# RESEARCH

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# Uniqueness of meromorphic functions concerning differential polynomials share one value

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# Abstract

In this paper, we study the uniqueness of meromorphic functions whose differential polynomial share a non-zero finite value. The results in this paper improve some results given by Fang (Math. Appl. **44**, 828-831, 2002), Banerjee (Int. J. Pure Appl. Math. **48**, 41-56, 2008) and Lahiri-Sahoo (Arch. Math. (Brno) **44**, 201-210, 2008). **2010 Mathematics Subject Classification**: 30D35

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# 1 Introduction and main results

In this paper, by meromorphic functions, we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [1-3]. It will be convenient to let *E* denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function *h*, we denote by T(r, h) the Nevanlinna characteristic of *h* and by S(r, h) any quantity satisfying  $S(r, h) = o\{T(r, h)\}$ , as  $r \to \infty$ ,  $r \notin E$ .

Let f and g be two non-constant meromorphic functions and let a be a finite complex value. We say that f and g share a CM, provided that f - a and g - a have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that f - a and g - a have the same zeros ignoring multiplicities. In addition, we say that f and g share  $\infty$  CM, if 1/f and 1/g share 0 CM, and we say that f and g share  $\infty$  IM, if 1/f and 1/g share 0 IM (see [3]). Suppose that f and g share a IM. Throughout this paper, we denote by  $\bar{N}_L\left(r, \frac{1}{f-a}\right)$  the reduced counting function of those common a-points of f and g in |z| < r, where the multiplicity of each such a-point of f is greater than that of the corresponding a-point of g, and denote by  $N_{11}\left(r, \frac{1}{f-a}\right)$  the counting function for common simple 1-point of both f and g. In addition, we need the following three definitions:

**Definition 1.1** Let *f* be a non-constant meromorphic function, and let *p* be a positive integer and  $a \in C \cup \{\infty\}$ . Then by  $N_{p}(r, 1/(f - a))$ , we denote the counting function of those *a*-points of *f* (counted with proper multiplicities) whose multiplicities are not



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greater than p, by  $\bar{N}_{p}(r, 1/(f-a))$  we denote the corresponding reduced counting function (ignoring multiplicities). By  $N_{(p}(r,1/(f-a)))$ , we denote the counting function of those *a*-points of f (counted with proper multiplicities) whose multiplicities are not less than p, by  $\bar{N}_{(p}(r, 1/(f-a)))$  we denote the corresponding reduced counting function (ignoring multiplicities), where and what follows,  $N_{p}(r,1/(f-a)), \bar{N}_{p}(r, 1/(f-a)), N_{(p}(r, 1/(f-a))), \bar{N}_{(p}(r, 1/(f-a)))$  mean  $N_{p}(r,f), \bar{N}_{p}(r,f), N_{(p}(r,f))$ , and  $\bar{N}_{(p}(r, f))$ , respectively, if  $a = \infty$ .

**Definition 1.2** Let f be a non-constant meromorphic function, and let a be any value in the extended complex plane, and let k be an arbitrary nonnegative integer. We define

$$\delta_k(a,f) = 1 - \lim_{n \to \infty} \frac{N_k\left(r, \frac{1}{f-a}\right)}{T(r,f)},\tag{1}$$

where

$$N_k\left(r,\frac{1}{f-a}\right) = \bar{N}\left(r,\frac{1}{f-a}\right) + \bar{N}_{\left(2\right)}\left(r,\frac{1}{f-a}\right) + \dots + \bar{N}_{\left(k\right)}\left(r,\frac{1}{f-a}\right).$$
(2)

**Remark 1.1.** From (1) and (2), we have  $0 \le \delta_k(a, f) \le \delta_{k-1}(a, f) \le \delta_1(a, f) \le \Theta(a, f) \le 1$ . **Definition 1.3** Let *f* be a non-constant meromorphic function, and let *a* be any value in the extended complex plane, and let *k* be an arbitrary nonnegative integer.

We define

$$\Theta_{k}(a,f) = 1 - \lim_{n \to \infty} \frac{\bar{N}_{k}\left(r, \frac{1}{f-a}\right)}{T(r,f)}.$$
(3)

**Remark 1.2.** From (3), we have  $0 \le \Theta(a, f) \le \Theta_{k}(a, f) \le \Theta_{k-1}(a, f) \le \Theta_{1}(a, f) \le 1$ .

**Definition 1.4** Let k be a positive integer. Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let  $z_0$  be a 1-point of f with multiplicity p, and a 1-point of g with multiplicity q. We denote by  $\bar{N}_{f>k}\left(r, \frac{1}{g-1}\right)$  the reduced counting function of those 1-points of f and g such that p > q = k.  $\bar{N}_{g>k}\left(r, \frac{1}{f-1}\right)$  is defined analogously.

It is natural to ask the following question:

**Question 1.1** What can be said about the relationship between two meromorphic functions f,g when two differential polynomials, generated by f and g, respectively, share certain values?

Regarding Question 1.1, we first recall the following result by Yang and Hua [4]:

**Theorem A.** Let f(z) and g(z) be two non-constant meromorphic functions,  $n \ge 11$  an integer and  $a \in \mathbb{C}$  - {0}. If  $f^n f$  and  $g^n g'$  share the value  $a \in \mathbb{C}$ , then either f = tg for a constant t with  $t^{n+1} = 1$  or  $g(z) = c_1 e^{cz}$  and  $f(z) = c_2 e^{-cz}$ , where c,  $c_1$  and  $c_2$  are constants satisfying  $(c_1 c_2)^{n+1} c^2 = -a^2$ .

Considering *k*th derivative instead of 1st derivative Fang [5] proved the following theorems.

**Theorem B.** Let f(z) and g(z) be two non-constant entire functions, and let *n*, *k* be two positive integers with n > 2k + 4. If  $[f^n]^{(k)}$  and  $[g^n]^{(k)}$  share 1 CM, then either f = tg for a constant *t* with  $t^n = 1$  or  $f(z) = c_1 e^{cz}$  and  $g(z) = c_2 e^{-cz}$ , where *c*,  $c_1$  and  $c_2$  are constants satisfying  $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ .

**Theorem C.** Let f(z) and g(z) be two non-constant entire functions, and let n, k be two positive integers with  $n \ge 2k + 8$ . If  $[f''(z)(f(z) - 1)]^{(k)}$  and  $[g''(z)(g(z) - 1)]^{(k)}$  share 1 CM, then  $f(z) \equiv g(z)$ .

In 2008, Banerjee [6] proved the following theorem.

**Theorem D.** Let *f* and *g* be two transcendental meromorphic functions, and let *n*, *k* be two positive integers with  $n \ge 9k + 14$ . Suppose that  $[f^n]^{(k)}$  and  $[g^n]^{(k)}$  share a non-zero constant *b* IM, then either f = tg for a constant *t* with  $t^n = 1$  or  $f(z) = c_1e^{cz}$  and *g*  $(z) = c_2e^{-cz}$ , where *c*,  $c_1$  and  $c_2$  are constants satisfying  $(-1)^k(c_1 c_2)^n(nc)^{2k} = b^2$ .

Recently, Lahiri and Sahoo [7] proved the following theorem.

**Theorem E.** Let *f* and *g* be two non-constant meromorphic functions, and  $\alpha (\neq 0, \infty)$  be a small function of *f* and *g*. Let *n* and  $m \geq 2$  be two positive integers with  $n > \max \{4, 4m + 22 - 5\Theta(\infty, f) - 5\Theta(\infty, g) - \min[\Theta(\infty, f), \Theta(\infty, g)]\}$ . If  $f^n(f^m - a)f$  and  $g^n(g^m - a)g'$  share  $\alpha$  IM for a non-zero constant *a*, then either  $f \equiv g$  or  $f \equiv -g$ .

Also, the possibility  $f \equiv -g$  does not arise if *n* and *m* are both even, both odd or *n* is even and *m* is odd.

One may ask, what can be said about the relationship between f and g, if we relax the nature of sharing values of Theorem D and Theorem E ? In this paper, we prove:

**Theorem 1.1.** Let f(z) and g(z) be two non-constant meromorphic functions, and let  $n(\ge 1)$ ,  $k(\ge 1)$  and  $m(\ge 0)$  be three integers. Let  $[f^n(f-1)^m]^{(k)}$  and  $[g^n(g-1)^m]^{(k)}$  share the value 1 IM. Then, one of the following holds:

(i) When m = 0 and n > 9k + 14, then either  $f(z) = c_1 e^{cz}$  and  $g(z) = c_2 e^{-cz}$ , where  $c_1$  and  $c_2$  are constants satisfying  $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$  or f = tg for a constant t with  $t^n = 1$ .

(ii) When m = 1, n > 9k + 18 and  $\Theta(\infty, f) > \frac{2}{n}$ , then  $f \equiv g$ .

(iii) When  $m \ge 2$ , n > 4m + 9k + 14, then  $f \equiv g$  or f and g satisfies the algebraic equation  $R(x, y) = x^n(x - 1)^m - y^n(y - 1)^m = 0$ .

**Theorem 1.2.** Let f(z) and g(z) be two non-constant meromorphic functions, and let  $m, n(\ge 2)$  and k be three positive integers such that n > 4m + 9k + 14. If  $[f^n(f^m - a)]^{(k)}$  and  $[g^n(g^m - a)]^{(k)}$  share the value 1 IM, where  $a(\ne 0)$  is a finite complex number, then either  $f \equiv g$  or  $f \equiv -g$ .

The possibility  $f \equiv -g$  does not arise if *n* and *m* are both odd or if *n* is even and *m* is odd or if *n* is odd and *m* is even.

**Remark 1.3**. If m = 0, m = 1, then the cases become Theorem 1.1 (i) (ii).

**Theorem 1.3.** Let f(z) and g(z) be two non-constant entire functions, and let  $n(\ge 1)$ ,  $k(\ge 1)$  and  $m(\ge 0)$  be three integers. Let  $[f^n(f - 1)^m]^{(k)}$  and  $[g^n(g - 1)^m]^{(k)}$  share the value 1 IM. Then, one of the following holds:

(i) When m = 0 and n > 5k + 7, then either  $f(z) = c_1 e^{cz}$  and  $g(z) = c_2 e^{-cz}$ , where c,  $c_1 and c_2$  are constants satisfying  $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$  or f = tg for a constant t with  $t^n = 1$ .

(ii) When  $m \ge 1$ , n > 4m + 5k + 7, then  $f \equiv g$  or f and g satisfies the algebraic equation  $R(x, y) = x^n(x - 1)^m - y^n(y - 1)^m = 0$ .

**Theorem 1.4.** Let f(z) and g(z) be two non-constant entire functions, and let m,  $n(\ge 1)$  and k be three positive integers such that n > 4m + 5k + 7. If  $[f^n(f^m - a)]^{(k)}$  and  $[g^n(g^m - a)]^{(k)}$  share the value 1 IM, where  $a(\ne 0)$  is a finite complex number, then either  $f \equiv g$  or  $f \equiv -g$ .

The possibility  $f \equiv -g$  does not arise if *n* and *m* are both odd or if *n* is even and *m* is odd or if *n* is odd and *m* is even.

**Remark 1.4.** If m = 0, then the cases becomes Theorem 1.3 (i).

## 2 Some lemmas

**Lemma 2.1**. (See [2,3].) Let f(z) be a non-constant meromorphic function, k a positive integer and let c be a non-zero finite complex number. Then,

$$T(r,f) \leq \bar{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-c}\right) - N\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f)$$

$$\leq \bar{N}(r,f) + N_{k+1}\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{f^{(k)}-c}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f).$$
(4)

where  $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$  is the counting function, which only counts those points such that  $f^{(k+1)} = 0$  but  $f(f^{(k)}-c) \neq 0$ 

**Lemma 2.2**. (See [8].) Let f(z) be a non-constant meromorphic function, and let k be a positive integer.

Suppose that  $f^{(k)} \neq 0$ , then

$$N\left(r,\frac{1}{f^{(k)}}\right) \le N\left(r,\frac{1}{f}\right) + k\bar{N}(r,f) + S(r,f)$$

**Lemma 2.3.** (See [9].) Let f(z) be a non-constant meromorphic function, *s*, *k* be two positive integers, then

$$N_s\left(r,\frac{1}{f^{(k)}}\right) \leq k\bar{N}(r,f) + N_{s+k}\left(r,\frac{1}{f}\right) + S(r,f).$$

Clearly,  $\overline{N}\left(r, \frac{1}{f^{(k)}}\right) = N_1\left(r, \frac{1}{f^{(k)}}\right).$ 

**Lemma 2.4**. (See [10].) Let *f*, *g* share (1,0). Then

(i) 
$$\bar{N}_{f>1}\left(r,\frac{1}{g-1}\right) \leq \bar{N}\left(r,\frac{1}{f}\right) + \bar{N}(r,f) - N_0\left(r,\frac{1}{f'}\right) + S(r,f),$$
  
(ii)  $\bar{N}_{g>1}\left(r,\frac{1}{f-1}\right) \leq \bar{N}\left(r,\frac{1}{g}\right) + \bar{N}(r,g) - N_0\left(r,\frac{1}{g'}\right) + S(r,g).$ 

**Lemma 2.5.** Let f(z) and g(z) be two non-constant meromorphic functions such that  $f^{(k)}$  and  $g^{(k)}$  share 1 IM, where k be a positive integer. If

$$\Delta = (2k+4)\Theta(\infty,g) + (2k+3)\Theta(\infty,f) + \delta_{k+2}(0,g) + \delta_{k+2}(0,f) + \delta_{k+1}(0,f) + 2\delta_{k+1}(0,g) > 4k+11$$

then either  $f^{(k)}g^{(k)} \equiv 1$  or  $f \equiv g$ . **Proof.** Let

$$\Phi(z) = \frac{f^{(k+2)}}{f^{(k+1)}} - 2\frac{f^{(k+1)}}{f^{(k)} - 1} - \frac{g^{(k+2)}}{g^{(k+1)}} + 2\frac{g^{(k+1)}}{g^{(k)} - 1}.$$
(5)

Clearly  $m(r, \Phi) = S(r, f) + S(r, g)$ . We consider the cases  $\Phi(z) \neq 0$  and  $\Phi(z) \equiv 0$ .

Let  $\Phi(z) \neq 0$ , then if  $z_0$  is a common simple 1-point of  $f^{(k)}$  and  $g^{(k)}$ , substituting their Taylor series at  $z_0$  into (5), we see that  $z_0$  is a zero of  $\Phi(z)$ . Thus, we have

$$N_{11}\left(r,\frac{1}{f^{(k)}-1}\right) = N_{11}\left(r,\frac{1}{g^{(k)}-1}\right) \le \bar{N}\left(r,\frac{1}{\Phi}\right) \le T(r,\Phi) + O(1) \le N(r,\Phi) + S(r,f) + S(r,g).$$
(6)

Our assumptions are that  $\Phi(z)$  has poles, all simple only at zeros of  $f^{(k+1)}$  and  $g^{(k+1)}$  and poles of f and g, and 1-points of f whose multiplicities are not equal to the multiplicities of the corresponding 1-points of g. Thus, we deduce from (5) that

$$N(r, \Phi) \leq \bar{N}(r, f) + \bar{N}(r, g) + \bar{N}_{(k+2)}\left(r, \frac{1}{f}\right) + \bar{N}_{(k+2)}\left(r, \frac{1}{g}\right) + N_0\left(r, \frac{1}{f^{(k+1)}}\right) + N_0\left(r, \frac{1}{g^{(k+1)}}\right) + \bar{N}_L\left(r, \frac{1}{f^{(k)} - 1}\right) + \bar{N}_L\left(r, \frac{1}{g^{(k)} - 1}\right).$$
(7)

here  $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$  has the same meaning as in Lemma 2.1. From Lemma 2.1, we have

$$T(r,g) \leq \bar{N}(r,g) + N_{k+1}\left(r,\frac{1}{g}\right) + \bar{N}\left(r,\frac{1}{g^{(k)}-1}\right) - N_0\left(r,\frac{1}{g^{(k+1)}}\right) + S(r,g).$$
(8)

Since

$$\bar{N}\left(r,\frac{1}{g^{(k)}-1}\right) = N_{11}\left(\left(r,\frac{1}{g^{(k)}-1}\right) + \bar{N}_{(2}\left(r,\frac{1}{f^{(k)}-1}\right) + \bar{N}_{g^{(k)}>1}\left(r,\frac{1}{f^{(k)}-1}\right).$$
(9)

Thus, we deduce from (6)-(9) that

$$T(r,g) \leq 2\tilde{N}(r,g) + \tilde{N}(r,f) + N_{k+1}\left(r,\frac{1}{g}\right) + \tilde{N}_{(k+2}\left(r,\frac{1}{f}\right) + \tilde{N}_{(k+2}\left(r,\frac{1}{g}\right) + N_0\left(r,\frac{1}{f^{(k+1)}}\right) + \tilde{N}_{(2}\left(r,\frac{1}{f^{(k)}-1}\right) + \tilde{N}_{L}\left(r,\frac{1}{g^{(k)}-1}\right) + \tilde{N}_{L}\left(r,\frac{1}{g^{(k)}-1}\right) + \tilde{N}_{g^{(k)}>1}\left(r,\frac{1}{f^{(k)}-1}\right) + S(r,f) + S(r,g).$$

$$(10)$$

From the definition of  $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ , we see that

$$N_0\left(r,\frac{1}{f^{(k+1)}}\right) + \bar{N}_{(2}\left(r,\frac{1}{f^{(k)}-1}\right) + N_{(2}\left(r,\frac{1}{f^{(k)}}\right) - \bar{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right) \le N\left(r,\frac{1}{f^{(k+1)}}\right).$$

The above inequality and Lemma 2.2 give

$$\begin{split} N_{0}\left(r,\frac{1}{f^{(k+1)}}\right) + \bar{N}_{(2}\left(r,\frac{1}{f^{(k)}-1}\right) &\leq N\left(r,\frac{1}{f^{(k+1)}}\right) - N_{(2}\left(r,\frac{1}{f^{(k)}}\right) + \bar{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right) \\ &\leq N\left(r,\frac{1}{f^{(k)}}\right) - N_{(2}\left(r,\frac{1}{f^{(k)}}\right) + \bar{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right) + \bar{N}(r,f) + S(r,f) \qquad (11) \\ &\leq \bar{N}\left(r,\frac{1}{f^{(k)}}\right) + \bar{N}(r,f) + S(r,f). \end{split}$$

Substituting (11) in (10), we get

$$\begin{split} T(r,g) &\leq 2\bar{N}(r,g) + \bar{N}(r,f) + N_{k+1}\left(r,\frac{1}{g}\right) + \bar{N}_{(k+2}\left(r,\frac{1}{f}\right) + \bar{N}_{(k+2}\left(r,\frac{1}{g}\right) + \bar{N}\left(r,\frac{1}{f^{(k)}}\right) + \bar{N}(r,f) \\ &+ \bar{N}_L\left(r,\frac{1}{f^{(k)}-1}\right) + \bar{N}_L\left(r,\frac{1}{g^{(k)}-1}\right) + \bar{N}_{g^{(k)}>1}\left(r,\frac{1}{f^{(k)}-1}\right) + S(r,f) + S(r,g) \\ &\leq 2\bar{N}(r,g) + 2\bar{N}(r,f) + N_{k+2}\left(r,\frac{1}{g}\right) + \bar{N}_{(k+2}\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{f^{(k)}}\right) + \bar{N}_L\left(r,\frac{1}{f^{(k)}-1}\right) \\ &+ \bar{N}_L\left(r,\frac{1}{g^{(k)}-1}\right) + \bar{N}_{g^{(k)}>1}\left(r,\frac{1}{f^{(k)}-1}\right) + S(r,f) + S(r,g). \end{split}$$
(12)

According to Lemma 2.3,

$$\bar{N}\left(r,\frac{1}{f^{(k)}}\right) = N_1\left(r,\frac{1}{f^{(k)}}\right) \le N_{k+1}\left(r,\frac{1}{f}\right) + k\bar{N}(r,f) + S(r,f).$$

$$\tag{13}$$

Therefore,

$$\begin{split} \bar{N}_{L}\left(r,\frac{1}{f^{(k)}-1}\right) &\leq N\left(r,\frac{1}{f^{(k)}-1}\right) - \bar{N}\left(r,\frac{1}{f^{(k)}-1}\right) \\ &\leq N\left(r,\frac{f^{(k)}}{f^{(k+1)}}\right) \leq N\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right) + S(r,f) \\ &\leq \bar{N}\left(r,\frac{1}{f^{(k)}}\right) + \bar{N}(r,f) + S(r,f) \\ &\leq N_{k+1}\left(r,\frac{1}{f}\right) + (k+1)\bar{N}(r,f) + S(r,f). \end{split}$$

similarly,

$$\bar{N}_L\left(r, \frac{1}{g^{(k)}-1}\right) \leq N_{k+1}\left(r, \frac{1}{g}\right) + (k+1)\bar{N}(r, g) + S(r, g).$$

Combining the above inequality, Lemma 2.4 and (12), we obtain

$$T(r,g) \leq (2k+4)\bar{N}(r,g) + (2k+3)\bar{N}(r,f) + N_{k+2}\left(r,\frac{1}{g}\right) + N_{k+2}\left(r,\frac{1}{f}\right) + N_{k+1}\left(r,\frac{1}{f}\right) + 2N_{k+1}\left(r,\frac{1}{g}\right) - N_0\left(r,\frac{1}{g^{(k+1)}}\right) + S(r,f) + S(r,g) \leq (2k+4)\bar{N}(r,g) + (2k+3)\bar{N}(r,f) + N_{k+2}\left(r,\frac{1}{g}\right) + N_{k+2}\left(r,\frac{1}{f}\right) + N_{k+1}\left(r,\frac{1}{f}\right) + 2N_{k+1}\left(r,\frac{1}{g}\right) + S(r,f) + S(r,g).$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that  $T(r, f) \leq T(r, g)$  for  $r \in I$ . Hence,

$$T(r,g) \le \{(2k+4)[1-\Theta(\infty,g)] + (2k+3)[1-\Theta(\infty,f)] + [1-\delta_{k+2}(0,g)] + [1-\delta_{k+2}(0,f)] + [1-\delta_{k+1}(0,f)] + 2[1-\delta_{k+1}(0,g)] + \varepsilon\}T(r,g) + S(r,g).$$

for  $\in$  *I* and 0 < $\varepsilon$  <  $\Delta$  - (4*k* +11)

Therefore, we can get  $T(r, g) \leq S(r, g), r \in I$ , by the condition, a contradiction. Hence, we get  $\Phi(z) \equiv 0$ . Then, by (5), we have

$$\frac{f^{(k+2)}}{f^{(k+1)}} - \frac{2f^{(k+1)}}{f^{(k)} - 1} \equiv \frac{g^{(k+2)}}{g^{(k+1)}} - \frac{2g^{(k+1)}}{g^{(k)} - 1}.$$

By integrating two sides of the above equality, we obtain

$$\frac{1}{f^{(k)} - 1} = \frac{bg^{(k)} + a - b}{g^{(k)} - 1}.$$
(14)

where  $a \neq 0$  and *b* are constants. We consider the following three cases:

**Case 1.**  $b \neq 0$  and a = b

(i) If b = -1, then from (14), we obtain that  $f^{(k)}g^{(k)} \equiv 1$ . (ii) If  $b \neq -1$ , then from (14), we get

$$f^{(k)} = \frac{(1+b)g^{(k)} - 1}{bg^{(k)}}.$$
(15)

From (15), we get

$$\bar{N}\left(r, \frac{1}{g^{(k)} - 1/(1+b)}\right) = \bar{N}\left(r, \frac{1}{f^{(k)}}\right).$$
(16)

Combing (13) (16) and Lemma 2.1, we have

$$T(r,g) \leq \bar{N}(r,g) + N_{k+1}\left(r,\frac{1}{g}\right) + \bar{N}\left(r,\frac{1}{g^{(k)} - 1/(b+1)}\right) - N_0\left(r,\frac{1}{g^{(k+1)}}\right) + S(r,g)$$

$$\leq \bar{N}(r,g) + N_{k+1}\left(r,\frac{1}{g}\right) + k\bar{N}(r,f) + N_{k+1}\left(r,\frac{1}{f}\right) + S(r,f) + S(r,g).$$
(17)

From (17), we get

$$\Theta(\infty,g) + k\Theta(\infty,f) + \delta_{k+1}(0,g) + \delta_{k+1}(0,f) \le k+2.$$

By the condition, we get a contradiction.

**Case 2**.  $b \neq 0$  and  $a \neq b$ . (i) If b = -1, then  $a \neq 0$ , from (14) we obtain

$$f^{(k)} = \frac{a}{a+1-g^{(k)}}.$$
(18)

From (18), we get

$$\bar{N}\left(r, \frac{1}{g^{(k)} - (a+1)}\right) = \bar{N}(r, f).$$
(19)

From (19) and Lemma 2.1 and in the same manner as in the proof of (17), we get

$$T(r,g) \leq \bar{N}(r,g) + N_{k+1}\left(r,\frac{1}{g}\right) + \bar{N}\left(r,\frac{1}{g^{(k)} - (a+1)}\right) + S(r,g)$$
  
$$\leq \bar{N}(r,g) + N_{k+1}\left(r,\frac{1}{g}\right) + \bar{N}(r,f) + S(r,g).$$

Using the argument as in case 1, we get a contradiction.

(ii) If  $b \neq -1$ , then from (14), we get

$$f^{(k)} - \left(1 + \frac{1}{b}\right) = \frac{-a}{b^2 [g^{(k)} + \frac{a - b}{b}]}.$$
(20)

From (20), we get

$$\bar{N}\left[r, \frac{1}{g^{(k)} + \left(\frac{a-b}{b}\right)}\right] = \bar{N}\left[f^{(k)} - \left(1 + \frac{1}{b}\right)\right] = \bar{N}(r, f^{(k)}) = \bar{N}(r, f).$$
(21)

Using the argument as in case 1, we get a contradiction.

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**Case 3**. b = 0. From (14), we obtain

$$f^{(k)} = \frac{1}{a}g^{(k)} + 1 - \frac{1}{a},$$
(22)

$$f = \frac{1}{a}g + p(z). \tag{23}$$

where p(z) is a polynomial with its degree  $\leq k$ . If  $p(z) \neq 0$ , then by second fundamental theorem for small functions, we have

$$T(r,g) \leq \bar{N}(r,g) + \bar{N}\left(r,\frac{1}{g}\right) + \bar{N}\left(r,\frac{1}{g+ap(z)}\right) + S(r,g)$$
  
$$\leq \bar{N}(r,g) + \bar{N}\left(r,\frac{1}{g}\right) + \bar{N}\left(r,\frac{1}{f}\right) + S(r,g).$$
(24)

Using the argument as in Case 1, we get a contradiction. Therefore,  $p(z) \equiv 0$ . So from (22) and (23), we obtain a = 1 and so  $f \equiv g$ . This proves the lemma.

**Lemma 2.6.** Let f(z) and g(z) be two non-constant entire functions such that  $f^{(k)}$  and  $g^{(k)}$  share 1 IM, where k be a positive integer. If

$$\Delta = \delta_{k+2}(0,g) + \delta_{k+2}(0,f) + \delta_{k+1}(0,f) + 2\delta_{k+1}(0,g) > 4$$

then either  $f^{(k)}g^{(k)} \equiv 1$  or  $f \equiv g$ .

**Proof**. Since *f* and *g* are entire functions, we have  $\overline{N}(r, f) = 0$  and  $\overline{N}(r, g) = 0$ . Proceeding as in the proof of Lemma 2.5, we obtain conclusion of Lemma 2.6.

**Lemma 2.7.** (See [11].) Let f(z) be a non-constant entire function, and let  $k(\ge 2)$  be a positive integer. If  $f^{(k)} \ne 0$ , then  $f = e^{az+b}$ , where  $a \ne 0$ , b are constants.

**Lemma 2.8**. (See [12].) Let f(z) be a non-constant meromorphic function. Let k be a positive integer, and let c be a non-zero finite complex number. Then,

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \dots + a_0) = nT(r, f) + S(r, f)$$

# **3 Proof of theorems**

## 3.1 Proof of Theorem 1.1

Let  $F = f^{n}(f - 1)^{m}$  and  $G = g^{n}(g - 1)^{m}$ .

By Lemma 2.8, we have

$$\Theta(\infty, F) = 1 - \overline{\lim_{n \to \infty} \frac{\bar{N}(r, F)}{T(r, F)}} = 1 - \overline{\lim_{n \to \infty} \frac{\bar{N}(r, f^n(f - 1)^m)}{(m + n)T(r, f)}}$$
  
$$\geq 1 - \overline{\lim_{n \to \infty} \frac{T(r, f)}{(m + n)T(r, f)}} \geq \frac{n + m - 1}{m + n},$$

$$\delta_{k+1}(0,F) = 1 - \lim_{n \to \infty} \frac{N_{k+1}\left(r,\frac{1}{F}\right)}{T(r,F)} = 1 - \lim_{n \to \infty} \frac{N_{k+1}\left(r,\frac{1}{f^n(f-1)^m}\right)}{(m+n)T(r,f)}$$
  
$$\geq 1 - \frac{(k+m+1)T(r,f)}{(m+n)T(r,f)} \geq \frac{n-k-1}{m+n},$$

Similarly,

$$\Theta(\infty, G) \ge \frac{n+m-1}{m+n}, \delta_{k+1}(0, G) \ge \frac{n-k-1}{m+n}, \delta_{k+2}(0, F) \ge \frac{n-k-2}{m+n}, \delta_{k+2}(0, G) \ge \frac{n-k-2}{m+n}.$$
  
Therefore,

$$\Delta = (2k+4)\Theta(\infty,G) + (2k+3)\Theta(\infty,F) + \delta_{k+2}(0,G) + \delta_{k+2}(0,F) + \delta_{k+1}(0,F) + 2\delta_{k+1}(0,G)$$

$$\geq (2k+4) \cdot \frac{m+n-1}{m+n} + (2k+3) \cdot \frac{m+n-1}{m+n} + \frac{n-k-2}{m+n} + \frac{n-k-2}{m+n} + \frac{n-k-1}{m+n} + 2 \cdot \frac{n-k-1}{m+n}$$

If n > 4m + 9k + 14, we obtain  $\Delta > 4k + 11$ . So by Lemma 2.5, we get either  $F^{(k)} G^{(k)} \equiv 1$  or  $F \equiv G$ .

**Case 1.** 
$$F^{(k)}G^{(k)} \equiv 1$$
, that is,  
 $(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} \equiv 1.$  (25)

**Case 1.1** when m = 0, that is,

$$(f^n)^{(k)}(g^n)^{(k)} \equiv 1.$$
<sup>(26)</sup>

Next, we prove  $f \neq 0$ ,  $\infty$  and  $g \neq 0$ ,  $\infty$ .

Suppose that *f* has *a* zero  $z_0$  of order *p*, then  $z_0$  is a pole of *g* of order *q*. By (26), we get np - k = nq + k, i.e., n(p - q) = 2k, which is impossible since n > 9k + 14.

Therefore, we conclude that  $f \neq 0$  and  $g \neq 0$ .

Similarly, Suppose that f has a pole  $z'_0$  of order p', then  $z'_0$  is a zero of g of order q'. By (26), we get np' + k = nq' - k, i.e., n(q' - p') = 2k, which is impossible since n > 9k + 14.

Therefore, we conclude that  $f \neq \infty$  oo and  $g \neq \infty$ . From (26), we get

$$(f^n)^{(k)} \neq 0 \quad \text{and} \quad (g^n)^{(k)} \neq 0.$$
 (27)

From (26)-(27) and Lemma 2.7, we get that  $f(z) = c_1 e^{cz}$  and  $g(z) = c_2 e^{-cz}$ , where  $c, c_1$  and  $c_2$  are three constants satisfying  $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ .

**Case 1.2** when  $m \ge 1$ 

Let *f* has a zero  $z_1$  of order  $p_1$ . From (25), we get  $z_1$  is a pole of *g*. Suppose that  $z_1$  is a pole of *g* of order  $q_1$ . Again by (25), we obtain  $np_1 - k = nq_1 + mq_1 + k$ , i.e.,  $n(p_1 - q_1) = mq_1 + 2k$ , which implies that  $p_1 \ge q_1 + 1$  and  $mq_1 + 2k \ge n$ . From n > 4m + 9k + 14, we can deduce  $p_1 \ge 6$ .

Let f - 1 has a zero  $z_2$  of order  $p_2$ , then  $z_2$  is a zero of  $[f^n(f - 1)^m]^{(k)}$  of order  $mp_2 - k$ . Therefore from (25), we obtain  $z_2$  is a pole of g of order  $q_2$ . Again by (25), we obtain  $mp_2 - k = (n + m)q_2 + k$ , i.e.,  $mp_2 = (n + m)q_2 + 2k$ , i.e.,  $p_2 \ge \frac{m + n}{m} + \frac{2k}{m}$ .

Let  $z_3$  be a zero of f' of order  $p_3$  that not a zero of f(f - 1), as above, we obtain from (25),  $p_3 - (k - 1) = (n + m)q_3 + k$ , i.e.,  $p_3 \ge n + m + 2k - 1$ .

Moreover, in the same manner as above, we have similar results for the zeros of  $[g^n(g-1)^m]^{(k)}$ .

On the other hand, Suppose  $z_4$  is a pole of f, from (25), we get  $z_4$  is a zero of  $[g^n(g-1)^m]^{(k)}$ .

Thus,

$$\begin{split} \bar{N}(r,f) &\leq \bar{N}\left(r,\frac{1}{g}\right) + \bar{N}\left(r,\frac{1}{g-1}\right) + \bar{N}\left(r,\frac{1}{g'}\right) \\ &\leq \frac{1}{6}N\left(r,\frac{1}{g}\right) + \frac{m}{m+n+2k}N\left(r,\frac{1}{g-1}\right) + \frac{1}{n+m+2k-1}N\left(r,\frac{1}{g'}\right). \end{split}$$

We get

$$\bar{N}(r,f) \leq \left(\frac{1}{6} + \frac{m}{m+n+2k} + \frac{1}{n+m+2k-1}\right)T(r,g) + S(r,g).$$

From this and the second fundamental theorem, we obtain

$$T(r,f) \leq \bar{N}(r,f) + \bar{N}\left(r,\frac{1}{f-1}\right) + \bar{N}\left(r,\frac{1}{f}\right) + S(r,f)$$
  
$$\leq \left(\frac{1}{6} + \frac{m}{m+n+2k} + \frac{1}{n+m+2k-1}\right)T(r,g) + \left(\frac{1}{6} + \frac{m}{m+n+2k}\right)T(r,f) + S(r,f) + S(r,g).$$

Similarly, we have

$$T(r,g) \leq \left(\frac{1}{6} + \frac{m}{m+n+2k} + \frac{1}{n+m+2k-1}\right)T(r,f) + \left(\frac{1}{6} + \frac{m}{m+n+2k}\right)T(r,g) + S(r,f) + S(r,g).$$

We can deduce from above

$$T(r,f)+T(r,g) \leq \left(\frac{1}{3} + \frac{2m}{m+n+2k} + \frac{1}{n+m+2k-1}\right) [T(r,f)+T(r,g)]+S(r,f)+S(r,g).$$

Since n > 4m + 9k + 14, we obtain

$$T(r,f) + T(r,g) \leq \left(\frac{1}{3} + \frac{2}{31} + \frac{1}{30}\right) \left[T(r,f) + T(r,g)\right] + S(r,f) + S(r,g).$$

i.e.,  $0.57[T(r, f) + T(r, g)] \le S(r, f) + S(r, g)$ , which is contradiction.

**Case 2.** 
$$F \equiv G$$
, i.e.,  
 $f^{n}(f-1)^{m} \equiv g^{n}(g-1)^{m}$ . (28)

Now we consider following three cases.

**Case 2.1** when m = 0, then from (28), we get f = tg for a constant t such that  $t^n =$ 1.

**Case 2.2** when m = 1, then from (28), we have

$$f^{n}(f-1) \equiv g^{n}(g-1).$$
 (29)

Suppose  $f \neq g$ . Let  $h = \frac{f}{g}$  be a constant. Then from (29), it follows that  $h \neq 1$ ,  $h^n \neq 1$ ,  $h^{n+1} \neq 1$  and  $g = \frac{1 - h^n}{1 - h^{n+1}} = \text{constant}$ , a contradiction. So we suppose h is not a con-

stant. Since  $f \neq g$ , we have  $h \neq 1$ .

From (29), we obtain  $g = \frac{1-h^n}{1-h^{n+1}}$  and  $f = \frac{h(1-h^n)}{1-h^{n+1}}$ . Hence, it follows that T(r, f) =

$$nT(r, h) + S(r, f).$$

By the second fundamental theorem, we have

$$\bar{N}(r,f) = \sum_{i=1}^{n} \bar{N}\left(r, \frac{1}{h-a_i}\right) \ge (n-2)T(r,h) + S(r,f)$$

where  $a_i \neq 1$  (*i* = 1, 2,..., *n*) are distinct roots of the equation  $h^{n+1} = 1$ . So we obtain

$$\Theta(\infty, f) = 1 - \frac{\bar{N}(r, f)}{T(r, f)} \le \frac{2}{n}$$

which contradicts the assumption  $\Theta(\infty, f) > \frac{2}{n}$ , thus  $f \equiv g$ .

**Case 2.3** when  $m \ge 2$ , then from (28), we obtain

$$f^{n}[f^{m} + \dots + (-1)^{i}C_{m}^{m-i}f^{m-i} + \dots + (-1)^{m}] \equiv g^{n}[g^{m} + \dots + (-1)^{i}C_{m}^{m-i}g^{m-i} + \dots + (-1)^{m}].$$
(30)

Let  $h = \frac{f}{a}$ , if h is a constant, then substituting f = gh into (30), we deduce

$$g^{n+m}(h^{n+m}-1)\cdots + (-1)^i C_m^{m-i} g^{m+n-i}(h^{n+m-i}-1) + \cdots + (-1)^m g^n(h^n-1) = 0,$$

which implies h = 1. Thus,  $f(z) \equiv g(z)$ . If h is not a constant, then we know by (30) that f and g satisfies the algebraic equation  $R(f, g) \equiv 0$ , where  $R(w_1, w_2) = w_1^n (w_1 - 1)^m - w_2^n (w_2 - 1)^m$ .

This completes the proof of Theorem 1.1.

## 3.2 Proof of Theorem 1.2

Consider  $F = f^n(f^m - a)$ ,  $G = g^n(g^m - a)$ , then  $F^{(k)}$  and  $G^{(k)}$  share 1 IM. By Lemma 2.8, we have

$$\Theta(\infty, F) = 1 - \overline{\lim_{n \to \infty} \frac{\bar{N}(r, F)}{T(r, F)}} = 1 - \overline{\lim_{n \to \infty} \frac{\bar{N}(r, f^n(f^m - a))}{(m + n)T(r, f)}}$$
  
$$\geq 1 - \overline{\lim_{n \to \infty} \frac{T(r, f)}{(m + n)T(r, f)}} \geq \frac{m + n - 1}{m + n},$$

and

$$\delta_{k+1}(0,F) = 1 - \overline{\lim_{n \to \infty}} \frac{N_{k+1}\left(r,\frac{1}{F}\right)}{T(r,F)} = 1 - \overline{\lim_{n \to \infty}} \frac{N_{k+1}\left(r,\frac{1}{f^n(f^m-a)}\right)}{(m+n)T(r,f)}$$
  
 
$$\geq 1 - \overline{\lim_{n \to \infty}} \frac{(k+m+1)T(r,f)}{(m+n)T(r,f)} \geq \frac{n-k-1}{m+n}.$$

Similarly,

$$\Theta(\infty,G) \ge \frac{m+n-1}{m+n}, \, \delta_{k+1}(0,G) \ge \frac{n-k-1}{m+n}, \, \delta_{k+2}(0,F) \ge \frac{n-k-2}{m+n}, \, \delta_{k+2}(0,G) \ge \frac{n-k-2}{m+n}, \, \delta_$$

*m* + *n* Therefore,

$$\begin{split} &\Delta = (2k+4)\Theta(\infty,G) + (2k+3)\Theta(\infty,F) + \delta_{k+2}(0,G) + \delta_{k+2}(0,F) + \delta_{k+1}(0,F) + 2\delta_{k+1}(0,G) \\ &\geq (2k+4) \cdot \frac{m+n-1}{m+n} + (2k+3) \cdot \frac{m+n-1}{m+n} + \frac{n-k-2}{m+n} + \frac{n-k-2}{m+n} + \frac{n-k-1}{m+n} + 2 \cdot \frac{n-k-1}{m+n} \end{split}$$

Since n > 4m + 9k + 14, we get  $\Delta > 4k + 11$ , then by Lemma 2.5, we obtain either  $F^{(k)}G^{(k)} \equiv 1$  or  $F \equiv G$ .

Let  $F^{(k)}G^{(k)} \equiv 1$ , i.e.,

$$[f^{n}(f^{m}-a)]^{(k)}[g^{n}(g^{m}-a)]^{(k)} \equiv 1,$$
(31)

We can rewrite (31) as

$$[f^{n}(f-a_{1})\cdots(f-a_{m})]^{(k)}[g^{n}(g-a_{1})\cdots(g-a_{m})]^{(k)} \equiv 1,$$
(32)

where  $a_1, a_2, \dots, a_m$  are roots of  $w^m - a = 0$ .

By the similar argument for (32) of case 1.2 of Theorem 1.1, the case  $F^{(k)}G^{(k)} \equiv 1$  does not arise.

Let  $F \equiv G$ , i.e.,

$$f^{n}(f^{m}-a) \equiv g^{n}(g^{m}-a).$$
 (33)

Obviously, if *m* and *n* are both odd or if *m* is odd and *n* is even or if *m* is even and *n* is odd, then  $f \equiv -g$  contradicts  $F \equiv G$ . Let  $f \neq g$  and  $f \neq -g$ . We put  $h = \frac{f}{g}$ , then  $h \neq 1$  and  $h \neq -1$ . So from (33), we get  $g^m = \frac{a(1-h^n)}{1-h^{n+m}}$ .

Since g is non-constant, we see that h is not a constant. Again since  $g^m$  has no simple pole,  $h - h_k$  has no simple zero, where  $h_k = \exp\left(\frac{2\pi ki}{n+m}\right)$  and k = 1, 2, ..., n + m - 1. Hence,  $\Theta(h_k, h) \ge \frac{1}{2}$  for k = 1, 2, ..., n + m - 1, which is impossible. Therefore either  $f \equiv g$  or  $f \equiv -g$ .

This completes the proof of Theorem 1.2.

#### 3.3 Proof of Theorem 1.3

Since *f* and *g* are entire functions, we have N(r, f) = N(r, g) = 0. Proceeding as in the proof of Theorem 1.1 and applying Lemma 2.6, we obtain that Theorem 1.3 holds.

### 3.4 Proof of Theorem 1.4

Since *f* and *g* are entire functions, we have N(r, f) = N(r, g) = 0. Proceeding as in the proof of Theorem 1.2 and applying Lemma 2.6, we can easily prove Theorem 1.4.

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CW drafted the manuscript and have made outstanding contributions to this paper. CM and JL made suggestions for revision. All authors read and approved the final manuscript.

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The authors declare that they have no competing interests.

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