab Lett*. **62**, 101–110 (2003)
10. Wang, XJ, Hu, SH, Yang, WZ, Ling, NX: Exponential inequalities and inverse moment for NOD sequence. *Stat Probab Lett*. **80**, 452–461 (2010). doi:10.1016/j.spl.2009.11.023
11. Hsu, PL, Robbins, H: Complete convergence and the law of large numbers. *Proc Natl Acad Sci USA*. **33**(2), 25–31 (1947). doi:10.1073/pnas.33.2.25
12. Erdős, P: On a theorem of Hsu and Robbins. *Ann Math Stat*. **20**(2), 286–291 (1949). doi:10.1214/aoms/1177730037

doi:10.1186/1029-242X-2011-142

**Cite this article as:** Shen et al.: Some exponential inequalities for acceptable random variables and complete convergence. *Journal of Inequalities and Applications* 2011 **2011**:142.