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On α-Šerstnev probabilistic normed spaces

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Abstract

In this article, the condition α-Š is defined for $\alpha \in]0, 1[\cup]1, +\infty[$and several classes of α-Šerstnev PN spaces, the relationship between α-simple PN spaces and α-Šerstnev PN spaces and a study of PN spaces of linear operators which are α-Šerstnev PN spaces are given.

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1. Introduction

Šerstnev introduced the first definition of a probabilistic normed (PN) space in a series of articles [1-4]; he was motivated by the problems of best approximation in statistics. His definition runs along the same path followed in order to probabilize the notion of metric space and to introduce Probabilistic Metric spaces (briefly, PM spaces).

For the reader's convenience, now we recall the most recent definition of a Probabilistic Normed space (briefly, a PN space) [5]. It is also the definition adopted in this article and became the standard one, and, to the best of the authors' knowledge, it has been adopted by all the researchers who, after them, have investigated the properties, the uses or the applications of PN spaces. This new definition is suggested by a result ([[5], Theorem 1]) that sheds light on the definition of a "classical" normed space. The notation is essentially fixed in the classical book by Schweizer and Sklar [6].

In the context of the PN spaces redefined in 1993, one introduces in this article a study of the concept of *α-Šerstnev PN spaces* (or generalized Šerstnev PN spaces, see [7]). This study, with $\alpha \in]0, 1[\cup]1, +\infty[$has never been carried out.

Some preliminaries

A *distribution function*, briefly a *d. f.*, is a function F defined on the extended reals $\overline{\mathbb{R}} := [-\infty, +\infty]$ that is non-decreasing, left-continuous on $\mathbb{R}$ and such that $F(-\infty) = 0$ and $F(+\infty) = 1$. The set of all d.f.'s will be denoted by Δ; the subset of those d.f.'s such that $F(0) = 0$ will be denoted by Δ^+ and by $\mathcal{D}^+$ the subset of the d.f.'s in Δ^+ such that $\lim_{x\to+\infty} F(x) = 1$. For every $a \in \mathbb{R}$, ε_a is the d.f. defined by

$$\varepsilon_a(x) := \begin{cases} 0, & x \leq a, \\ 1, & x > a. \end{cases}$$

The set Δ, as well as its subsets, can partially be ordered by the usual pointwise order; in this order, ε_0 is the maximal element in Δ^+. The subset $\mathcal{D}^+ \subset \Delta^+$ is the subset of the *proper* d.f.'s of Δ^+.

Definition 1.1. [8,9] A triangle function is a mapping τ from $\Delta^+ \times \Delta^+$ into Δ^+ such that, for all F, G, H, K in Δ^+,

(1) $\tau(F, \varepsilon_0) = F$,
(2) $\tau(F, G) = \tau(G, F)$,
(3) $\tau(F, G) \le \tau(H, K)$ whenever $F \le H$, $G \le K$,
(4) $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$.

Typical continuous triangle functions are the operations τ_T and τ_{T^*}, which are, respectively, given by

$$\tau_T(F,\ G)(x) := \sup_{s+t=x} T(F(s),\ G(t)),$$

and

$$\tau_{T*}(F,\ G)(x)\ :=\ \inf_{s+t=x} T^*(F(s),\ G(t)).$$

for all $F, G \in \Delta^+$ and all $x \in \mathbb{R}$ [6]. Here, T is a continuous t-norm and T^* is the corresponding continuous t-conorm, i.e., both are continuous binary operations on [0, 1] that are commutative, associative, and nondecreasing in each place; T has 1 as identity and T^* has 0 as identity. If T is a t-norm and T^* is defined on $[0, 1] \times [0, 1]$ via $T^*(x, y) := 1 - T(1 - x, 1 - y)$, then T^* is a t-conorm, specifically the t-conorm of T.

Definition 1.2. A PM space is a triple $(S, \mathcal{F}, \tau)$ where S is a nonempty set (whose elements are the points of the space), $\mathcal{F}$ is a function from $S \times S$ into Δ^+, τ is a triangle function, and the following conditions are satisfied for all p, q, r in S:

(PM1) $\mathcal{F}(p, p) = \varepsilon_0$.
(PM2) $\mathcal{F}(p, q) \neq \varepsilon_0$ if $p \neq q$.
(PM3) $\mathcal{F}(p, q) = \mathcal{F}(q, p)$.
(PM4) $\mathcal{F}(p,\ r) \ge \tau(\mathcal{F}(p,\ q),\ \mathcal{F}(q,\ r))$.

Definition 1.3. (introduced by Šerstnev [1] about PN spaces: it was the first definition) A PN space is a triple (V, ν, τ), where V is a (real or complex) linear space, ν is a mapping from V into Δ^+ and τ is a continuous triangle function and the following conditions are satisfied for all p and q in V:

(N1) $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in V);
(N3) $\nu_{p+q} \ge \tau\ (\nu_p, \nu_q)$;

$$\left(\check{S}\right)\ \forall\alpha \in \mathbb{R}\backslash\{0\} \quad \forall x \in \overline{\mathbb{R}}_+ \quad \nu_{\alpha p}(x) = \nu_p\left(\tfrac{x}{\alpha}\right).$$

Notice that condition (Š) implies
(N2) $\forall p \in V\ \nu_{-p} = \nu_p$.

Definition 1.4. (PN spaces redefined: [5]) A *PN* space is a quadruple (V, ν, τ, τ^*), where V is a real linear space, τ and τ^* are continuous triangle functions such that $\tau \le \tau^*$, and the mapping $\nu : V \to \Delta^+$ satisfies, for all p and q in V, the conditions:

(N1) $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in V);
(N2) $\forall p \in V\ \nu_{-p} = \nu_{\mathrm{p}}$;
(N3) $\nu_{\mathrm{p}+q} \ge \tau\ (\nu_{\mathrm{p}}, \nu_q)$;

(N4) $\forall\ \alpha \in [0, 1]\ \nu_p \leq \tau^* (\nu_{\alpha\ p}, \nu_{(1-\alpha)\ p})$.

The function ν is called the *probabilistic norm*. If ν satisfies the condition, weaker than (N1),

$$\nu_\theta = \varepsilon_0,$$

then (V,ν, τ, τ^*) is called a *Probabilistic Pseudo-Normed* space (briefly, a PPN space). If ν satisfies the conditions (N1) and (N2), then (V,ν, τ, τ^*) is said to be a *Probabilistic seminormed space* (briefly, PSN space). If $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some continuous t-norm T and its t-conorm T^*, then $(V, \nu, \tau_T, \tau_{T^*})$ is denoted by (V, ν, T) and is called a *Menger* PN space. A PN space is called a *Šerstnev space* if it satisfies (N1), (N3) and condition (Š).

Definition 1.5. [6] Let (V,ν, τ, τ^*) be a PN space. For every $\lambda > 0$, the strong λ-neighborhood $N_p(\lambda)$ at a point p of V is defined by

$$N_p(\lambda) := \{q \in V : \nu_{q-p}(\lambda) > 1 - \lambda\}.$$

The system of neighborhoods $\{N_p(\lambda): p \in V, \lambda > 0\}$ determines a Hausdorff topology on V, called the strong topology.

Definition 1.6. [6] Let (V, ν, τ, τ^*) be a PN space. A sequence $\{p_n\}_n$ of points of V is said to be a strong Cauchy sequence in V if it has the property that given $\lambda > 0$, there is a positive integer N such that

$$\nu_{p_n - p_m}(\lambda) > 1 - \lambda \quad \text{whenever } m,\ n > N.$$

A PN space (V,ν, τ, τ^*) is said to be strongly complete if every strong Cauchy sequence in V is strongly convergent.

Definition 1.7. [10] A subset A of a PN space (V,ν, τ, τ^*) is said to be $\mathcal{D}$-compact if every sequence of points of A has a convergent subsequence that converges to a member of A.

The *probabilistic radius* R_A of a nonempty set A in PN space (V,ν, τ, τ^*) is defined by

$$R_A(x) := \begin{cases} l^-\phi_A(x), & x \in [0, +\infty[, \\ 1, & x = \infty, \end{cases}$$

where $l^- f(x)$ denotes the left limit of the function f at the point x and $\varphi_A(x) := \inf\{\nu_p(x): p \in A\}$.

Definition 1.8. [11] Definition 2.1] A nonempty set A in a PN space (V,ν, τ, τ^*) is said to be:

(a) certainly bounded, if $R_A(x_0) = 1$ for some $x_0 \in]0, +\infty[$;
(b) perhaps bounded, if one has $R_A(x) < 1$ for every $x \in]0, \infty[$, and $l^- R_A(+\infty) = 1$.

Moreover, the set A will be said to be $\mathcal{D}$-bounded if either (a) or (b) holds, i.e., if $R_A \in \mathcal{D}^+$.

Definition 1.9. [12] A subset A of a topological vector space (briefly, TV space) E is topologically bounded, if for every sequence $\{\lambda_n\}_n$ of real numbers that converges to 0 as $n \to \infty$ and for every sequence $\{p_n\}_n$ of elements of A, one has $\lambda_n p_n \to \theta$ in the topology of E. Also by Rudin [[13], Theorem 1.30], A is topologically bounded if, and only if, for every neighborhood U of θ, we have $A \subseteq tU$ for all sufficiently large t.

From the point of view of topological vector spaces, the most interesting PN spaces are those that are not Šerstnev (or 1-Šerstnev) spaces. In these cases vector addition is still continuous (provided the triangle function is determined by a continuous t-norm), while scalar multiplication, in general, is not continuous with respect to the strong topology [14].

We recall from [15]: for $0 < b \leq +\infty$, let M_b be the set of *m-transforms* consisting of all continuous and strictly increasing functions from $[0, b]$ onto $[0, +\infty]$. More generally, let $\widetilde{M}$ be the set of non-decreasing left-continuous functions $\varphi : [0, +\infty]\ [0, +\infty]$, with $\varphi(0) = 0$, $\varphi(+\infty) = +\infty$ and $\varphi(x) > 0$ for $x > 0$. Then $M_b \subseteq \widetilde{M}$ once m is extended to $[0, +\infty]$ by $m(x) = +\infty$ for all $x \geq b$. Note that a function $\phi \in \widetilde{M}$ is bijective if, and only if, $\varphi \in M_{+\infty}$. Sometimes, the probabilistic norms ν and ν' of two given PN spaces satisfy $\nu' = \nu\varphi$ for some $\varphi \in M_{+\infty}$. not necessarily bijective. Let $\hat{\phi}$ be the (unique) quasi-inverse of φ which is left-continuous. Recall from [[6], p. 49] that $\hat{\phi}$ is defined by $\hat{\phi}(0) = 0$, $\hat{\phi}(+\infty) = +\infty$ and $\hat{\phi}(t) = sup\{u : \phi(u) < t\}$ for all $0 < t < +\infty$. It follows that $\hat{\phi}(\phi(x)) \leq x$ and $\phi(\hat{\phi}(y)) \leq y$ for all x and y.

Definition 1.10. A quadruple (V, ν, τ, τ^*) is said to satisfy the φ-Šerstnev condition if

$(\phi - \check{S})\nu_{\lambda p}(x) = \nu_p\left(\widehat{\phi}\left(\frac{\phi(x)}{|\lambda|}\right)\right)$ for every $p \in V$, for every $x > 0$ and $\lambda \in \mathbb{R}\setminus\{0\}$.

A PN space (V, ν, τ, τ^*) which satisfies the φ-Šerstnev condition is called a φ-Šerstnev PN space.

Example 1.1. If $\varphi(x) = x^{1/\alpha}$ for a fixed positive real number α, the condition (φ-Š) takes the form

$(\alpha - \check{S})\nu_{\lambda p}(x) = \nu_p\left(\frac{x}{|\lambda|^\alpha}\right)$ for every $p \in V$, for every $x > 0$ and $\lambda \in \mathbb{R}\setminus\{0\}$.

PN spaces satisfying the condition (α-Š) are called α-Šerstnev PN spaces. For $\alpha = 1$ one has a Šerstnev (or 1-Šerstnev) PN space.

Definition 1.11. Let $(V, ||\cdot||)$ be a normed space and let G be a d.f. of Δ^+ different from ε_0 and $\varepsilon_{+\infty}$; define $\nu : V \to \Delta^+$ by $\nu_\theta = \varepsilon_0$ and

$$\nu_p(t) := G\left(\frac{t}{\| p \|^{\alpha}}\right) \quad (p \neq \theta,\ t > 0),$$

where $\alpha \geq 0$. Then the pair (V, ν) will be called the α-simple space generated by $(V, ||\cdot||)$ and G.

The α-simple space generated by $(V, ||\cdot||)$ and G is, as immediately checked, a PSN space; it will be denoted by $(V, ||\cdot||, G; \alpha)$.

A PSN space (V, ν) is said to be equilateral if there is d.f. $F \in \Delta^+$, different from ε_0 and from ε_∞, such that, for every $p \neq \theta$, $\nu_p = F$. In Definition 1.11, if $\alpha = 0$ and $\alpha = 1$, one obtains the equilateral and simple space, respectively.

Definition 1.12. [16] The PN space (V, ν, τ, τ^*) is said to satisfy the double infinity-condition *(briefly, DI-condition)* if the probabilistic norm ν is such that, for all $\lambda \in \mathbb{R}\setminus\{0\}$, $x \in \mathbb{R}$ and $p \in V$,

$$\nu_{\lambda p}(x) = \nu_p(\varphi(\lambda,\ x)),$$

where $\phi : \mathbb{R} \times [0, +\infty[\to [0, +\infty[$satisfies

$$\lim_{x\to+\infty} \varphi(\lambda, x) = +\infty \quad \text{and} \quad \lim_{\lambda\to 0} \varphi(\lambda, x) = +\infty.$$

Definition 1.13. Let $(S, \leq)$ be a partially ordered set and let f and g be commutative and associative binary operations on S with common identity e. Then, f *dominates* g, and one writes $f \gg g$, if, for all x_1, x_2, y_1, y_2 in S,

$$f(g(x_1, y_1), g(x_2, y_2)) \geq g(f(x_1, x_2), f(y_1, y_2)).$$

It is easily shown that the dominance relation is reflexive and antisymmetric. However, although not, in general, transitive, as examples due to Sherwood [17] and Sarkoci [18] show.

2. Main results (I)–α-simple PN space and some classes of α-Šerstnev PN spaces

In this section, we give several classes of α-Šerstnev PN spaces and characterize them. Also, we investigate the relationship between α-simple PN spaces and α-Šerstnev PN spaces.

Theorem 2.1. ([[16], Theorem 2.1]) *Let* (V, ν, τ, τ^*) *be a* PN *space which satisfies the DI-condition. Then for a subset* $A \subseteq V$, *the following statements are equivalent:*

(a) *A is* $\mathcal{D}$ *-bounded.*
(b) *A is bounded, namely, for every* $n \in \mathbf{N}$ *and for every* $p \in A$, *there is* $k \in \mathbf{N}$ *such that* $\nu_{p/k}(1/n) > 1 - 1/n$.
(c) *A is topologically bounded.*

Example 2.1. Let (V, ν, τ, τ^*) be an α-Šerstnev PN space. It is easy to see that (V, ν, τ, τ^*) satisfies the *DI*-condition, where

$$\varphi(\lambda, x) = \frac{x}{|\lambda|^\alpha}.$$

Theorem 2.2. *Let* (V, ν, τ, τ^*) *be an* α*-Šerstnev* PN *space. Then, for a subset* $A \subseteq V$, *the same statements as in Theorem 2.1 are equivalent.*

Definition 2.1. The PN space (V, ν, τ, τ^*) is called strict whenever $\nu(V) \subseteq \mathcal{D}^+$.

Corollary 2.1. *Let* $W_1 = (V, \nu, \tau, \tau^*)$ *and* $W_2 = (V, \nu', \tau', (\tau^*)')$ *be two PN spaces with the same base vector space and suppose that* $\nu' = \nu\varphi$ *for some* $\phi \in \widetilde{M}$. *Then the following statement holds:*

- *If the scalar multiplication* $\eta : \mathbb{R} \times V \to V$ *is continuous at the first place with respect to* ν, *then it is with respect to* ν'. *If* W_1 *is a TV PN space. then it is with* W_2.

It was proved in [[14], Theorem 4] that, if the triangle function τ^* is Archimedean, i.e., if τ^* admits no idempotents other than ε_0 and ε_∞ [6], and $\nu_p \neq \varepsilon_\infty$ for all $p \in V$, then for every $p \in V$ the map from $\mathbb{R}$ into V defined by $\lambda \; \alpha \; \lambda p$ is continuous and, as a consequence of [14] the PN space (V, ν, τ, τ^*) is a TV space.

Theorem 2.3. [7]*Let* $\phi \in \widetilde{M}$ *such that* $\lim_{x\to\infty} \hat{\phi}(x) = \infty$. *A* φ*-Šerstnev PN space is a TV space if, and only if, it is strict.*

Corollary 2.2. *An α-Šerstnev PN space (V,ν, τ, τ^*) is a TV space if, and only if, it is strict.*

Corollary 2.3. *Let (V,ν, τ, τ^*) be an α-Šerstnev PN space and τ^* be Archimedean and $\nu_p \neq \varepsilon_\infty$ for all $p \in V$. Then the probabilistic norm ν is strict.*

Theorem 2.4. *Every equilateral PN space (V, F, Π_M) with $F = \beta\varepsilon_0$ and $\beta \in]0, 1[$ satisfies the following statements:*

(i) *It is an α-Šerstnev PN space.*
(ii) *It is an α-simple PN space.*

Theorem 2.5. *Every α-simple space satisfies the (α-Š) condition for $\alpha \in]0, 1[\cup]1, +\infty[$.*

Proof. Let $(V, ||\cdot||, G; \alpha)$ be an α-simple PN space with $\alpha \in]0, 1[\cup]1, +\infty[$. From $\nu_p(t) = G\left(\frac{t}{\|p\|^\alpha}\right)$ for every $t \in [0, \infty]$, one has $\nu_{\lambda p}(t) = G\left(\frac{t}{\|\lambda p\|^\alpha}\right) = G\left(\frac{t}{|\lambda|^\alpha\|p\|^\alpha}\right)$ and $\nu_p\left(\frac{t}{|\lambda|^\alpha}\right) = G\left(\frac{\frac{t}{|\lambda|^\alpha}}{\|p\|\alpha}\right) = G\left(\frac{t}{|\lambda|^\alpha\|p\|\alpha}\right)$. Then $\nu_{\lambda p}(t) = \nu_p\left(\frac{t}{|\lambda|^\alpha}\right)$ and hence $(V, ||\cdot||, G; \alpha)$ is an α- Šerstnev PN space.

An α-simple space with $a \neq 1$ does not satisfy the condition (Š) as seen in the following theorem.

Theorem 2.6. *Let $(V, ||\cdot||)$ be a normed space, G a d. f. different from ε_0 and ε_∞, and let α be a positive real number different from 1. Then the α-simple space $(V, ||\cdot||, G; \alpha)$ satisfies the condition (Š) only when G = constant in $(0, +\infty)$.*

Proof. It is immediately checked that the α-simple space $(V, ||\cdot||, G; \alpha)$ satisfies (N1) and (N2). Hence, it is a PSN space. It is well known that the condition (Š) holds if, and only if, for every $p \in V$ and $\beta \in [0, 1]$, one has

$$\nu_p = \tau_M(\nu_{\beta p}, \nu_{(1-\beta)p}).$$

To see G has to be constant: for every $p \neq \theta$ and $x \in]0, +\infty[$, one has

$$G\left(\frac{x}{\| p \|^{\alpha}}\right) = \sup_{x=s+t} \min\left\{G\left(\frac{s}{\beta^\alpha \| p \|^{\alpha}}\right), G\left(\frac{t}{(1-\beta)^\alpha \| p \|^{\alpha}}\right)\right\}.$$

Since G is non-decreasing, the lower upper bound is reached when

$$\frac{s}{\beta^\alpha \| p \|^{\alpha}} = \frac{t}{(1-\beta)^\alpha \| p \|^{\alpha}},$$

equivalent to $s = \frac{\beta^\alpha}{\beta^\alpha+(1-\beta)^\alpha}x$. Hence the lower upper bound is

$$G\left(\frac{x}{[\beta^\alpha + (1-\beta)^\alpha] \| p \|^{\alpha}}\right).$$

Finally, since the function of β given by $\beta^\alpha+(1-\beta)^\alpha$, being continuous in the compact set $[0, 1]$, takes all values between 1 and $2^{1-\alpha}$, and $\frac{x}{\|p\|^\alpha}$ takes any value in $(0, \infty)$, one concludes that $G(x) = G(\lambda x)$ for every $\lambda \in [1, 2^{\alpha-1}]$ (if $\alpha > 1$) or for every $\lambda \in [2^{\alpha-1}, 1]$ (if $\alpha < 1$). Then G = constant in $(0, +\infty)$ and the proof is concluded.

Notice that if G = constant in $(0, +\infty)$, then $(V, ||\cdot||, G; \alpha)$ is a PN space of Šerstnev under any triangle function τ.

Among all α-simple spaces $(V, ||\cdot||, G; \alpha)$ one has the α-simple PN spaces considered in Theorem 3.2 in [19], i.e., the Menger PN space given by $\left(V,\ \nu,\ \tau_{T_{G^*}},\ \tau_{T^*_{G^*}}\right)$, and in Theorem 3.1 in [19], i.e., the Menger PN space given by $\left(V,\ \nu,\ \tau_{T_{G^*}},\ \tau_{T^*_{G}}\right)$. From Theorems 3.1 and 3.2 in [19] the following result holds:

Corollary 2.4. *Every α-simple PN spaces of the type considered in Theorems 3.1 and 3.2 in* [19]*are (α-Š) PN spaces of Menger.*

Next, we give an example of an α-Šerstnev PN space which is also an α-simple PN space.

Example 2.2. Let $(\mathbb{R}, \nu, \tau, \tau^*)$ be an α-Šerstnev PN space. Let $\nu_1 = G$ with $G \in \Delta^+$ different from ε_0 and $\varepsilon_{+\infty}$. Since $(\mathbb{R}, \nu, \tau, \tau^*)$ is an α-Šerstnev PN space, for every $p \in \mathbb{R}$, one has

$$\nu_p(x) = \nu_{p\cdot 1}(x) = \nu_1\left(\frac{x}{|\,p\,|^{\alpha}}\right) = G\left(\frac{x}{|\,p\,|^{\alpha}}\right).$$

The preceding example suggests the following theorem.

Theorem 2.7. *Let* $(V, ||\cdot||)$ *be a normed space and* $\dim V = 1$. *Then every α-Šerstnev PN space is an α-simple PN space.*

Proof. Let $x \in V$ and $||x|| = 1$. Then $V = \{\lambda x : \lambda \in \mathbb{R}\}$. Now if $p \in V$, there is a $\lambda \in \mathbb{R}$ such that $p = \lambda x$. Therefore, one has

$$\nu_p(t) = \nu_{\lambda x}(t) = \nu_x\left(\frac{t}{|\,\lambda|^{\alpha}}\right) = G\left(\frac{t}{\|\,p\,\|^{\alpha}}\right),$$

and (V, ν, τ, τ^*) is an α-simple PN space.

The converse of Theorem 2.5 fails as is shown in the following examples.

Example 2.3. Let $\beta \in]0, 1]$. For $p = (p_1, p_2) \in \mathbb{R}^2$, one defines the probabilistic norm ν by $\nu_\theta = \varepsilon_0$ and

$$\nu_p(x) = \begin{cases} \varepsilon_\infty(x),\ p_1 \neq 0, \\ \beta\varepsilon_0(x) \text{ otherwise} \end{cases}$$

We show that $(\mathbb{R}^2, \nu, \Pi_M, \Pi_M)$ is an α-Šerstnev PN space, but it is not an α-simple PN space. It is easily ascertained that (N1) and (N2) hold. Now assume that $p = (p_1, p_2)$ and $q = (q_1, q_2)$ belong to $\mathbb{R}^2$, hence $p + q = (p_1 + q_1, p_2 + q_2)$. If $p_1 + q_1 = 0$, then $\nu_{p+q} = \beta\varepsilon_0$. So $\Pi_M(\nu_p, \nu_q) \leq \nu_{p+q}$. Let $p_1 + q_1 \neq 0$. Then, $p_1 \neq 0$ or $q_1 \neq 0$. Without loss of generality, suppose that $p_1 \neq 0$. Then $\Pi_M(\nu_p, \nu_q) = \nu_{p+q} = \varepsilon_\infty$. As a consequence (N3) holds. Similarly, (N4) holds. Let $p = (p_1, p_2)$ and $\lambda \in \mathbb{R}\backslash\{0\}$. If $p_1 \neq 0$, then

$$\nu_{\lambda p}(x) = \varepsilon_\infty \quad \text{and} \quad \nu_p\left(\frac{x}{|\,\lambda|^{\alpha}}\right) = \varepsilon_\infty\left(\frac{x}{|\,\lambda|^{\alpha}}\right).$$

In the other direction, if $p_1 = 0$, and $p_2 \neq 0$, then

$$\nu_{\lambda p}(x) = \beta\varepsilon_0(x) \quad \text{and} \quad \nu_p\left(\frac{x}{|\,\lambda|^{\alpha}}\right) = \beta\varepsilon_0\left(\frac{x}{|\,\lambda|^{\alpha}}\right).$$

Therefore, $(\mathbb{R}^2, \nu, \Pi_M, \Pi_M)$ is an α-Šerstnev PN space.

Now we show that it is not an α-simple PN space. Assume, if possible, $(\mathbb{R}^2, \nu, \Pi_M, \Pi_M)$ is an α-simple PN space. Hence, there is $G \in \Delta^+\backslash\{\varepsilon_0, \varepsilon_\infty\}$ such that

$\varepsilon_\infty(x) = \nu_{(1,0)}(x) = G(x)$, for every $p \in \mathbb{R}^2$. So

$$\varepsilon_\infty(x) = \nu_{(1,0)}(x) = G(x),$$

and

$$\beta\varepsilon_0(x) = \nu_{(0,1)}(x) = G(x),$$

which is a contradiction.

Example 2.4. Let $0 < \alpha \leq 1$. For $p = (p_1, p_2) \in \mathbb{R}^2$, define ν by $\nu_\theta = \varepsilon_0$ and

$$\nu_p(x) := \begin{cases} \varepsilon_\infty(x), & p_2 \neq 0, \\ e^{\frac{-\|p\|^\alpha}{x}}, & \text{otherwise.} \end{cases}$$

It is not difficult to show that $(\mathbb{R}^2, \nu, \Pi_\Pi, \Pi_M)$ is an α-Šerstnev PN space, but it is not an α-simple PN space.

Let V be a normed space with dim $V > 1$ (finite or infinite dimensional) and $\{e_i\}_{i\in I}$ be a basis for V, where $||e_i|| = 1$. We can construct some examples on V, similar to Examples 2.3 and 2.4, of α-Šerstnev PN spaces which are not α-simple PN spaces.

Example 2.5. (a) Let $\beta \in\ ]0, 1]$ and $i_0 \in I$. For $p \in V$, we define the probabilistic norm ν by $\nu_\theta = \varepsilon_0$ and

$$\nu_p(x) := \begin{cases} \beta\varepsilon_0(x), & p = \lambda e_{i_0} (\lambda \in \mathbb{R}\backslash\{0\}), \\ \varepsilon_\infty(x), & \text{otherwise.} \end{cases}$$

Then, (V, ν, Π_M, Π_M) is an α-Šerstnev PN space, but it is not an α-simple PN space.

(b) Let $0 < \alpha = 1$. For $p \in V$, define ν by $\nu_\theta = \varepsilon_0$ and

$$\nu_p(x) := \begin{cases} e^{\frac{-|\lambda|^\alpha}{x}} & p = \lambda e_{i_0} (\lambda \in \mathbb{R}\backslash\{0\}), \\ \varepsilon_\infty(x) & \text{otherwise} \end{cases}$$

Then (V, ν, Π_Π, Π_M) is an α-Šerstnev PN space, but it is not an α-simple PN space.

Proposition 2.1. *Let (V, ν, τ, τ^*) be an α-Šerstnev PN space. Then, its completion $(\hat{V}, \nu, \tau, \tau^*)$ is also an α-Šerstnev PN space.*

Proof. By [[20], Theorem 3], the completion of a PN space is a PN space.

Then we only have to check that the α-Šerstnev condition holds for $\hat{V}$. Indeed if $p = \lim_{n\to\infty} p_n$, where $p_n \in V$, and $\lambda > 0$, then for all $x \in \mathbb{R}^+$,

$$\nu_{\lambda p}(x) = \lim_{n\to\infty} \nu_{\lambda p_n}(x) = \lim_{n\to\infty} \nu_{p_n}\left(\frac{x}{|\lambda|^\alpha}\right) = \nu_p\left(\frac{x}{|\lambda|^\alpha}\right).$$

The following result concerns finite products of PN spaces [21]. In a given PN space (V, ν, τ, τ^*) the value of the probabilistic norm of $p \in V$ at the point x will be denoted by $\nu(p)(x)$ or by $\nu_p(x)$.

Proposition 2.2. *Let* $(V_i, \nu_i, \tau, \tau^*)$ *be* α*-Šerstnev PN spaces for* $i = 1, 2$, *and let* τ_T *be a triangle function. Suppose that* $\tau^* \gg \tau_T$ *and* $\tau_T \gg \tau$. *Let* $\nu : V_1 \times V_2 \to \Delta^+$ *be defined for all* $p = (p_1, p_2) \in V_1 \times V_2$ *via*

$$\nu(p_1, p_2) := \tau_T(\nu_1(p_1), \nu_2(p_2)).$$

Then the τ_T *-product* $(V_1 \times V_2, \nu, \tau, \tau^*)$ *is an* α*-Šerstnev PN space under* τ *and* τ^*.

Proof. For every $\lambda \in \mathbb{R}\backslash\{0\}$ and for every left-continuous t-norm T, one has

$$\begin{aligned}\nu_{\lambda p} &= \tau_T(\nu_1(\lambda p_1), \nu_2(\lambda p_2))(x) \\ &= \sup\{T(\nu_1(\lambda p_1)(u), \nu_2(\lambda p_2)(x-u))\} \\ &= \sup\left\{T\left(\nu_1(p_1)\left(\frac{u}{|\lambda|^\alpha}\right), \nu_2(p_2)\left(\frac{x-u}{|\lambda|^\alpha}\right)\right)\right\} \\ &= \tau_T(\nu_1(p_1), \nu_2(p_2))\left(\frac{x}{|\lambda|^\alpha}\right) = \nu_p\left(\frac{x}{|\lambda|^\alpha}\right)\end{aligned}$$

for every $\alpha \in]0, 1[\cup]1, +\infty[$. It is easy to check the axioms (N1) and (N2) hold.

(N3) Let $p = (p_1, p_2)$ and $q = (q_1, q_2)$ be points in $V_1 \times V_2$. Then since $\tau_T \gg \tau$, one has

$$\begin{aligned}\nu_{p+q} &= \tau_T(\nu_1(p_1 + q_1), \nu_2(p_2 + q_2)) \\ &\geq \tau_T(\tau(\nu_1(p_1), \nu_1(q_1)), \tau(\nu_2(p_2), \nu_2(q_2))) \\ &\geq \tau(\tau_T(\nu_1(p_1), \nu_2(p_2)), \tau_T(\nu_1(q_1), \nu_2(q_2))) = \tau(\nu_p, \nu_q).\end{aligned}$$

(N4) Next, for any $\beta \in [0, 1]$, we have

$$\nu_1(p_1) \leq \tau^*(\nu_1(\beta p_1), \nu_1((1-\beta)p_1))$$

and

$$\nu_2(p_2) \leq \tau^*(\nu_2(\beta p_2), \nu_2((1-\beta)p_2)).$$

Whence since $\tau^* \gg \tau_T$, we have

$$\begin{aligned}\nu_p &= \tau_T(\nu_1(p_1), \nu_2(p_2)) \\ &\leq \tau_T(\tau^*(\nu_1(\beta p_1), \nu_1((1-\beta)p_1)), \tau^*(\nu_2(\beta p_2), \nu_2((1-\beta)p_2))) \\ &\leq \tau^*(\nu_{\beta p}, \nu_{(1-\beta)p}),\end{aligned}$$

which concludes the proof.

Example 2.6. Assume that in Proposition 2.2 choose $V_1 \equiv V_2 \equiv \mathbb{R}^2$ and $\tau_T \equiv \Pi_M$. Let $0 < \alpha \leq 1$. For $p = (p_1, p_2) \in \mathbb{R}^2$, define ν_1 and ν_2 by $\nu_1(\theta) = \nu_2(\theta) = \varepsilon_0$ and

$$\nu_1(p)(x) \equiv \nu_2(p)(x) := \begin{cases}\varepsilon_\infty(x), & p_2 \neq 0, \\ e^{\frac{-\|p\|^\alpha}{X}}, & \text{otherwise.}\end{cases}$$

Then $(\mathbb{R}^2 \times \mathbb{R}^2, \nu, \Pi_\Pi, \Pi_M)$, with

$$\nu(p, q) = \tau_T(\nu_1(p), \nu_2(q))$$

is the Π_M -product and it is an α-Šerstnev PN space under Π_Π and Π_M.

Proof. The conclusion follows from Lemma 2.1 in [22].

3. Main results (II)–PN spaces of linear operators which are α-Šerstnev PN spaces

Let $(V_1, \nu, \tau_1, \tau_1^*)$ and $(V_2, \nu', \tau_2, \tau_2^*)$ be two PN spaces and let $L = L(V_1, V_2)$ be the vector space of linear operators $T : V_1 \to V_2$.

As was shown in [14], PN spaces are not necessarily topological linear spaces.

We recall that for a given linear map $T \in L$, the map $\nu^A : L \to \mathcal{D}^+$ is defined via $\nu^A(T) := R'_{TA}$.

We recall also [23,24] that a subset H of a space V is said to be a *Hamel basis* (or algebraic basis) if every vector x of V can be represented in a unique way as a *finite* sum

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n,$$

where $\alpha_1, \alpha_2, ..., \alpha_n$ are scalars and $u_1, u_2, ..., u_n$ belong to H; a subset H of V is a Hamel basis if, and only if, it is a maximal linear independent set [25]. This condition ensures that $(L(V_1, V_2), \nu^A, \tau, \tau^*)$ is a PN space as we can see in [[26], Theorem 3.2].

Theorem 3.1. *Let A be a subset of a PN space $(V_1, \nu, \tau_1, \tau_1^*)$ that contains a Hamel basis for V_1. Let $(V_2, \nu', \tau_2, \tau_2^*)$ be an α-Šerstnev PN space. Then $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ is an α-Šerstnev PN space whose topology is stronger than that of simple convergence for operators, i.e.,*

$$\nu^A(T_n - T) \to \varepsilon_0 \Rightarrow \forall p \in V_1 \quad \nu'_{T_n p - Tp} \to \varepsilon_0.$$

Proof. By [[26], Theorem 3.2], it suffices to check that it is an α-Šerstnev space. Let $\lambda > 0$ and $x \in \mathbb{R}^+$. Then

$$\begin{aligned}
\nu^A_{\lambda T}(x) &= R'_{\lambda TA}(x) = l^- \inf_{p \in A} \nu'_{\lambda Tp}(x) \\
&= l^- \inf_{p \in A} \nu'_{Tp}\left(\frac{x}{\| \lambda \mid |^\alpha}\right) = R'_{TA}\left(\frac{x}{\| \lambda \mid |^\alpha}\right) \\
&= \nu^A_T\left(\frac{x}{\| \lambda \mid |^\alpha}\right).
\end{aligned}$$

Corollary 3.1. *Let A be an absorbing subset of a PN space $(V_1, \nu, \tau_1, \tau_1^*)$. If $(V_2, \nu', \tau_2, \tau_2^*)$ is an α-Šerstnev PN space, then $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ is an α-Šerstnev PN space; convergence in the probabilistic norm ν^A is equivalent to uniform convergence of operators on A.*

Proof. See Theorem 3.1 and [[26], Corollary 3.1].

Corollary 3.2. *If V_2 is α complete α-Šerstnev PN space, then $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ is also a complete α-Šerstnev PN space.*

Proof. See Theorem 3.1 and [[26], Theorem 4.1].

In the remainder of this section, we study some classes of α-Šerstnev PN spaces of linear operators. We investigate the relationship between $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$, and $(V_1, \nu, \tau_1, \tau_1^*)$ or $(V_2, \nu', \tau_2, \tau_2^*)$ and we set some conditions such that $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ becomes a TV space.

Theorem 3.2. *Let A be a subset of a PN space $(V_1, \nu, \tau_1, \tau_1^*)$ that contains a Hamel basis for V_1 and $(V_2, \nu', \tau_2, \tau_2^*)$ be an α-Šerstnev PN space. If $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ is a TV space, then $(V_2, \nu', \tau_2, \tau_2^*)$ is a TV space.*

Proof. Assume, if possible, $(V_2, \nu', \tau_2, \tau_2^*)$ is not a TV space. Hence, by Corollary 2.2, there is a $q \in V_2$ such that $\nu'_q \in \Delta^+ \backslash \mathcal{D}^+$. Let $p_0 \neq \theta$ and $p_0 \in A$. Now, we define $T : V_1 \to V_2$ by

$$T(p) := \begin{cases} \lambda q, \, p = \lambda p_0 (\lambda \in \mathbb{R}), \\ 0, \quad \text{otherwise}. \end{cases}$$

Then, $\nu^A(T) = \lim_{x\to\infty} \inf\{\nu'_{Tp}(x) \mid p \in A\} \leq \lim_{x\to\infty} \nu'_{\lambda q}(x) < 1$. So $\nu^A(T) \in \Delta^+ \backslash \mathcal{D}^+$ and $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ is not a TV space, which is a contradiction.

The following theorem shows that the converse of the preceding theorem does not hold.

Theorem 3.3. *Let A be a subset of a PN space $(V_1, \nu, \tau_1, \tau_1^*)$ that contains a Hamel basis for V_1 and $(V_2, \nu', \tau_2, \tau_2^*)$ be an α-Šerstnev PN space. Then the following statements hold:*

(i) *If* $\sup\{|\lambda| : \lambda \in \mathbb{R}, \lambda p \in A\} = \infty$ *for some $p \in A$ and $p \neq \theta$, then $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ is not a TV space.*
(ii) *If $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ is a TV space, then* $\sup\{|\lambda| : \lambda \in \mathbb{R}, \lambda p \in A\} < \infty$ *for every $p \in A$ and $p \neq \theta$.*

Proof. Since statement (ii) is the contrapositive of statement (i), it suffices to prove (i). By Corollary 2.2, it is enough to show that $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ is not strict. Let $p \neq \theta$ and $\sup\{|\lambda| : \lambda \in \mathbb{R}, \lambda p \in A\} = \infty$. We define $T \in L(V_1, V_2)$ such that $T(p) \neq \theta$. Let $\{\lambda_n\}_n \subseteq \{|\lambda| : \lambda \in \mathbb{R}, \lambda p \in A\}$ and $|\lambda_n| \to \infty$ as $n \to \infty$. Since $\nu'_{T(p)} \neq \varepsilon_0$, one has

$$\lim_{n\to\infty} \nu'_{\lambda_n T(p)}(x) = \lim_{n\to\infty} \nu'_{T(p)} \left(\frac{x}{|\lambda_n|^\alpha} \right) = \beta < 1$$

for every $x \in \mathbb{R}$. Hence $\inf\{\nu'_{T(p)}(x) : p \in A\} \leq \beta < 1$ for every $x \in \mathbb{R}$, so

$$\lim_{x\to\infty} \inf\{\nu'_{T(p)}(x) : p \in A\} < 1.$$

Then $\nu^A(T) \in \Delta^+ \backslash \mathcal{D}^+$.

Corollary 3.3. *Let $(V_1, \nu, \tau_1, \tau_1^*)$ be a PN space and $(V_2, \nu', \tau_2, \tau_2^*)$ be an α-Šerstnev PN space. Then $(L(V_1, V_2), \nu^{V_1}, \tau_2, \tau_2^*)$ is not a TV space.*

Example 3.1. Suppose that A is a subset of a PN space $(V_1, \nu, \tau_1, \tau_1^*)$ that contains a Hamel basis for V_1. Let $\alpha \in]0, 1]$ and V_2 be a normed space. If we define $\nu : V_2 \to \Delta^+$ by $\nu_\theta = \varepsilon_0$ and $\nu_p(x) := e^{\frac{-\|p\|^\alpha}{x}}$ for $p \neq \theta$ and $x > 0$, then $(V_2, \nu, \Pi_\Pi, \Pi_M)$ is a TV space. If $\sup\{|\lambda| : \lambda \in \mathbb{R}, \lambda p \in A\} = \infty$ for some $p \in A$ and $p \neq \theta$, then $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ is not a TV space.

Lemma 3.1. [[27], p. 105]

(a) *If V is a finite-dimensional PN space and* $\mathcal{T}_1$, $\mathcal{T}_2$ *are two topologies on V that make it into a TV space, then* $\mathcal{T}_1 = \mathcal{T}_2$.
(b) *If V is a TV PN space and M is a finite-dimensional linear manifold in V, then M is closed.*

If $(X, ||\cdot||)$ is a normed space, we say that $A \subseteq X$ is classically bounded if, and only if, there is an $M \in \mathbb{R}$ such that for each $a \in A$, $||a|| \leq M$. Now, we state the following theorem that we will use it frequently in the rest of this section.

Theorem 3.4. *If* dim $V = n < \infty$ *and* (V, ν, τ, τ^*) *is a PN space that is also a TV space and A is a subspace of V, then:*

(a) *V is normable.*
(b) *V is complete.*
(c) *A is* $\mathcal{D}$*-compact if, and only if, it is compact.*

Also if $(V, \nu, \tau_1, \tau_1^*)$ *is an α-Šerstnev PN space, then:*

(d) *A is* $\mathcal{D}$*-bounded if, and only if, it is topologically bounded if, and only if, it is classically bounded.*
(e) *A is* $\mathcal{D}$*-compact if, and only if, it is compact if, and only if, it is closed and* $\mathcal{D}$*-bounded.*

Proof. (a) Let $\{e_1, e_2, ..., e_n\}$ be a Hamel basis for V. Then, for every p in V, there are $\alpha_1, \alpha_2, ..., \alpha_n$ in $\mathbb{R}$ such that $p = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$. If $\| p \| := \sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}$, then $||\cdot||$ defines a norm on V. It is easy to check that $(V, ||\cdot||)$ is a TV space. By Lemma 3.1, if $\mathcal{T}_1$ is the strong topology and $\mathcal{T}_2$ is the norm topology on V which is defined as above, then $\mathcal{T}_1 = \mathcal{T}_2$. So V is normable.

Before proving the other parts, we notice the following fact:

(i) A sequence $\{p_n\}_n$ is a strong Cauchy sequence if, and only if, it is Cauchy sequence in the norm topology.
(ii) A sequence $\{p_n\}_n$ is a strongly convergent to $p \in V$ if, and only if, it is convergent to p in the norm topology.

(b) Let $\{p_n\}_n$ be a strong Cauchy sequence. Then $\{p_n\}_n$ is a Cauchy sequence in the norm topology. Since $(V, \mathcal{T}_2)$ is complete, there is $p \in V$ such that $p_n \to p$ in $(V, \mathcal{T}_2)$ as $n \to \infty$. So $p_n \to p$ in $(V, \mathcal{T}_1)$ as $n \to \infty$. Hence, the result follows.
(c) Since $\mathcal{T}_1 = \mathcal{T}_2$, the identity map $I : (V, \mathcal{T}_1) \to (V, \mathcal{T}_2)$ is a homeomorphism. Hence, [[28], Theorem 28.2] and the arguments before part (b) give the desired conclusion.
(d) By the fact that $\mathcal{T}_1 = \mathcal{T}_2$ and Theorem 2.2, the results follow.
(e) Let $(\mathbb{R}^n, ||\cdot||)$ be Euclidean space and $\{e_1, e_2, ..., e_n\}$ be a Hamel basis for V. We define $f : (V, \mathcal{T}_2) \to (\mathbb{R}^n, \|\cdot\|)$ by $f(\alpha_1 e_1 + a_2 e_2 + \cdots + \alpha_n e_n) = (a_1, a_2, ..., a_n)$. It is clear that f is a homeomorphism. Since a subset in $\mathbb{R}^n$ is compact if, and only if, it

is closed and bounded, A is compact in the strong topology if, and only if, it is closed and $\mathcal{D}$-bounded.

Theorem 3.5. *Let A be a subset of a PN space $(V_1, \nu, \tau_1, \tau_1^*)$ that contains a Hamel basis for V_1 and $(V_2, \nu', \tau_2, \tau_2^*)$ be an α-Šerstnev PN space. Then one has:*

(a) $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ *is a TV space if, and only if, TA is $\mathcal{D}$-bounded for every $T \in L(V_1, V_2)$.*

(b) *Let $V_1 = V_2 = V$. If $(L(V, V), \nu^A, \tau_2, \tau_2^*)$ is a TV space, then A is $\mathcal{D}$-bounded.*

Moreover, if $(V_1, \nu, \tau_1, \tau_1^*)$ and $(V_2, \nu', \tau_2, \tau_2^*)$ are α-Šerstnev PN spaces that are TV spaces, then the following statements hold:

(c) *Let* dim $V_1 < \infty$. *If A is $\mathcal{D}$-bounded, then $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ is a TV space.*

(d) *Let* dim $V_1 < \infty$ *and* dim $V_1 \leq$ dim V_2. *Then $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ is a TV space if, and only if, A is $\mathcal{D}$-bounded.*

Proof. Parts (a) and (b) infer immediately from Corollary 2.2. We just prove parts (c) and (d).

(c) It is enough to show that TA is $\mathcal{D}$-bounded for every $T \in L(V_1, V_2)$. Since dim $V_1 < \infty$, Theorem 3.4 and [[27], p. 70] imply that $T : V_1 \to \mathrm{Rang}T$ is continuous for every $T \in L(V_1, V_2)$. Also by [[11], Theorem 2.2], $\bar{A}$ is $\mathcal{D}$-bounded. Hence, Theorem 3.4 concludes that $\bar{A}$ is compact. Then, $T\bar{A}$ is compact. Invoking Theorem 3.4, it follows that TA is $\mathcal{D}$-bounded.

(d) Let $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ be a TV space. Since dim $V_1 < \infty$ and dim $V_1 \leq$ dim V_2, we can define a one-to-one linear operator T: $V_1 \to V_2$. Then, by Theorem 3.4 and [[27], p. 70], $T : V_1 \to \mathrm{Rang}T$ is a homeomorphism. Since TA is $\mathcal{D}$-bounded, $\overline{TA}$ is compact. So, $T^{-1}(\overline{TA})$ is compact and therefore A is $\mathcal{D}$-bounded.

Conversely, it follows from part (c).

Theorem 3.6. *Let $(V_1, \nu, \tau_1, \tau_1^*)$ and $(V_2, \nu', \tau_2, \tau_2^*)$ be α-Šerstnev PN spaces and A be a subset of V_1 that contains a Hamel basis for V_1. If* dim $V_1 < \infty$, $(V_1, \nu, \tau_1, \tau_1^*)$ *is a TV space and A is $\mathcal{D}$-bounded, then $(V_2, \nu', \tau_2, {\tau_2}^*)$ is a TV space if, and only if, $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ is a TV space.*

Proof. By Theorems 3.1 and 3.5(c), the proof is obvious.

Example 3.2. Let $\alpha \in]0, 1]$ and $n > m$. We define $\nu : \mathbb{R}^n \to \Delta^+$ by $\nu_\theta = \varepsilon_0$ and $\nu_p(x) := e^{\frac{-\|p\|^\alpha}{x}}$ for $p \in \mathbb{R}^n$ and $x > 0$. Also we define ν': $\mathbb{R}^m \to \Delta^+$ by $\nu'_\theta = \varepsilon_0$ and $\nu_p(x) := e^{\frac{-\|p\|^\alpha}{x}}$ for $p \in \mathbb{R}^m$ and $x > 0$. Hence $(\mathbb{R}^n, \nu, \Pi_\Pi, \Pi_M)$ and $(\mathbb{R}^m, \nu, \Pi_\Pi, \Pi_M)$ are α-Šerstnev PN spaces; furthermore, they are TV spaces. Then A is classically bounded in $\mathbb{R}^n$ if, and only if, $(L(\mathbb{R}^n, \mathbb{R}^m), \nu^A, \Pi_\Pi, \Pi_M)$ is a TV space.

The following example shows that the converse of Theorem 3.3 is not true.

Example 3.3. Let $\alpha \in]0, 1]$ and $n > m$. We define $(\mathbb{R}^n, \nu, \Pi_\Pi, \Pi_M)$ and $(\mathbb{R}^m, \nu, \Pi_\Pi, \Pi_M)$ in a similar way to the earlier example. If $A = \{(k, k^2, 0, ..., 0): k \in \mathbf{N}\} \cup \{(1, 0, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, 0, ..., 0, 1)\}$, then A is a subset of $\mathbb{R}^n$ that contains a Hamel basis for $\mathbb{R}^n$. Although $\sup\{|\lambda| : \lambda \in \mathbb{R}, \lambda p \in A\} < \infty$ for every $p \in A$ and $p \neq \theta$, $(L(\mathbb{R}^n, \mathbb{R}^m), \nu^A, \Pi_\Pi, \Pi_M)$ is not a TV space, because A is not $\mathcal{D}$ -bounded.

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