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A new Hilbert-type integral inequality in the whole plane with the non-homogeneous kernel

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Abstract

By using the way of weight functions and the technique of real analysis, a new Hilbert-type integral inequality with the non-homogeneous kernel in the whole plane with the best constant factor is given. As applications, the equivalent inequalities with the best constant factors, the reverses and some particular cases are obtained.

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1 Introduction

If $f(x)$, $g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x)\mathrm{d}x < \infty$ and $0 < \int_0^\infty g^2(x)\mathrm{d}x < \infty$, then we have (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \pi\left(\int_0^\infty f^2(x)\mathrm{d}x \int_0^\infty g^2(x)\mathrm{d}x\right)^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible. Inequality (1) is well known as Hilbert's integral inequality, which is important in Mathematical Analysis and its applications [2].

If $p, r > 1$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{r}+\frac{1}{s}=1\lambda >0$, $f(x)$, $g(x) \geq 0$, such that $0 < \int_0^\infty x^{p(1+\frac{\lambda}{r})} f^p(x)\mathrm{d}x < \infty$ and, $0 < \int_0^\infty x^{q(1+\frac{\lambda}{s})} g^q(x)\mathrm{d}x < \infty$, then we have [3]:

$$\begin{aligned}&\int_0^\infty \int_0^\infty (\min\{x,y\})^\lambda f(x)g(y)\mathrm{d}x\mathrm{d}y \\ &< \frac{rs}{\lambda}\left(\int_0^\infty x^{p(1+\frac{\lambda}{r})-1} f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_0^\infty x^{q(1+\frac{\lambda}{s})-1} g^q(x)\mathrm{d}x\right)^{\frac{1}{q}},\end{aligned} \tag{2}$$

where the constant factor $\frac{rs}{\lambda}$ is the best possible. By using the way of weight functions, we can get two Hilbert-type integral inequalities with non-homogeneous kernels similar to (1) and (2) as follows [4,5]:

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{|1+xy|^{\lambda}}\mathrm{d}x\mathrm{d}y \\ < \quad & B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\int_{-\infty}^{\infty}x^{p(1-\frac{\lambda}{2})-1}f^{p}(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\int_{-\infty}^{\infty}x^{q(1-\frac{\lambda}{2})-1}g^{q}(x)\mathrm{d}x\right\}^{\frac{1}{q}}(\lambda>0), \end{aligned} \tag{3}$$

$$\begin{aligned} &\int_{0}^{\infty}\int_{0}^{\infty}(\min\{1,xy\})^{\lambda}f(x)g(y)\mathrm{d}x\mathrm{d}y \\ < \quad & \frac{\lambda}{\alpha(\lambda-\alpha)}\left\{\int_{0}^{\infty}x^{p(1+\alpha)-1}f^{p}(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\int_{0}^{\infty}x^{p(1+\alpha)-1}g^{q}(x)\mathrm{d}x\right\}^{\frac{1}{q}}(0<\alpha<\lambda). \end{aligned} \tag{4}$$

Some inequalities with the non-homogenous kernels have been studied in [6-8].

In this paper, by using the way of weight functions and the technique of real analysis, a new Hilbert-type integral inequality in the whole plane with the non-homogenous kernel and a best constant factor is built. As applications, the equivalent forms, the reverses and some particular cases are obtained.

2 Some lemmas

Lemma 1 *If* $0<\alpha_1<\alpha_2<\pi$*, define the weight functions* $\omega(y)$ *and* $\tilde{\omega}(x)$ *as follow:*

$$\omega(y):=\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{\min\{1,|xy|\}}{\sqrt{1+2xy\cos\alpha_i+(xy)^2}}\right\}\frac{1}{|x|}\mathrm{d}x,\ (y\in(-\infty,\infty)), \tag{5}$$

$$\tilde{\omega}(x):=\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{\min\{1,|xy|\}}{\sqrt{1+2xy\cos\alpha_i+(xy)^2}}\right\}\frac{1}{|y|}\mathrm{d}y,\ (x\in(-\infty,\infty)). \tag{6}$$

Then, we have $\omega(y)=\tilde{\omega}(x)=k(x,y\neq 0)$ *where*

$$k:=2\ln\left[\left(1+\sec\frac{\alpha_1}{2}\right)\left(1+\csc\frac{\alpha_2}{2}\right)\right]. \tag{7}$$

Proof. *Setting* $u=x\cdot|y|$ *in (5), we find*

$$\omega(y)=\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{\min\{|u|,1\}}{\sqrt{u^2+2u(y/|y|)\cos\alpha_i+1}}\right\}\frac{1}{|u|}\mathrm{d}u. \tag{8}$$

For $y\in(0,\infty)$*, we have*

$$\begin{aligned} \omega(y)=&\int_{-\infty}^{-1}\frac{1}{\sqrt{u^2+2u\cos\alpha_2+1}}\frac{-1}{u}\mathrm{d}u+\int_{-1}^{0}\frac{\mathrm{d}u}{\sqrt{u^2+2u\cos\alpha_2+1}} \\ &+\int_{0}^{1}\frac{\mathrm{d}u}{\sqrt{u^2+2u\cos\alpha_1+1}}+\int_{1}^{\infty}\frac{1}{\sqrt{u^2+2u\cos\alpha_1+1}}\frac{\mathrm{d}u}{u}. \end{aligned} \tag{9}$$

Setting

$$\omega_1 := \int_{-\infty}^{-1} \frac{1}{\sqrt{u^2 + 2u\cos\alpha_2 + 1}} \frac{-1}{u} \mathrm{d}u,$$

$$\omega_2 := \int_{-1}^{0} \frac{1}{\sqrt{u^2 + 2u\cos\alpha_2 + 1}} \mathrm{d}u,$$

$$\omega_3 := \int_{0}^{1} \frac{1}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} \mathrm{d}u,$$

$$\omega_4 := \int_{1}^{\infty} \frac{1}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} \frac{1}{u} \mathrm{d}u,$$

we find

$$\omega_1 \overset{v=-u}{=} \int_{1}^{\infty} \frac{\mathrm{d}v}{v\sqrt{v^2 + 2v\cos(\pi - \alpha_2) + 1}}$$

$$\overset{z=1/v}{=} \int_{0}^{1} \frac{\mathrm{d}z}{\sqrt{z^2 + 2z\cos(\pi - \alpha_2) + 1}},$$

$$\omega_2 \overset{v=-u}{=} \int_{0}^{1} \frac{\mathrm{d}v}{\sqrt{v^2 + 2v\cos(\pi - \alpha_2) + 1}} = \omega_1,$$

$$\omega_4 \overset{z=1/u}{=} \int_{0}^{1} \frac{\mathrm{d}z}{\sqrt{z^2 + 2z\cos\alpha_1 + 1}} = \omega_3.$$

Then, we have

$$\begin{aligned}
\omega(y) &= 2(\omega_1 + \omega_3) \\
&= 2\left\{\int_{0}^{1} \frac{\mathrm{d}u}{\sqrt{u^2 + 2u\cos(\pi - \alpha_2) + 1}} + \int_{0}^{1} \frac{\mathrm{d}u}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}}\right\} \\
&= 2\{\int_{0}^{1} \frac{1}{\sqrt{[u + \cos(\pi - \alpha_2)]^2 + \sin^2(\pi - \alpha_2)}} \mathrm{d}u \\
&+ \int_{0}^{1} \frac{1}{\sqrt{(u + \cos\alpha_1)^2 + \sin^2\alpha_1}} \mathrm{d}u\} \\
&= 2\left\{\ln(1 + \csc\frac{\alpha_2}{2}) + \ln(1 + \sec\frac{\alpha_1}{2})\right\} \\
&= 2\ln\left[(1 + \sec\frac{\alpha_1}{2})(1 + \csc\frac{\alpha_2}{2})\right] = k.
\end{aligned}$$

For $y \in (-\infty, 0)$, *we can obtain*

$$\begin{aligned}
\omega(y) = &\int_{-\infty}^{-1} \frac{1}{\sqrt{u^2 - 2ucos\alpha_1 + 1}} \frac{-\mathrm{d}u}{u} + \int_{-1}^{0} \frac{\mathrm{d}u}{\sqrt{u^2 - 2ucos\alpha_1 + 1}} \\
&+ \int_{0}^{1} \frac{\mathrm{d}u}{\sqrt{u^2 - 2ucos\alpha_2 + 1}} + \int_{1}^{\infty} \frac{1}{\sqrt{u^2 - 2ucos\alpha_2 + 1}} \frac{\mathrm{d}u}{u} \\
= &\; 2(\omega 1 + \omega 3) = k.
\end{aligned}$$

By the same way, we still can find that $\tilde{\omega}(x) = \omega(y) = k(x, y \neq 0)$. The lemma is proved. □

Lemma 2 *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \alpha_1 < \alpha_2 < \pi$, *f* (*x*) *is a nonnegative measurable function in* (-∞,∞), *then we have*

$$\begin{aligned} J \quad : \quad &= \int_{-\infty}^{\infty} |y|^{-1} \left[\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{min\{1, |xy|\}}{\sqrt{1 + 2xycos\alpha_i + (xy)^2}} \right\} f(x)\mathrm{d}x \right]^p \mathrm{d}y \\ &\leq \quad k^p \int_{-\infty}^{\infty} |x|^{p-1} f^p(x) dx. \end{aligned} \tag{10}$$

Proof. By Lemma 1 and Hölder's inequality [9], *we have*

$$\begin{aligned} &\left(\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{\min\{1, |xy|\}}{\sqrt{1 + 2xy\cos\alpha_i + (xy)^2}} \right\} f(x)\mathrm{d}x \right)^p \\ = \quad &\left(\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{\min\{1, |xy|\}}{\sqrt{1 + 2xy\cos\alpha_i + (xy)^2}} \right\} \left[\frac{|x|^{1/q}}{|y|^{1/p}} f(x) \right] \left[\frac{|y|^{1/p}}{|x|^{1/q}} \right] \mathrm{d}x \right)^p \\ \leq \quad &\left[\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{\min\{1, |xy|\}}{\sqrt{1 + 2xy\cos\alpha_i + (xy)^2}} \right\} \frac{|x|^{p-1}}{|y|} f^p(x)\mathrm{d}x \right] \\ &\times \left[\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{\min\{1, |xy|\}}{\sqrt{1 + 2xy\cos\alpha_i + (xy)^2}} \right\} \frac{|y|^{q-1}}{|x|} \mathrm{d}x \right]^{p-1} \\ = \quad &\omega(y)^{p-1} |y| \left[\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{\min\{1, |xy|\}}{\sqrt{1 + 2xy\cos\alpha_i + (xy)^2}} \right\} \frac{|x|^{p-1}}{|y|} f^p(x)\mathrm{d}x \right] \end{aligned} \tag{11}$$

Then, by (6), (11) and Fubini theorem [10], *it follows*

$$\begin{aligned} J &\leq k^{p-1} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{\min\{1, | xy |\}}{\sqrt{1 + 2xy\cos\alpha_i + (xy)^2}} \right\} \frac{| x|^{p-1}}{| y |} f^p(x)\mathrm{d}x \right] \mathrm{d}y \\ &= k^{p-1} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{\min\{1, | xy |\}}{\sqrt{1 + 2xy\cos\alpha_i + (xy)^2}} \right\} \frac{1}{| y |} \mathrm{d}y \right] | x|^{p-1} f^p(x)\mathrm{d}x \\ &= k^{p-1} \int_{-\infty}^{\infty} \tilde{\omega}(x) \; | x|^{p-1} f^p(x)\mathrm{d}x \\ &= k^p \int_{-\infty}^{\infty} | x|^{p-1} f^p(x) dx \end{aligned}$$

The lemma is proved. □

3 Main results and applications

Theorem 3 *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \alpha_1 < \alpha_2 < \pi$, $f(x)$, $g(x) \geq 0$, *satisfying* $0 < \int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y < \infty$ *and* $0 < \int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y < \infty$, *then we have*

$$\begin{aligned} I &:= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{\min\{1, |xy|\}}{\sqrt{1 + 2xy\cos\alpha_i + (xy)^2}} \right\} f(x)g(y)\mathrm{d}x\mathrm{d}y \\ &< k\left(\int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y\right)^{\frac{1}{q}}, \end{aligned} \tag{12}$$

$$\begin{aligned} J &= \int_{-\infty}^{\infty} |y|^{-1} \left[\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{\min\{1, |xy|\}}{\sqrt{1 + 2xy\cos\alpha_i + (xy)^2}} \right\} f(x)\mathrm{d}x\right]^p \mathrm{d}y \\ &< k^p \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)dx, \end{aligned} \tag{13}$$

where the constant factors k and k^p are the best possible (k is defined by (7)). Inequality (12) and (13) are equivalent.

Proof. If (11) takes the form of equality for a $y \in (-\infty, 0) \cup (0, \infty)$, then there exists constants M and N, such that they are not all zero, and

$$M\frac{|x|^{p/q}}{|y|} f^p(x) = N\frac{|y|^{q/p}}{|x|} a.e. \text{ in } (-\infty, \infty).$$

Hence, there exists a constant C, such that

$$M|x|^p f^p(x) = N|y|^q = C a.e. \ in \ (-\infty, \infty).$$

We suppose $M \neq 0$ (otherwise $N = M = 0$). Then, it follows

$$|x|^{p-1} f^p(x) = \frac{C}{M|x|} a.e. \ in \ (-\infty, \infty),$$

which contradicts the fact that $0 < \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x < \infty$. Hence, (11) takes the form of strict sign-inequality; so does (10), and we have (13).

By Hölder's inequality [9], we have

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \left[|y|^{\frac{-1}{p}} \int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{\min\{1, |xy|\}}{\sqrt{1 + 2xy\cos\alpha_i + (xy)^2}} \right\} f(x)\mathrm{d}x \right] \left[|y|^{\frac{1}{p}} g(y)\mathrm{d}y \right] \\ &\leq J^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y\right)^{\frac{1}{q}}. \end{aligned} \tag{14}$$

By (13), we have (12). On the other hand, suppose that (12) is valid. Setting

$$g(y) = |y|^{-1}\left[\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{\min\{1,|xy|\}}{\sqrt{1+2xy\cos\alpha_i+(xy)^2}}\right\}f(x)\mathrm{d}x\right]^{p-1},$$

then $J = \int_{-\infty}^{\infty}|y|^{q-1}g^q(y)dy$. By (10), it follows $J < \infty$. If $J = 0$, then (13) is naturally valid. Assuming that $0 < J < \infty$, by (12), we obtain

$$\int_{-\infty}^{\infty}|y|^{q-1}g^q(y)\mathrm{d}y = J = I < k\left(\int_{-\infty}^{\infty}|x|^{p-1}f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{-\infty}^{\infty}|y|^{q-1}g^q(y)\mathrm{d}y\right)^{\frac{1}{q}}, \tag{15}$$

$$J^{\frac{1}{p}} = \left(\int_{-\infty}^{\infty}|y|^{q-1}g^q(y)\mathrm{d}y\right)^{\frac{1}{p}} < k\left(\int_{-\infty}^{\infty}|x|^{p-1}f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}. \tag{16}$$

Hence, we have (13), which is equivalent to (12).

If the constant factor k in (12) is not the best possible, then there exists a positive constant K with $K < k$, such that (12) is still valid as we replace k by K, then we have

$$I < K\left(\int_{-\infty}^{\infty}|x|^{p-1}f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{-\infty}^{\infty}|y|^{q-1}g^q(y)\mathrm{d}y\right)^{\frac{1}{q}}. \tag{17}$$

For $\varepsilon > 0$, define functions $\tilde{f}(x), \tilde{g}(y)$ as follows:

$$\tilde{f}(x) := \begin{cases} x^{-\frac{2\varepsilon}{p}-1}, & x \in (1,\infty), \\ 0, & x \in [-1,1], \\ (-x)^{-\frac{2\varepsilon}{p}-1}, & x \in (-\infty,-1), \end{cases}$$

$$\tilde{g}(y) := \begin{cases} y^{\frac{2\varepsilon}{q}-1}, & y \in (0,1), \\ 0, & y \in (-\infty,-1]\cup[1,\infty), \\ (-y)^{\frac{2\varepsilon}{q}-1}, & y \in (-1,0). \end{cases}$$

Replacing $f(x)$, $g(y)$ by $\tilde{f}(x)$, $\tilde{g}(y)$ in (17), we obtain

$$\begin{aligned} \tilde{I} &:= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{\min\{1,|xy|\}}{\sqrt{1+2xy\cos\alpha_i+(xy)^2}}\right\}\tilde{f}(x)\tilde{g}(y)\mathrm{d}x\mathrm{d}y \\ &< K\left(\int_{-\infty}^{\infty}|x|^{p-1}\tilde{f}^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{-\infty}^{\infty}|y|^{q-1}\tilde{g}^q(y)\mathrm{d}y\right)^{\frac{1}{q}} \\ &= \frac{K}{\varepsilon}, \end{aligned} \tag{18}$$

$$\tilde{I} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{\min\{1, |xy|\}}{\sqrt{1 + 2xy\cos\alpha_i + (xy)^2}} \right\} \tilde{f}(x)\tilde{g}(y)\mathrm{d}x\mathrm{d}y = \sum_{i=1}^{4} I_i, \tag{19}$$

where,

$$I_1 := \int_{-1}^{0} (-y)^{\frac{2\varepsilon}{q}-1} \left[\int_{-\infty}^{-1} \min_{i\in\{1,2\}} \left\{ \frac{\min\{1, |xy|\}}{\sqrt{1 + 2xy\cos\alpha_i + (xy)^2}} \right\} (-x)^{-\frac{2\varepsilon}{p}-1}\mathrm{d}x \right] \mathrm{d}y,$$

$$I_2 := \int_{-1}^{0} (-y)^{\frac{2\varepsilon}{q}-1} \left[\int_{1}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{\min\{1, |xy|\}}{\sqrt{1 + 2xy\cos\alpha_i + (xy)^2}} \right\} x^{-\frac{2\varepsilon}{p}-1}\mathrm{d}x \right] \mathrm{d}y,$$

$$I_3 := \int_{0}^{1} y^{\frac{2\varepsilon}{q}-1} \left[\int_{-\infty}^{-1} \min_{i\in\{1,2\}} \left\{ \frac{\min\{1, |xy|\}}{\sqrt{1 + 2xy\cos\alpha_i + (xy)^2}} \right\} (-x)^{-\frac{2\varepsilon}{p}-1}\mathrm{d}x \right] \mathrm{d}y,$$

$$I_4 := \int_{0}^{1} y^{\frac{2\varepsilon}{q}-1} \left[\int_{1}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{\min\{1, |xy|\}}{\sqrt{1 + 2xy\cos\alpha_i + (xy)^2}} \right\} x^{-\frac{2\varepsilon}{p}-1}\mathrm{d}x \right] \mathrm{d}y.$$

By Fubini theorem [10], we obtain

$$\begin{aligned}
I_1 = I_4 &= \int_{0}^{1} y^{\frac{2\varepsilon}{q}-1} \left[\int_{1}^{\infty} \frac{min\{1, xy\}}{\sqrt{1 + 2xycos\alpha_1 + (xy)^2}} x^{-\frac{2\varepsilon}{p}-1}\mathrm{d}x \right] \mathrm{d}y \\
&\overset{u=xy}{=} \int_{0}^{1} y^{2\varepsilon-1} \left[\int_{y}^{\infty} \frac{min\{u, 1\}}{\sqrt{u^2 + 2ucos\alpha_1 + 1}} u^{-\frac{2\varepsilon}{p}-1}\mathrm{d}u \right] \mathrm{d}y \\
&= \int_{0}^{1} y^{2\varepsilon-1} \left[\int_{y}^{1} \frac{u^{-\frac{2\varepsilon}{p}}}{\sqrt{u^2 + 2ucos\alpha_1 + 1}}\mathrm{d}u \right] \mathrm{d}y \\
&\quad + \int_{0}^{1} y^{2\varepsilon-1} \left[\int_{1}^{\infty} \frac{u^{-\frac{2\varepsilon}{p}-1}}{\sqrt{u^2 + 2ucos\alpha_1 + 1}}\mathrm{d}u \right] \mathrm{d}y \\
&= \int_{0}^{1} \left(\int_{0}^{u} y^{2\varepsilon-1}\mathrm{d}y \right) \frac{u^{-\frac{2\varepsilon}{p}}\mathrm{d}u}{\sqrt{u^2 + 2ucos\alpha_1 + 1}} + \frac{1}{2\varepsilon} \int_{1}^{\infty} \frac{u^{-\frac{2\varepsilon}{p}-1}\mathrm{d}u}{\sqrt{u^2 + 2ucos\alpha_1 + 1}} \\
&= \frac{1}{2\varepsilon} \left[\int_{0}^{1} \frac{u^{\frac{2\varepsilon}{q}}}{\sqrt{u^2 + 2ucos\alpha_1 + 1}}\mathrm{d}u + \int_{1}^{\infty} \frac{u^{-\frac{2\varepsilon}{p}-1}}{\sqrt{u^2 + 2ucos\alpha_1 + 1}}\mathrm{d}u \right],
\end{aligned}$$

$$\begin{aligned}
I_2 = I_3 &= \int_{0}^{1} y^{\frac{2\varepsilon}{q}-1} \left[\int_{1}^{\infty} \frac{min\{1, xy\}}{\sqrt{1 - 2xycos\alpha_2 + (xy)^2}} x^{-\frac{2\varepsilon}{p}-1}\mathrm{d}x \right] \mathrm{d}y \\
&= \frac{1}{2\varepsilon} \left[\int_{0}^{1} \frac{u^{\frac{2\varepsilon}{q}}}{\sqrt{u^2 - 2ucos\alpha_2 + 1}}\mathrm{d}u + \int_{1}^{\infty} \frac{u^{-\frac{2\varepsilon}{p}-1}}{\sqrt{u^2 - 2ucos\alpha_2 + 1}}\mathrm{d}u \right].
\end{aligned}$$

In view of the above results, by using (18) and (19), it follows

$$\begin{aligned}&\int_0^1 \frac{u^{\frac{2\varepsilon}{q}}}{\sqrt{u^2+2u\cos\alpha_1+1}}\mathrm{d}u+\int_1^\infty \frac{u^{-\frac{2\varepsilon}{p}-1}}{\sqrt{u^2+2u\cos\alpha_1+1}}\mathrm{d}u\\&+\int_0^1 \frac{u^{\frac{2\varepsilon}{q}}}{\sqrt{u^2-2u\cos\alpha_2+1}}\mathrm{d}u+\int_1^\infty \frac{u^{-\frac{2\varepsilon}{p}-1}}{\sqrt{u^2-2u\cos\alpha_2+1}}\mathrm{d}u\\&=\varepsilon\tilde{I}<\varepsilon\cdot\frac{K}{\varepsilon}=K.\end{aligned} \tag{20}$$

By Fatou lemma [10] and (20), we find

$$\begin{aligned}k=\omega(y)&=\int_0^\infty \frac{min\{u,1\}\mathrm{d}u}{u\sqrt{u^2+2ucos\alpha_1+1}}+\int_0^\infty \frac{min\{u,1\}\mathrm{d}u}{u\sqrt{u^2-2ucos\alpha_2+1}}\\&=\int_0^1 \lim_{\varepsilon\to 0^+}\frac{u^{\frac{2\varepsilon}{q}}\mathrm{d}u}{\sqrt{u^2+2ucos\alpha_1+1}}+\int_1^\infty \lim_{\varepsilon\to 0^+}\frac{u^{-\frac{2\varepsilon}{p}-1}\mathrm{d}u}{\sqrt{u^2+2ucos\alpha_1+1}}\\&\quad+\int_0^1 \lim_{\varepsilon\to 0^+}\frac{u^{\frac{2\varepsilon}{q}}\mathrm{d}u}{\sqrt{u^2-2ucos\alpha_2+1}}+\int_1^\infty \lim_{\varepsilon\to 0^+}\frac{u^{-\frac{2\varepsilon}{p}-1}\mathrm{d}u}{\sqrt{u^2-2ucos\alpha_2+1}}\\&\le \underline{lim}_{\varepsilon\to 0^+}\Big[\int_0^1 \frac{u^{\frac{2\varepsilon}{q}}\mathrm{d}u}{\sqrt{u^2+2ucos\alpha_1+1}}+\int_1^\infty \frac{u^{-\frac{2\varepsilon}{p}-1}\mathrm{d}u}{\sqrt{u^2+2ucos\alpha_1+1}}\\&\quad+\int_0^1 \frac{u^{\frac{2\varepsilon}{q}}\mathrm{d}u}{\sqrt{u^2-2ucos\alpha_2+1}}+\int_1^\infty \frac{u^{-\frac{2\varepsilon}{p}-1}\mathrm{d}u}{\sqrt{u^2-2ucos\alpha_2+1}}\Big]\le K,\end{aligned} \tag{21}$$

which contradicts the fact that $K < k$. Hence, the constant factor k in (12) is the best possible.

If the constant factor in (13) is not the best possible, then by (14), we may get a contradiction that the constant factor in (12) is not the best possible. Thus, the theorem is proved. □

Theorem 4 *As the assumptions of Theorem 3, replacing* $p >1$ *by* $0 < p <1$, *we have the equivalent reverse of (12) and (13) with the best constant factors.*

Proof. The way of proving of Theorem 4 is similar to Theorem 3. By the reverse Hölder's inequality [9], we have the reverse of (10) and (14). It is easy to obtain the reverse of (13). In view of the reverses of (13) and (14), we obtain the reverse of (12). On the other hand, suppose that the reverse of (12) is valid. Setting the same $g(y)$ as theorem 3, by the reverse of (10), we have $J >0$. If $J = \infty$, then the reverse of (13) is obvious value; if $J <\infty$, then by the reverse of (12), we obtain the reverses of (15) and (16). Hence, we have the reverse of (13), which is equivalent to the reverse of (12).

If the constant factor k in the reverse of (12) is not the best possible, then there exists a positive constant $\tilde{K}(\text{with } \tilde{K} > k)$, such that the reverse of (12) is still valid as we replace k by $\tilde{K}$. By the reverse of (20), we have

$$\begin{aligned}&\int_0^1\left[\frac{1}{\sqrt{u^2+2u\cos\alpha_1+1}}+\frac{1}{\sqrt{u^2-2u\cos\alpha_2+1}}\right]u^{\frac{2\varepsilon}{q}}\mathrm{d}u\\&+\int_1^\infty\left[\frac{1}{\sqrt{u^2+2u\cos\alpha_1+1}}+\frac{1}{\sqrt{u^2-2u\cos\alpha_2+1}}\right]u^{-\frac{2\varepsilon}{p}-1}\mathrm{d}u\\&>\tilde{K}.\end{aligned}\tag{22}$$

For $0<\varepsilon_0<\frac{|q|}{2}$, we have $\frac{2\varepsilon_0}{q}>-1$. For $0<\varepsilon\le\varepsilon_0$, we obtain $u^{\frac{2\varepsilon}{q}}\le u^{\frac{2\varepsilon_0}{q}}\,(u\in(0,1])$ and

$$\begin{aligned}&\int_0^1\left[\frac{1}{\sqrt{u^2+2u\cos\alpha_1+1}}+\frac{1}{\sqrt{u^2-2u\cos\alpha_2+1}}\right]u^{\frac{2\varepsilon_0}{q}}\mathrm{d}u\\&\le\left(\frac{1}{\sin\alpha_1}+\frac{1}{\sin\alpha_2}\right)\int_0^1u^{\frac{2\varepsilon_0}{q}}\mathrm{d}u=\left(\frac{1}{\sin\alpha_1}+\frac{1}{\sin\alpha_2}\right)\frac{1}{1+(2\varepsilon_0)/q}<\infty.\end{aligned}$$

Then, by Lebesgue control convergence theorem [10], we have for $\varepsilon\to0^+$ that

$$\begin{aligned}&\int_0^1\left[\frac{1}{\sqrt{u^2+2u\cos\alpha_1+1}}+\frac{1}{\sqrt{u^2-2u\cos\alpha_2+1}}\right]u^{\frac{2\varepsilon}{q}}\mathrm{d}u\\&=\int_0^1\left[\frac{1}{\sqrt{u^2+2u\cos\alpha_1+1}}+\frac{1}{\sqrt{u^2-2u\cos\alpha_2+1}}\right]\mathrm{d}u+o(1).\end{aligned}\tag{23}$$

By Levi's theorem [10], we find for $\varepsilon\to0^+$ that

$$\begin{aligned}&\int_1^\infty\left[\frac{1}{\sqrt{u^2+2u\cos\alpha_1+1}}+\frac{1}{\sqrt{u^2-2u\cos\alpha_2+1}}\right]u^{-\frac{2\varepsilon}{p}-1}\mathrm{d}u\\&=\int_1^\infty\left[\frac{1}{\sqrt{u^2+2u\cos\alpha_1+1}}+\frac{1}{\sqrt{u^2-2u\cos\alpha_2+1}}\right]u^{-1}\mathrm{d}u+\tilde{o}(1).\end{aligned}\tag{24}$$

By (22), (23) and (24), for $\varepsilon\to0^+$ in (22), we have $k\ge\tilde{K}$, which contradicts the fact that $k<\tilde{K}$. Hence, the constant factor k in the reverse of (12) is the best possible.

If the constant factor in reverse of (13) is not the best possible, then by the reverse of (14), we may get a contradiction that the constant factor in the reverse of (12) is not the best possible. Thus, the theorem is proved. □

Remark 1 *For $\alpha_1=\alpha_2=\alpha\in(0,\pi)$ in (12) and (13), we have the following equivalent inequalities:*

$$\begin{aligned}&\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{\min\{1,|xy|\}}{\sqrt{1+2xy\cos\alpha+(xy)^2}}f(x)g(y)\mathrm{d}x\mathrm{d}y\\&<k_0\left(\int_{-\infty}^\infty|x|^{p-1}f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{-\infty}^\infty|x|^{q-1}g^q(y)\mathrm{d}y\right)^{\frac{1}{q}},\end{aligned}\tag{25}$$

$$\int_{-\infty}^{\infty} |y|^{-1}\left[\int_{-\infty}^{\infty} \frac{\min\{1, |xy|\}}{\sqrt{1 + 2xy\cos\alpha + (xy)^2}} f(x)\mathrm{d}x\right]^p \mathrm{d}y < k_0^p \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)dx, \tag{26}$$

where the constant factors $k_0 := 2\ln[(1 + \sec\frac{\alpha}{2})(1 + \csc\frac{\alpha}{2})]$ *and* k_0^p *are the best possible.*

Remark 2 *For* $\alpha_1 = \alpha_2 = \frac{\pi}{3}$, $p = q = 2$ *in (12) and (13), we have the following equivalent inequalities:*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\min\{1, |xy|\}}{\sqrt{1 + xy + (xy)^2}} f(x)g(y)\mathrm{d}x\mathrm{d}y < 2\ln(3 + 2\sqrt{3})\left(\int_{-\infty}^{\infty} |x| f^2(x)\mathrm{d}x \int_{-\infty}^{\infty} |y| g^2(y)\mathrm{d}y\right)^{\frac{1}{2}}, \tag{27}$$

$$\int_{-\infty}^{\infty} |y|^{-1}\left[\int_{-\infty}^{\infty} \frac{\min\{1, |xy|\}}{\sqrt{1 + xy + (xy)^2}} f(x)\mathrm{d}x\right]^2 \mathrm{d}y < 2^2[\ln(3 + 2\sqrt{3})]^2 \int_{-\infty}^{\infty} |x| f^2(x)dx. \tag{28}$$

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