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Higher order Hermite-Fejér interpolation polynomials with Laguerre-type weights

Heesun Jung[1*] and Ryozi Sakai[2]

* Correspondence: hsun90@skku.edu
[1]Department of Mathematics Education, Sungkyunkwan University Seoul 110-745, Republic of Korea
Full list of author information is available at the end of the article

Abstract

Let $\mathbb{R}^+ = [0, \infty)$ and $R : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function which is the Laguerre-type exponent, and $p_{n,\rho}(x)$, $\rho > -\frac{1}{2}$ be the orthonormal polynomials with the weight $w_\rho(x) = x^\rho e^{-R(x)}$. For the zeros $\{x_{k,n,\rho}\}_{k=1}^n$ of $p_{n,\rho}(x) = p_n(w_\rho^2; x)$, we consider the higher order Hermite-Fejér interpolation polynomial $L_n(l, m, f; x)$ based at the zeros $\{x_{k,n,\rho}\}_{k=1}^n$, where $0 \le l \le m - 1$ are positive integers.

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1. Introduction and main results

Let $\mathbb{R} = [-\infty, \infty)$ and $\mathbb{R}^+ = [0, \infty)$. Let $R : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, non-negative, and increasing function. Consider the exponential weights $w_\rho(x) = x^\rho \exp(-R(x))$, $\rho > -1/2$, and then we construct the orthonormal polynomials $\{p_{n,\rho}(x)\}_{n=0}^\infty$ with the weight $w_\rho(x)$. Then, for the zeros $\{x_{k,n,\rho}\}_{k=1}^n$ of $p_{n,\rho}(x) = p_n(w_\rho^2; x)$, we obtained various estimations with respect to $p_{n,\rho}^{(j)}(x_{k,n,\rho})$, $k = 1, 2, ..., n$, $j = 1, 2, ..., \nu$, in [1]. Hence, in this article, we will investigate the higher order Hermite-Fejér interpolation polynomial $L_n(l, m, f; x)$ based at the zeros $\{x_{k,n,\rho}\}_{k=1}^n$, using the results from [1], and we will give a divergent theorem. This article is organized as follows. In Section 1, we introduce some notations, the weight classes $\mathcal{L}_2$, $\tilde{\mathcal{L}}_\nu$ with $\mathcal{L}(C^2)$, $\mathcal{L}(C^2+)$, and main results. In Section 2, we will introduce the classes $\mathcal{F}(C^2)$ and $\mathcal{F}(C^2+)$, and then, we will obtain some relations of the factors derived from the classes $\mathcal{F}(C^2)$, $\mathcal{F}(C^2+)$ and the classes $\mathcal{L}(C^2+)$, $\mathcal{L}(C^2+)$. Finally, we will prove the main theorems using known results in [1-5], in Section 3.

We say that $f : \mathbb{R} \to \mathbb{R}^+$ is quasi-increasing if there exists $C > 0$ such that $f(x) \le Cf(y)$ for $0 < x < y$. The notation $f(x) \sim g(x)$ means that there are positive constants C_1, C_2 such that for the relevant range of x, $C_1 \le f(x)/g(x) \le C_2$. The similar notation is used for sequences, and sequences of functions. Throughout this article, $C, C_1, C_2, ...$ denote positive constants independent of n, x, t or polynomials $P_n(x)$. The same symbol does not necessarily denote the same constant in different occurrences. We denote the class of polynomials with degree n by $\mathcal{P}_n$.

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First, we introduce classes of weights. Levin and Lubinsky [5,6] introduced the class of weights on $\mathbb{R}^+$ as follows. Let $I = [0, d)$, where $0 < d \leq \infty$.

Definition 1.1. [5,6] We assume that $R : I \to [0, \infty)$ has the following properties: Let $Q(t) = R(x)$ and $x = t^2$.

(a) $\sqrt{x}R(x)$ is continuous in I, with limit 0 at 0 and $R(0) = 0$;

(b) $R''(x)$ exists in $(0, d)$, while Q'' is positive in $(0, \sqrt{d})$;

(c)

$$\lim_{x \to d-} R(x) = \infty;$$

(d) The function

$$T(x) := \frac{xR'(x)}{R(x)}$$

is quasi-increasing in $(0, d)$, with

$$T(x) \geq \Lambda > \frac{1}{2}, \quad x \in (0, d);$$

(e) There exists $C_1 > 0$ such that

$$\frac{|R''(x)|}{R'(x)} \leq C_1 \frac{R'(x)}{R(x)}, \quad \text{a.e.} \quad x \in (0, d).$$

Then, we write $w \in \mathcal{L}(C^2)$. If there also exist a compact subinterval $J^* \ni 0$ of $I^* = (-\sqrt{d}, \sqrt{d})$ and $C_2 > 0$ such that

$$\frac{Q''(t)}{|Q'(t)|} \geq C_2 \frac{|Q'(t)|}{Q(t)}, \quad \text{a.e.} \quad t \in I^* \backslash J^*,$$

then we write $w \in \mathcal{L}(C^2+)$.

We consider the case $d = \infty$, that is, the space $\mathbb{R}^+ = [0, \infty)$, and we strengthen Definition 1.1 slightly.

Definition 1.2. We assume that $R : \mathbb{R}^+ \to \mathbb{R}^+$ has the following properties:

(a) $R(x)$, $R'(x)$ are continuous, positive in $\mathbb{R}^+$, with $R(0) = 0$, $R'(0) = 0$;

(b) $R''(x) > 0$ exists in $\mathbb{R}^+ \backslash \{0\}$;

(c)

$$\lim_{x \to \infty} R(x) = \infty;$$

(d) The function

$$T(x) := \frac{xR'(x)}{R(x)}$$

is quasi-increasing in $\mathbb{R}^+\backslash\{0\}$, with

$$T(x) \geq \Lambda > \frac{1}{2}, \quad x \in \mathbb{R}^+\backslash\{0\};$$

(e) There exists $C_1 > 0$ such that

$$\frac{R''(x)}{R'(x)} \leq C_1 \frac{R'(x)}{R(x)}, \quad \text{a.e.} \quad x \in \mathbb{R}^+\backslash\{0\}.$$

There exist a compact subinterval $J \ni 0$ of $\mathbb{R}^+$ and $C_2 > 0$ such that

$$\frac{R''(x)}{R'(x)} \geq C_2 \frac{R'(x)}{R(x)}, \quad \text{a.e.} \quad t \in \mathbb{R}^+\backslash J,$$

then we write $w \in \mathcal{L}_2$.

To obtain estimations of the coefficients of higher order Hermite-Fejér interpolation polynomial based at the zeros $\{x_{k,n,\rho}\}_{k=1}^{n}$, we need to focus on a smaller class of weights.

Definition 1.3. Let $w = \exp(-R) \in \mathcal{L}_2$ and let $\nu \geq 2$ be an integer. For the exponent R, we assume the following:

(a) $R^{(j)}(x) > 0$, for $0 \leq j \leq \nu$ and $x > 0$, and $R^{(j)}(0) = 0$, $0 \leq j \leq \nu - 1$.

(b) There exist positive constants $C_i > 0$, $i = 1, 2, ..., \nu - 1$ such that for $i = 1, 2, ..., \nu - 1$

$$R^{(i+1)}(x) \leq C_i R^{(i)}(x) \frac{R'(x)}{R(x)}, \quad \text{a.e.} \quad x \in \mathbb{R}^+\backslash\{0\}.$$

(c) There exist positive constants C, $c_1 > 0$ and $0 \leq \delta < 1$ such that on $x \in (0, c_1)$

$$R^{(\nu)}(x) \leq C\left(\frac{1}{x}\right)^{\delta}. \tag{1.1}$$

(d) There exists $c_2 > 0$ such that we have one among the following

(d1) $T(x)/\sqrt{x}$ is quasi-increasing on (c_2, ∞),

(d2) $R^{(\nu)}(x)$ is nondecreasing on (c_2, ∞).

Then we write $w(x) = e^{-R(x)} \in \tilde{\mathcal{L}}_\nu$.

Example 1.4. [6,7] Let $\nu \geq 2$ be a fixed integer. There are some typical examples satisfying all conditions of Definition 1.3 as follows: Let $\alpha > 1$, $l \geq 1$, where l is an integer. Then we define

$$R_{l,\alpha}(x) = \exp_l(x^\alpha) - \exp_l(0),$$

where $\exp_l(x) = \exp(\exp(\exp \ldots \exp(x)) \ldots)$ is the l-th iterated exponential.

(1) If $\alpha > v$, $w(x) = e^{-R_{l,\alpha}(x)} \in \tilde{\mathcal{L}}_v$.

(2) If $\alpha \leq v$ and α is an integer, we define

$$R^*_{l,\alpha}(x) = \exp_l(x^\alpha) - \exp_l(0) - \sum_{j=1}^{r} \frac{R^{(j)}_{l,\alpha}(0)}{j!} x^j.$$

Then $w(x) = e^{-R^*_{l,\alpha}(x)} \in \tilde{\mathcal{L}}_v$.

In the remainder of this article, we consider the classes $\mathcal{L}_2$ and $\tilde{\mathcal{L}}_v$; Let $w \in \mathcal{L}_2$ or $w \in \tilde{\mathcal{L}}_v$ $v \geq 2$. For $\rho > -\frac{1}{2}$, we set $w_\rho(x) := x^\rho w(x)$. Then we can construct the orthonormal polynomials $p_{n,\rho}(x) = p_n(w_\rho^2; x)$ of degree n with respect to $w_\rho^2(x)$. That is,

$$\int_0^\infty p_{n,\rho}(u) p_{m,\rho}(u) w_\rho^2(u) du = \delta_{nm} (\text{Kronecker's delta}) \quad n, m = 0, 1, 2, \ldots.$$

Let us denote the zeros of $p_{n,\rho}(x)$ by

$$0 < x_{n,n,\rho} < \cdots < x_{2,n,\rho} < x_{1,n,\rho} < \infty.$$

The Mhaskar-Rahmanov-Saff numbers a_v is defined as follows:

$$v = \frac{1}{\pi} \int_0^1 \frac{a_v t R'(a_v t)}{\sqrt{t(1-t)}} \mathrm{d}t, \quad v > 0.$$

Let l, m be non-negative integers with $0 \leq l < m \leq v$. For $f \in C^{(l)}(\mathbb{R})$, we define the (l, m)-order Hermite-Fejér interpolation polynomials $L_n(l, m, f; x) \in \mathcal{P}_{mn-1}$ as follows: For each $k = 1, 2, \ldots, n$,

$$L_n^{(j)}(l, m, f; x_{k,n,\rho}) = f^{(j)}(x_{k,n,\rho}), \quad j = 0, 1, 2, \ldots, l,$$
$$L_n^{(j)}(l, m, f; x_{k,n,\rho}) = 0, \quad j = l+1, l+2, \ldots, m-1.$$

For each $P \in \mathcal{P}_{mn-1}$, we see $L_n(m-1, m, P; x) = P(x)$. The fundamental polynomials $h_{s,k,n,\rho}(m; x) \in \mathcal{P}_{mn-1}$, $k = 1, 2, \ldots, n$, of $L_n(l, m, f; x)$ are defined by

$$h_{s,k,n,\rho}(l, m; x) = l^m_{k,n,\rho}(x) \sum_{i=s}^{m-1} e_{s,i}(l, m, k, n)(x - x_{k,n,\rho})^i. \tag{1.2}$$

Here, $l_{k,n,\rho}(x)$ is a fundamental Lagrange interpolation polynomial of degree $n - 1$ [[8], p. 23] given by

$$l_{k,n,\rho}(x) = \frac{p_n(w_\rho^2; x)}{(x - x_{k,n,\rho}) p'_n(w_\rho^2; x_{k,n,\rho})}$$

and $h_{s,k,n,\rho}(l, m; x)$ satisfies

$$h^{(j)}_{s,k,n,\rho}(l, m; x_{p,n,\rho}) = \delta_{s,j}\delta_{k,p} \quad j, s = 0, 1, \ldots, m-1, \; p = 1, 2, \ldots, n. \tag{1.3}$$

Then

$$L_n(l,m,f;x) = \sum_{k=1}^{n}\sum_{s=0}^{l} f^{(s)}(x_{k,n,\rho})h_{s,k,n,\rho}(l,m;x).$$

In particular, for $f \in C(\mathbb{R})$, we define the m-order Hermite-Fejér interpolation polynomials $L_n(m,f;x) \in \mathcal{P}_{mn-1}$ as the (0, m)-order Hermite-Fejér interpolation polynomials $L_n(0, m, f; x)$. Then we know that

$$L_n(m,f;x) = \sum_{k=1}^{n} f(x_{k,n,\rho})h_{k,n,\rho}(m;x),$$

where $e_i(m, k, n)$: = $e_{0,i}(0, m, k, n)$ and

$$h_{k,n,\rho}(m;x) = l_{k,n,\rho}^{m}(x)\sum_{i=0}^{m-1} e_i(m,k,n)(x - x_{k,n,\rho})^i. \tag{1.4}$$

We often denote $l_{k,n}(x)$: = $l_{k,n,\rho}(x)$, $h_{s,k,n}(x)$: = $h_{s,k,n,\rho}(x)$, and $x_{k,n}$: = $x_{k,n,\rho}$ if they do not confuse us.

Theorem 1.5. *Let* $w(x) = \exp(-R(x)) \in \mathcal{L}(C^2+)$ *and* $\rho > -1/2$.

(a) *For each* $m \geq 1$ *and* $j = 0, 1, \ldots$, *we have*

$$|\,(l_{k,n}^{m})^{(j)}(x_{k,n})\,| \leq C\left(\frac{n}{\sqrt{a_{2n} - x_{k,n}}}\right)^{j} x_{k,n}^{-\frac{j}{2}}. \tag{1.5}$$

(b) *For each* $m \geq 1$ *and* $j = s, \ldots, m - 1$, *we have* $e_{s,s}(l, m, k, n) = 1/s!$ *and*

$$|\,e_{s,j}(l,m,k,n)\,| \leq C\left(\frac{n}{\sqrt{a_{2n} - x_{k,n}}}\right)^{j-s} x_{k,n}^{-\frac{j-s}{2}}. \tag{1.6}$$

We remark $\mathcal{L}_2 \subset \mathcal{L}(C^2+)$.

Theorem 1.6. *Let* $w(x) = \exp(-R(x)) \in \tilde{\mathcal{L}}_\nu$, $\nu \geq 2$ *and* $\rho > -1/2$. *Assume that* $1 + 2\rho - \delta/2 \geq 0$ *for* $\rho < -1/4$ *and if* $T(x)$ *is bounded, then assume that*

$$a_n \leq Cn^{2/(1+\nu-\delta)}, \tag{1.7}$$

where $0 \leq \delta < 1$ *is defined in (1.1). Then we have the following:*

(a) *If* j *is odd, then we have for* $m \geq 1$ *and* $j = 0, 1, \ldots, \nu - 1$,

$$\begin{aligned} |\,(l_{k,n}^{m})^{(j)}(x_{k,n})\,| \leq\; & C\left(\frac{T(a_n)}{\sqrt{a_n x_{k,n}}} + R'(x_{k,n}) + \frac{1}{x_{k,n}}\right) \\ & \times \left(\frac{n}{\sqrt{a_{2n}} - \sqrt{x_{k,n}}} + \frac{T(a_n)}{\sqrt{a_n}}\right)^{j-1} x_{k,n}^{-\frac{j-1}{2}}. \end{aligned} \tag{1.8}$$

(b) *If $j - s$ is odd, then we have for $m \geq 1$ and $0 \leq s \leq j \leq m - 1$,*

$$
\begin{aligned}
|\, e_{s,j}(l,m,k,n)\,| \leq & C\left(\frac{T(a_n)}{\sqrt{a_n x_{k,n}}} + R'(x_{k,n}) + \frac{1}{x_{k,n}}\right) \\
& \times \left(\frac{n}{\sqrt{a_{2n}} - \sqrt{x_{k,n}}} + \frac{T(a_n)}{\sqrt{a_n}}\right)^{j-s-1} x_{k,n}^{-\frac{j-s-1}{2}}.
\end{aligned}
\tag{1.9}
$$

Theorem 1.7. *Let $0 < \varepsilon < 1/4$. Let $\frac{1}{\varepsilon}\frac{a_n}{n^2} \leq x_{k,n} \leq \varepsilon a_n$. Let s be a positive integer with $2 \leq 2s \leq \nu$. Then under the same conditions as the assumptions of Theorem 1.6, there exists $\mu_1(\varepsilon, n) > 0$ such that*

$$
\left|p_{n,\rho}^{(2s)}(x_{k,n})\right| \leq C\delta(\varepsilon,n)\left(\frac{n}{\sqrt{a_n}}\right)^{2s-1} \left|p_n'(x_{k,n})\right| x_{k,n}^{-\frac{(2s-1)}{2}}
$$

and $\delta(\varepsilon, n) \to 0$ as $n \to \infty$ and $\varepsilon \to 0$.

Theorem 1.8. *[4, Lemma 10] Let $0 < \varepsilon < 1/4$. Let $\frac{1}{\varepsilon}\frac{a_n}{n^2} \leq x_{k,n} \leq \varepsilon a_n$. Let s be a positive integer with $2 \leq 2s \leq \nu - 1$. Suppose the same conditions as the assumptions of Theorem 1.6. Then*

(a) *for $1 \leq 2s - 1 \leq \nu - 1$,*

$$
\left|(l_{k,n}^m)^{(2s-1)}(x_{k,n})\right| \leq C\delta(\varepsilon,n)\left(\frac{n}{\sqrt{a_n}}\right)^{2s-1} x_{k,n}^{-\frac{2s-1}{2}},
\tag{1.10}
$$

where $\delta(\varepsilon, n)$ is defined in Theorem 1.7.

(b) *there exists $\beta(n, k)$ with $0 < D_1 \leq \beta(n, k) \leq D_2$ for absolute constants D_1, D_2 such that the following holds:*

$$
(l_{k,n}^m)^{(2s)}(x_{k,n}) = (-1)^s \phi_s(m)\beta^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s} x_{k,n}^{-s}(1 + \xi_s(m,\varepsilon,x_{k,n},n))
\tag{1.11}
$$

and $|\xi_s(m, \varepsilon, x_{k,n}, n)| \to 0$ as $n \to \infty$ and $\varepsilon \to 0$.

Theorem 1.9. *[4, (4.16)],* [9]*Let $0 < \varepsilon < 1/4$. Let $\frac{1}{\varepsilon}\frac{a_n}{n^2} \leq x_{k,n} \leq \varepsilon a_n$. Let s be a positive integer with $2 \leq 2s \leq m - 1$. Suppose the same conditions as the assumptions of Theorem 1.6. Then for $j = 0, 1, 2, \ldots$, there is a polynomial $\Psi_j(x)$ of degree j such that $(-1)^j \psi_j(-m) > 0$ for $m = 1, 3, 5, \ldots$ and the following relation holds:*

$$
e_{2s}(m,k,n) = \frac{(-1)^s}{(2s)!}\Psi_s(-m)\beta^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s} x_{k,n}^{-s}\left(1 + \eta_s(m,\varepsilon,x_{k,n},n)\right)
\tag{1.12}
$$

and $|\eta_s(m, \varepsilon, x_{k,n}, n)| \to 0$ as $n \to \infty$ and $\varepsilon \to 0$.

Theorem 1.10. *Let m be an odd positive integer. Suppose the same conditions as the assumptions of Theorem 1.6. Then there is a function f in $C(\mathbb{R}^+)$ such that for any fixed* *interval* $[a, b]$, $a > 0$,

$$\limsup_{n\to\infty} \max_{a\le x\le b} |L_n(m, f; x)| = \infty.$$

2. Preliminaries

Levin and Lubinsky introduced the classes $\mathcal{L}(C^2)$ and $\mathcal{L}(C^2+)$ as analogies of the classes $\mathcal{F}(C^2)$ and $\mathcal{F}(C^2+)$ defined on $I^* = (-\sqrt{d}, \sqrt{d})$. They defined the following:

Definition 2.1. [10] We assume that $Q : I^* \to [0, \infty)$ has the following properties:

(a) $Q(t)$ is continuous in I^*, with $Q(0) = 0$;
(b) $Q''(t)$ exists and is positive in $I^*\backslash\{0\}$;
(c)

$$\lim_{t\to\sqrt{d}-} Q(t) = \infty;$$

(d) The function

$$T^*(t) := \frac{tQ'(t)}{Q(t)}$$

is quasi-increasing in $(0, \sqrt{d})$, with

$$T^*(t) \ge \Lambda^* > 1, \quad t \in I^*\backslash\{0\};$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(t)}{|Q'(t)|} \le C_1 \frac{|Q'(t)|}{Q(t)}, \quad \text{a.e.} \quad t \in I^*\backslash\{0\}.$$

Then we write $W \in \mathcal{F}(C^2)$. If there also exist a compact subinterval $J^* \ni 0$ of I^* and $C_2 > 0$ such that

$$\frac{Q''(t)}{|Q'(t)|} \ge C_2 \frac{Q'(t)}{|Q(t)|}, \quad \text{a.e.} \quad t \in I^*\backslash J^*,$$

then we write $W \in \mathcal{F}(C^2+)$.

Then we see that $w \in \mathcal{L}(C^2) \Leftrightarrow W \in \mathcal{F}(C^2)$ and $w \in \mathcal{L}(C^2+) \Leftrightarrow W \in \mathcal{F}(C^2+)$ where $W(t) = w(x)$, $x = t^2$, from [6, Lemma 2.2]. In addition, we easily have the following:

Lemma 2.2. [1]*Let* $Q(t) = R(x)$, $x = t^2$. *Then we have*

$$w \in \mathcal{L}_2 \Rightarrow W \in \mathcal{F}(C^2+),$$

where $W(t) = w(x)$; $x = t^2$.

On $\mathbb{R}$, we can consider the corresponding class to $\tilde{\mathcal{L}}_\nu$ as follows:

Definition 2.3. [11] Let $W = \exp(-Q) \in \mathcal{F}(C^2+)$ and $\nu \ge 2$ be an integer. Let Q be a continuous and even function on $\mathbb{R}$. For the exponent Q, we assume the following:

(a) $Q^{(j)}(x) > 0$, for $0 \le j \le \nu$ and $t \in \mathbb{R}^+/\{0\}$.

(b) There exist positive constants $C_i > 0$ such that for $i = 1, 2, ..., \nu - 1$,

$$Q^{(i+1)}(t) \leq C_i Q^{(i)}(t)\frac{Q'(t)}{Q(t)}, \quad \text{a.e.} \quad x \in \mathbb{R}^+\backslash\{0\}.$$

(c) There exist positive constants $C, c_1 > 0$, and $0 \leq \delta^* < 1$ such that on $t \in (0, c_1)$,

$$Q^{(\nu)}(t) \leq C\left(\frac{1}{t}\right)^{\delta^*}. \tag{2.1}$$

(d) There exists $c_2 > 0$ such that we have one among the following:
(d1) $T^*(t)/t$ is quasi-increasing on (c_2, ∞),
(d2) $Q^{(\nu)}(t)$ is nondecreasing on (c_2, ∞).

Then we write $W(t) = e^{-Q(t)} \in \tilde{\mathcal{F}}_\nu$.

Let $W \in \tilde{\mathcal{F}}_\nu$, and $\nu \geq 2$. For $\rho^* > -\frac{1}{2}$, we set

$$W_{\rho^*}(t) := |t|^{\rho^*}W(t).$$

Then we can construct the orthonormal polynomials $P_{n,\rho^*}(t) = P_n(W_{\rho^*}^2; t)$ of degree n with respect to $W_{\rho^*}(t)$. That is,

$$\int_{-\infty}^{\infty} P_{n,\rho^*}(v)P_{m,\rho^*}(v)W_{\rho^*}^2(v)\mathrm{d}t = \delta_{nm}, \quad n, m = 0, 1, 2, \ldots.$$

Let us denote the zeros of $P_{n,\rho^*}(t)$ by

$$-\infty < t_{nn} < \cdots < t_{2n} < t_{1n} < \infty.$$

There are many properties of $P_{n,\rho^*}(t) = P_n(W_{\rho^*}; t)$ with respect to $W_{\rho^*}(t)$, $W \in \tilde{\mathcal{F}}_\nu, \nu = 2, 3, \ldots$ of Definition 2.3 in [2,3,7,11-13]. They were obtained by transformations from the results in [5,6]. Jung and Sakai [2, Theorem 3.3 and 3.6] estimate $P_{n,\rho^*}^{(j)}(t_{k,n})$, $k = 1, 2, ..., n$, $j = 1, 2, ..., \nu$ and Jung and Sakai [1, Theorem 3.2 and 3.3] obtained analogous estimations with respect to $p_{n,\rho}^{(j)}(x_{k,n})$, $k = 1, 2, ..., n$, $j = 1, 2, ..., \nu$. In this article, we consider $w = \exp(-R) \in \tilde{\mathcal{L}}_\nu$ and $p_{n,\rho}(x) = p_n(w_\rho; x)$. In the following, we give the transformation theorems.

Theorem 2.4. *[13, Theorem 2.1] Let $W(t) = W(x)$ with $x = t^2$. Then the orthonormal polynomials $P_{n,\rho^*}(t)$ on $\mathbb{R}$ can be entirely reduced to the orthonormal polynomials $p_{n,\rho}(x)$ in $\mathbb{R}^+$ as follows: For $n = 0, 1, 2, ...$,*

$$P_{2n,2\rho+\frac{1}{2}}(t) = p_{n,\rho}(x) \quad \textit{and} \quad P_{2n+1,2\rho-\frac{1}{2}}(t) = tp_{n,\rho}(x).$$

In this article, we will use the fact that $w_\rho(x) = x^\rho \exp(-R(x))$ is transformed into $W_{2\rho+1/2}(t) = |t|^{2\rho+1/2} \exp(-Q(t))$ as meaning that

$$\begin{aligned}\int_0^\infty p_{n,\rho}(x)p_{m,\rho}(x)w_\rho^2(x)\mathrm{d}x &= 2\int_0^\infty p_{n,\rho}(t^2)p_{m,\rho}(t^2)t^{4\rho+1}W^2(t)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} P_{2n,2\rho+1/2}(t)P_{2m,2\rho+1/2}(t)W_{2\rho+1/2}^2(t)\mathrm{d}t.\end{aligned}$$

Theorem 2.5. *[1, Theorem 2.5] Let $Q(t) = R(x)$, $x = t^2$. Then we have*

$$w(x) = \exp(-R(x)) \in \tilde{\mathcal{L}}_\nu \Rightarrow W(t) = \exp(-Q(t)) \in \tilde{\mathcal{F}}_\nu. \tag{2.2}$$

In particular, we have

$$Q^{(\nu)}(t) \le C\left(\frac{1}{t}\right)^{\delta},$$

where $0 \le \delta < 1$ is defined in (1.1).

For convenience, in the remainder of this article, we set as follows:

$$\rho^* := 2\rho + \frac{1}{2} \text{ for } \rho > -\frac{1}{2}, \quad p_n(x) := p_{n,\rho}(x), \quad P_n(t) := P_{n,\rho^*}(t), \tag{2.3}$$

and $x_{k,n} = x_{k,n,\rho}$, $t_{kn} = t_{k,n,\rho^*}$. Then we know that $\rho^* > -\frac{1}{2}$ and

$$p_n(x) = P_{2n,\rho^*}(t),\ x = t^2, \quad x_{k,n} = t_{k,2n}^2, \quad t_{k,2n} > 0,\ k = 1, 2, \ldots, n. \tag{2.4}$$

In the following, we introduce useful notations:

(a) The Mhaskar-Rahmanov-Saff numbers a_ν and a_u^* are defined as the positive roots of the following equations, that is,

$$\nu = \frac{1}{\pi}\int_0^1 a_\nu t R'(a_\nu t)\{t(1-t)\}^{-\frac{1}{2}} dt,\ \nu > 0$$

and

$$u = \frac{2}{\pi}\int_0^1 a_u^* t Q'(a_u^* t)(1-t^2)^{-\frac{1}{2}} dt,\ u > 0.$$

(b) Let

$$\eta_n = \{nT(a_n)\}^{-\frac{2}{3}} \quad \text{and} \quad \eta_n^* = \{nT^*(a_n^*)\}^{-\frac{2}{3}}.$$

Then we have the following:

Lemma 2.6. *[6, (2.5),(2.7),(2.9)]*

$$a_n = a_{2n}^{*\,2}, \quad \eta_n = 4^{2/3}\eta_{2n}^*, \quad T(a_n) = \frac{1}{2}T^*(a_{2n}^*).$$

To prove main results, we need some lemmas as follows:

Lemma 2.7. *[13, Theorem 2.2, Lemma 3.7] For the minimum positive zero, $t_{[n/2],n}$ ($[n/2]$ is the largest integer $\le n/2$), we have*

$$t_{[n/2],n} \sim a_n^* n^{-1},$$

and for the maximum zero t_{1n} we have for large enough n,

$$1 - \frac{t_{1n}}{a_n^*} \sim \eta_n^*, \quad \eta_n^* = (nT^*(a_n^*))^{-\frac{2}{3}}.$$

Moreover, for some constant $0 < \varepsilon \le 2$ *we have*

$$T^*(a_n^*) \le Cn^{2-\varepsilon}.$$

Remark 2.8. (a) Let $W(t) \in \mathcal{F}(C^2+)$. Then

(a-1) $T(x)$ is bounded $\Leftrightarrow$ $T^*(t)$ is bounded.
(a-2) $T(x)$ is unbounded $\Rightarrow$ $a_n \le C(\eta)n^\eta$ for any $\eta > 0$.
(a-3) $T(a_n) \le Cn^{2-\varepsilon}$ for some constant $0 < \varepsilon \le 2$.

(b) Let $w(x) \in \tilde{\mathcal{L}}_\nu$. Then
(b-1) $\rho > -1/2 \Rightarrow \rho^* > -1/2$.
(b-2) $1 + 2\rho - \delta/2 \ge 0$ for $\rho < -1/4 \Rightarrow 1 + 2\rho^* - \delta^* \ge 0$ for $\rho^* < 0$.
(b-3) $a_n \le Cn^{2/(1+\nu-\delta)} \Rightarrow a_n^* \le Cn^{1/(1+\nu-\delta^*)}$.

Proof of Remark 2.8. (a) (a-1) and (a-3) are easily proved from Lemma 2.6. From [11, Theorem 1.6], we know the following: When $T^*(t)$ is unbounded, for any $\eta > 0$ there exists $C(\eta) > 0$ such that

$$a_t^* \le C(\eta)t^\eta, \quad t \ge 1.$$

In addition, since $T(x) = T^*(t)/2$ and $a_n = {a_{2n}^*}^2$, we know that (a-2).

(b) Since $w(x) \in \tilde{\mathcal{L}}_\nu$, we know that $W(t) \in \tilde{\mathcal{F}}_\nu$ and $\delta^* = \delta$ by Theorem 2.5. Then from (2.3) and Lemma 2.6, we have (b-1), (b-2), and (b-3). □

Lemma 2.9. *[1, Lemma 3.6] For* $j = 1, 2, 3, \ldots$, *we have*

$$p_n^{(j)}(x) = \sum_{i=1}^{j} (-1)^{j-i} c_{j,i} P_{2n}^{(i)}(t) t^{-2j+i},$$

where $c_{j,i} > 0 (1 \le i \le j, j = 1, 2, \ldots)$ *satisfy the following relations: for* $k = 1, 2, \ldots$,

$$c_{k+1,1} = \frac{2k-1}{2} c_{k,1}, \quad c_{k+1,k+1} = \frac{1}{2^{k+1}}, \quad c_{1,1} = \frac{1}{2},$$

and for $2 \le i \le k$,

$$c_{k+1,i} = \frac{c_{k,i-1} + (2k-i)c_{k,i}}{2}.$$

3. Proofs of main results

Our main purpose is to obtain estimations of the coefficients $e_{s,i}(l, m, k, n)$, $k = 1, 2, \ldots, 0 \le s \le l$, $s \le i \le m - 1$.

Theorem 3.1. *[1, Theorem 1.5] Let* $w(x) = \exp(-R(x)) \in \mathcal{L}(C^2+)$ *and let* $\rho > -1/2$. *For each* $k = 1, 2, \ldots, n$ *and* $j = 0, 1, \ldots$, *we have*

$$|\, p_{n,\rho}^{(j)}(x_{k,n}) \,| \le C\left(\frac{n}{\sqrt{a_{2n} - x_{k,n}}}\right)^{j-1} x_{k,n}^{-\frac{j-1}{2}} \,|\, p'_{n,\rho}(x_{k,n}) \,|.$$

Proof of Theorem 1.5. (a) From Theorem 3.1 we know that

$$|l_{k,n}^{(j)}(x_{k,n})| = \left|\frac{p_n^{(j+1)}(x_{k,n})}{(j+1)p_n'(x_{k,n})}\right| \le C\left(\frac{n}{\sqrt{a_{2n}-x_{k,n}}}\right)^j x_{k,n}^{-\frac{j}{2}}.$$

Then, assuming that (a) is true for $1 \le m' < m$, we have

$$\begin{aligned} |(l_{k,n}^m)^{(j)}(x_{k,n})| &= \left|\sum_{s=0}^{j}\binom{j}{s}(l_{k,n}^{m-1})^{(s)}(x_{k,n})l_{k,n}{}^{(j-s)}(x_{k,n})\right| \\ &\le C\sum_{s=0}^{j}\left(\frac{n}{\sqrt{a_{2n}-x_{k,n}}}\right)^s x_{k,n}^{-\frac{s}{2}}\left(\frac{n}{\sqrt{a_{2n}-x_{k,n}}}\right)^{j-s} x_{k,n}^{-\frac{j-s}{2}} \\ &\le \left(\frac{n}{\sqrt{a_{2n}-x_{k,n}}}\right)^j x_{k,n}^{-\frac{j}{2}}. \end{aligned}$$

Therefore, the result is proved by induction with respect to m.

(b) From (2) and (3), we know $e_{s,\ s}(l, m, k, n) = 1/s!$ and the following recurrence relation: for $s + 1 \le i \le m - 1$,

$$e_{s,i}(l,m,k,n) = -\sum_{p-s}^{i-1}\frac{1}{(i-p)!}e_{s,p}(l,m,k,n)(l_{k,n})^{(i-p)}(x_{k,n}). \tag{3.5}$$

Therefore, we have the result by induction on i and (3.5).

Theorem 3.2. *[1, Theorem 1.6] Let* $w(x) = \exp(-R(x)) \in \tilde{\mathcal{L}}_v$ *and let* $\rho > -1/2$. *Suppose the same conditions as the assumptions of Theorem 1.6. For each* $k = 1, 2, ..., n$ *and* $j = 1, ..., \nu$, *we have*

$$|p_{n,\rho}^{(j)}(x_{k,n})| \le C\left(\frac{n}{\sqrt{a_n}-\sqrt{x_{k,n}}} + \frac{T(a_n)}{\sqrt{a_n}}\right)^{j-1} x_{k,n}^{-\frac{j-1}{2}} |p_{n,\rho}'(x_{k,n})|$$

and in particular, if j is even, then we have

$$\begin{aligned} |\ p_{n,\rho}^{(j)}(x_{k,n})| \ &\le C\left(\frac{T(a_n)}{\sqrt{a_n x_{k,n}}} + R'(x_{k,n}) + \frac{1}{x_{k,n}}\right) \\ &\times\left(\frac{n}{\sqrt{a_n}-\sqrt{x_{k,n}}} + \frac{T(a_n)}{\sqrt{a_n}}\right)^{j-2} x_{k,n}^{-\frac{j-2}{2}} |p_{n,\rho}'(x_{k,n})|. \end{aligned}$$

Proof of Theorem 1.6. We use the induction method on m.

(a) For $m = 1$, we have the result because of

$$l_{k,n}^{(j)}(x_{k,n}) = \frac{p_n^{(j+1)}(x_{k,n})}{(j+1)p_n'(x_{k,n})}, \quad j = 1, 2, 3, \ldots,$$

and Theorem 3.2. Now we assume the theorem for $1 \le m' < m$. Then, we have the following: For $1 \le 2s - 1 \le \nu - 1$,

$$(l_{k,n}^{m})^{(2s-1)}(x_{k,n}) = \sum_{r=0}^{s}\binom{2s-1}{2r}(l_{k,n}^{m-1})^{(2r)}(x_{k,n})l_{k,n}^{(2s-2r-1)}(x_{k,n}) + \sum_{r=0}^{s}\binom{2s-1}{2r+1}(l_{k,n}^{m-1})^{(2r+1)}(x_{k,n})l_{k,n}^{(2s-2r-2)}(x_{k,n}).$$

Since

$$\frac{n}{\sqrt{a_{2n}-x_{k,n}}} \le \frac{n}{\sqrt{a_{2n}}-\sqrt{x_{k,n}}},$$

we have

$$\begin{aligned}
&\left|(l_{k,n}^{m-1})^{(2r)}(x_{k,n})l_{k,n}^{(2s-2r-1)}(x_{k,n})\right| \\
&\le C\left(\frac{T(a_n)}{\sqrt{a_n x_{k,n}}} + R'(x_{k,n}) + \frac{1}{x_{k,n}}\right) \\
&\quad \times \left(\frac{n}{\sqrt{a_{2n}-x_{k,n}}}\right)^{2r}\left(\frac{n}{\sqrt{a_{2n}}-\sqrt{x_{k,n}}} + \frac{T(a_n)}{\sqrt{a_n}}\right)^{2s-2r-2} x_{k,n}^{-s+1} \\
&\le C\left(\frac{T(a_n)}{\sqrt{a_n x_{k,n}}} + R'(x_{k,n}) + \frac{1}{x_{k,n}}\right) \\
&\quad \times \left(\frac{n}{\sqrt{a_{2n}}-\sqrt{x_{k,n}}} + \frac{T(a_n)}{\sqrt{a_n}}\right)^{2s-2} x_{k,n}^{-s+1},
\end{aligned}$$

and similarly

$$\begin{aligned}
\left|(l_{k,n}^{m-1})^{(2r+1)}(x_{k,n})l_{k,n}^{(2s-2r-2)}(x_{k,n})\right| \le &C\left(\frac{T(a_n)}{\sqrt{a_n x_{k,n}}} + R'(x_{k,n}) + \frac{1}{x_{k,n}}\right) \\
&\times \left(\frac{n}{\sqrt{a_{2n}}-\sqrt{x_{k,n}}} + \frac{T(a_n)}{\sqrt{a_n}}\right)^{2s-2} x_{k,n}^{-s+1}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\left|(l_{k,n}^{m})^{(2s-1)}(x_{k,n})\right| \le &C\left(\frac{T(a_n)}{\sqrt{a_n x_{k,n}}} + R'(x_{k,n}) + \frac{1}{x_{k,n}}\right) \\
&\times \left(\frac{n}{\sqrt{a_{2n}}-\sqrt{x_{k,n}}} + \frac{T(a_n)}{\sqrt{a_n}}\right)^{2s-2} x_{k,n}^{-s+1}.
\end{aligned}$$

(b) To prove the result, we proceed by induction on i. From (1.2) and (1.3) we know $e_{s,s}(l, m, k, n) = 1/s!$ and the following recurrence relation: for $s + 1 \le i \le m - 1$,

$$e_{s,i}(l,m,k,n) = -\sum_{p=s}^{i-1}\frac{1}{(i-p)!}e_{s,p}(l,m,k,n)(l_{k,n}^{m})^{(i-p)}(x_{k,n}). \tag{3.6}$$

When $i - s$ is odd, we know that

$$\begin{cases} i-p: \text{odd}, & \text{if } p-s: \text{even} \\ i-p: \text{even}, & \text{if } p-s: \text{odd}. \end{cases}$$

Then, we have (1.9) from (1.5), (1.8), (3.6), and the assumption of induction on i. □

Theorem 3.3. *[1, Theorem 1.7] Let* $0 < \varepsilon < 1/4$. *Let* $\frac{1}{\varepsilon}\frac{a_n}{n^2} \le x_{k,n} \le \varepsilon a_n$ *and let s be a positive integer with* $2 \le 2s \le \nu - 1$. *Suppose the same conditions as the assumptions of Theorem 1.6. Then there exists* $\beta(n, k)$, $0 < D_1 \le \beta(n, k) \le D_2$ *for absolute constants* D_1, D_2 *such that the following equality holds:*

$$p_{n,\rho}^{(2s+1)}(x_{k,n}) = (-1)^s \beta^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}(1+\rho_s(\varepsilon, x_{k,n}, n))p_n'(x_{k,n})x_{k,n}^{-s}$$

and $|\rho_s(\varepsilon, x_{k,n}, n)| \to 0$ *as* $n \to \infty$ *and* $\varepsilon \to 0$.

Lemma 3.4. *[3, Theorem 2.5] Let* $W \in \mathcal{F}(C^2+)$ *and* $r = 1, 2, \ldots$ *Then, uniformly for* $1 \le k \le n$,

$$\left|\frac{P_n^{(r)}(t_{k,n})}{P'_n(t_{k,n})}\right| \le C\left(\frac{n}{\sqrt{{a_{2n}^*}^2 - t_{k,n}^2}}\right)^{r-1}.$$

Lemma 3.5. *[2, Theorem 3.3] Let* $\rho^* > -1/2$ *and* $W(x) = \exp(-Q(x)) \in \tilde{\mathcal{F}}_\nu$, $\nu \ge 2$. *Assume that* $1 + 2\rho^* - \delta^* \ge 0$ *for* $\rho^* < 0$ *and if* $T^*(t)$ *is bounded, then assume*

$$a_n^* \le C n^{1/(1+\nu-\delta^*)},$$

where $0 \le \delta^* < 1$ *is defined in (2.1). Let* $0 < \alpha < 1/2$. *Let* $\frac{1}{\varepsilon}\frac{a_n^*}{n} \le |t_{kn}| \le \varepsilon a_n^*$ *and let s be a positive integer with* $2 \le 2s \le \nu$. *Then there exists* $\mu(\varepsilon, n) > 0$ *such that*

$$\left|P_n^{(2s)}(t_{k,n})\right| \le C\mu(\varepsilon, n)\left(\frac{n}{a_n}\right)^{2s-1}\left|P_n'(t_{k,n})\right|$$

and $\mu(\varepsilon, n) \to 0$ *as* $n \to \infty$ *and* $\varepsilon \to 0$.

Proof of Theorem 1.7. By Lemma 2.9, we have

$$\begin{aligned}\left|p_n^{(2s)}(x_{k,n})\right| &= \left|\sum_{i=1}^{2s}(-1)^{2s-i}c_{2s,i}P_{2n}^{(i)}(t_{k,n})t_{k,n}^{-4s+i}\right| \\ &\le C\left|c_{2s,2s}P_{2n}^{(2s)}(t_{k,n})t_{k,n}^{-2s}\right| + \left|\sum_{i=1}^{2s-1}(-1)^{2s-i}c_{2s,i}P_{2n}^{(i)}(t_{k,n})t_{k,n}^{-4s+i}\right|.\end{aligned}$$

Since, we have by Lemma 3.5,

$$\left|c_{2s,2s}P_{2n}^{(2s)}(t_{k,n})t_{k,n}^{-2s}\right| \le C\mu(\varepsilon, 2n)\left(\frac{n}{a_{2n}^*}\right)^{2s-1}\left|P_{2n}'(t_{k,n})\right|\left|t_{k,n}\right|^{-2s}$$

and by Lemma 3.4,

$$\begin{aligned}&\left|\sum_{i=1}^{2s-1}(-1)^{2s-i}c_{2s,i}P_{2n}^{(i)}(t_{k,n})t_{k,n}^{-4s+i}\right| \\ \le\ & C\left(\frac{n}{a_{2n}^*}\right)^{2s-1}\left|P_{2n}'(t_{k,n})\right|\left|t_{k,n}\right|^{-2s}\sum_{i=1}^{2s-1}\left(\frac{n}{a_{2n}^*}\mid t_{k,n}\mid\right)^{-2s+i} \\ \le\ & C\varepsilon\left(\frac{n}{a_{2n}^*}\right)^{2s-1}\left|P_{2n}'(t_{k,n})\right|\left|t_{k,n}\right|^{-2s},\end{aligned}$$

we have

$$\left|p_n^{(2s)}(x_{k,n})\right| \le C\delta(\varepsilon,n)\left(\frac{n}{a_{2n}^*}\right)^{2s-1}\left|P_{2n}'(t_{k,n})\right|\left|t_{k,n}\right|^{-2s}$$
$$\le C\delta(\varepsilon,n)\left(\frac{n}{\sqrt{a_n}}\right)^{2s-1}\left|p_n'(x_{k,n})\right|x_{k,n}^{-\frac{(2s-1)}{2}},$$

where $\delta(\varepsilon, n) = \mu(\varepsilon, 2n) + \varepsilon$. □

Here we can estimate the coefficients $e_i(v, k, n)$ of the fundamental polynomials $h_{kn}(v; x)$. For $j = 0, 1, \ldots$, define $\varphi_j(1){:} = (2j + 1)^{-1}$ and for $k \ge 2$,

$$\varphi_j(k) := \sum_{r=0}^{j}\frac{1}{2j-2r+1}\binom{2j}{2r}\varphi_r(k-1). \tag{3.7}$$

Proof of Theorem 1.8. In a manner analogous to the proof of Theorem 1.6 (a), we use mathematical induction with respect to m.

(a) From Theorem 1.7, we know that for $1 \le 2s -1 \le v - 1$,

$$\left|l_{k,n}^{(2s-1)}(x_{k,n})\right| = \left|\frac{p_n^{(2s)}(x_{k,n})}{2sp_n'(x_{k,n})}\right| \le C\delta(\varepsilon,n)\left(\frac{n}{\sqrt{a_n}}\right)^{2s-1}x_{k,n}^{-\frac{2s-1}{2}}.$$

From Theorem 1.5, we know that for $x_{k,\ n} \le a_n/4$,

$$\left|(l_{k,n}^m)^{(j)}(x_{k,n})\right| \le C\left(\frac{n}{\sqrt{a_n}}\right)^j x_{k,n}^{-\frac{j}{2}}. \tag{3.8}$$

Then, we have by mathematical induction on m,

$$\begin{aligned}\left|(l_{k,n}^m)^{(2s-1)}(x_{k,n})\right| \le& C\sum_{r=0}^{s}\binom{2s-1}{2r}\left|(l_{k,n}^{m-1})^{(2r)}(x_{k,n})l_{k,n}^{(2s-2r-1)}(x_{k,n})\right| \\ &+\sum_{r=0}^{s}\binom{2s-1}{2r+1}\left|(l_{k,n}^{m-1})^{(2r+1)}(x_{k,n})l_{k,n}^{(2s-2r-2)}(x_{k,n})\right| \\ \le& C\delta(\varepsilon,n)\left(\frac{n}{\sqrt{a_n}}\right)^{2s-1}x_{k,n}^{-\frac{2s-1}{2}}.\end{aligned}$$

(b) From Theorem 3.3, we know that for $0 \le 2s \le v - 1$,

$$\begin{aligned}l_{k,n}^{(2s)}(x_{k,n}) &= \frac{p_n^{(2s+1)}(x_{k,n})}{(2s+1)p_n'(x_{k,n})} \\ &= (-1)^s\phi_s(1)\beta^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}x_{k,n}^{-s}(1+\rho_s(\varepsilon,x_{k,n},n)).\end{aligned} \tag{3.9}$$

If we let $\xi_s(1, \varepsilon, x_{k,\ n}, n) = \rho_s\ (\varepsilon, x_{k,\ n}, n)$, then (1.11) holds for $m = 1$ because $|\xi_s(1, \varepsilon, x_{k,\ n}, n)| \to 0$ as $n \to \infty$ and $\varepsilon \to 0$. Now, we split $(l_{k,n}^m)^{(2s)}(x_{k,n})$ into two terms as follows:

$$\begin{aligned}(l_{k,n}^m)^{(2s)}(x_{k,n}) =& \sum_{0\le 2r\le 2s}\binom{2s}{2r}(l_{k,n}^{m-1})^{(2r)}(x_{k,n})l_{k,n}^{(2s-2r)}(x_{k,n}) \\ &+\sum_{1\le 2r-1\le 2s}\binom{2s}{2r-1}(l_{k,n}^{m-1})^{(2r-1)}(x_{k,n})l_{k,n}^{(2s-2r+1)}(x_{k,n}).\end{aligned} \tag{3.10}$$

For the second term, we have from (1.10),

$$\left|\sum_{1\le 2r-1\le 2s}\binom{2s}{2r-1}(l_{k,n}^{m-1})^{(2r-1)}(x_{k,n})l_{k,n}^{(2s-2r+1)}(x_{k,n})\right|\le C\delta^2(\varepsilon,n)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}x_{k,n}^{-s}\quad(3.11)$$

For the first term, we let $\xi_s(m)=\xi_s(m,\varepsilon,x_{k,n},n)$ for convenience. Then we know that

$$l_{k,n}^{(2s-2r)}(x_{k,n})=(-1)^{s-r}\phi_{s-r}(1)\beta^{s-r}(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s-r}x_{k,n}^{-(s-r)}(1+\xi_{s-r}(1))$$

and $|\xi_{s-r}(1)|\to 0$ as $n\to\infty$ and $\varepsilon\to 0$. By mathematical induction, we assume for $0\le 2r\le 2s$;

$$(l_{k,n}^{m-1})^{(2r)}(x_{k,n})=(-1)^r\phi_r(m-1)\beta^r(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2r}x_{k,n}^{-r}(1+\xi_r(m-1))$$

and $|\xi_r(m-1)|\to 0$ as $n\to\infty$ and $\varepsilon\to 0$. Then, since

$$\begin{aligned}(l_{k,n}^{m-1})^{(2r)}(x_{k,n})l_{k,n}^{(2s-2r)}(x_{k,n})&=(-1)^s\beta^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}x_{k,n}^{-s}\\&\times\phi_r(m-1)\phi_{s-r}(1)(1+\xi_r(m-1))\left(1+\xi_{s-r}(1)\right),\end{aligned}$$

we have for $0\le 2r\le 2s$, using the definition of (3.7),

$$\begin{aligned}&\sum_{0\le 2r\le 2s}\binom{2s}{2r}(l_{k,n}^{m-1})^{(2r)}(x_{k,n})l_{k,n}^{(2s-2r)}(x_{k,n})\\&=(-1)^s\beta^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}x_{k,n}^{-s}\\&\quad\times\sum_{0\le 2r\le 2s}\binom{2s}{2r}\phi_r(m-1)\phi_{s-r}(1)(1+\xi_r(m-1))(1+\xi_{s-r}(1))\\&=(-1)^s\phi_s(m)\beta^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}x_{k,n}^{-s}+(-1)^s\beta^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}x_{k,n}^{-s}\\&\quad\times\sum_{0\le 2r\le 2s}\binom{2s}{2r}\phi_r(m-1)\phi_{s-r}(1)(\xi_r(m-1)+\xi_{s-r}(1)+\xi_r(m-1)\xi_{s-r}(1)).\end{aligned}$$

Here, we consider (3.10). If we let

$$\begin{aligned}&\xi_s(m,\varepsilon,x_{k,n},n)=\xi_s(m)=\\&\sum_{0\le 2r\le 2s}\binom{2s}{2r}\frac{\phi_r(m-1)\phi_{s-r}(1)}{\phi_s(m)}(\xi_r(m-1)+\xi_{s-r}(1)+\xi_r(m-1)\xi_{s-r}(1))\\&\quad+\sum_{1\le 2r-1\le 2s}\binom{2s}{2r-1}\frac{(l_{k,n}^{m-1})^{(2r-1)}(x_{k,n})l_{k,n}^{(2s-2r+1)}(x_{k,n})}{(-1)^s\phi_s(m)\beta^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}x_{k,n}^{-s}},\end{aligned}$$

then we have

$$\begin{aligned}
\xi_s(m) &\le \sum_{0\le 2r\le 2s} \binom{2s}{2r} \frac{\phi_r(m-1)\phi_{s-r}(1)}{\phi_s(m)} (\xi_r(m-1)+\xi_{s-r}(1)+\xi_r(m-1)\xi_{s-r}(1)) \\
&\quad + C\frac{\delta^2(\varepsilon,n)(\frac{n}{\sqrt{a_n}})^{2s}x_{k,n}^{-s}}{(-1)^s\phi_s(m)\beta^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}x_{k,n}^{-s}} \quad \text{by the definition of (3.11)} \\
&\le \sum_{0\le 2r\le 2s} \binom{2s}{2r} \frac{\phi_r(m-1)\phi_{s-r}(1)}{\phi_s(m)} (\xi_r(m-1)+\xi_{s-r}(1)+\xi_r(m-1)\xi_{s-r}(1)) \\
&\quad + C'\delta^2(\varepsilon,n).
\end{aligned}$$

Then, we know that (1.11) holds and $|\xi_i(j)| \to 0$ as $n \to \infty$ and $\varepsilon \to 0$, using mathematical induction on m. Therefore, we have the result.

We rewrite the relation (3.7) in the form for $\nu = 1, 2, 3 ...,$

$$\phi_0(\nu) := 1$$

and for $j = 1, 2, 3 ..., \nu = 2, 3, 4, ...,$

$$\phi_j(\nu) - \phi_j(\nu-1) = \frac{1}{2j+1}\sum_{r=0}^{j-1}\binom{2j+1}{2r}\phi_r(\nu-1).$$

Now, for every j we will introduce an auxiliary polynomial determined by $\{\Psi_j(y)\}_{j=1}^{\infty}$ as the following lemma:

Lemma 3.6. *[4, Lemma 11]* (i) *For* $j = 0, 1, 2 ...,$ *there exists a unique polynomial* $\Psi_j(y)$ *of degree* j *such that*

$$\Psi_j(\nu) = \phi_j(\nu), \quad \nu = 1, 2, 3, \ldots .$$

(ii) $\Psi_0(y) = 1$ *and* $\Psi_j(0) = 0, j = 1, 2, ...$

Since $\Psi_j(y)$ is a polynomial of degree j, we can replace $\varphi_j(v)$ in (3.7) with $\Psi_j(y)$, that is,

$$\Psi_j(y) = \sum_{r=0}^{j}\frac{1}{2j-2r+1}\binom{2j}{2r}\Psi_r(y-1), \quad j = 0, 1, 2, ...,$$

for an arbitrary y and $j = 0, 1, 2, ...$ We use the notation $F_{kn}(x, y) = (l_{k,n}(x))^y$ which coincides with $l_{k,n}^y(x)$ if y is an integer. Since $l_{k,n}(x_{k,n}) = 1$, we have $F_{kn}(x, t) > 0$ for x in a neighborhood of $x_{k,n}$ and an arbitrary real number t.

We can show that $(\partial/\partial x)^j F_{kn}(x_{k,n}, y)$ is a polynomial of degree at most j with respect to y for $j = 0, 1, 2, ...,$ where $(\partial/\partial x)^j F_{kn}(x_{k,n}, y)$ is the jth partial derivative of $F_{kn}(x, y)$ with respect to x at $(x_{k,n}, y)$ [14, p. 199]. We prove these facts by induction on j. For $j = 0$ it is trivial. Suppose that it holds for $j \ge 0$. To simplify the notation, let $F(x) = F_{kn}(x, y)$ and $l(x) = l_{k,n}(x)$ for a fixed y. Then $F'(x)l(x) = yl'(x)F(x)$. By Leibniz's rule, we easily see that

$$F^{(j+1)}(x_{k,n}) = -\sum_{s=0}^{j-1}\binom{j}{s}F^{(s+1)}(x_{k,n})l^{(j-s)}(x_{k,n}) + y\sum_{s=0}^{j}\binom{j}{s}l^{(s+1)}(x_{k,n})F^{(j-s)}(x_{k,n}),$$

which shows that $F^{(j+1)}(x_{k,\ n})$ is a polynomial of degree at most $j + 1$ with respect to y. Let $P_{kn}^{[j]}(y)$, $j = 0, 1, 2, \ldots$ be defined by

$$(\partial/\partial x)^{2j}F_{kn}(x_{k,n}, y) = (-1)^j\beta^j(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2j}x_{k,n}^{-j}\Psi_j(y) + P_{kn}^{[j]}(y). \tag{3.12}$$

Then $P_{kn}^{[j]}(y)$ is a polynomial of degree at most $2j$.

By Theorem 1.8 (1.11), we have the following:

Lemma 3.7. *[4, Lemma 12] Let* $j = 0, 1, 2, \ldots$, *and* M *be a positive constant. Let* $0 < \varepsilon < 1/4$, $\frac{1}{\varepsilon}\frac{a_n}{n^2} \le x_{k,n} \le \varepsilon a_n$, *and* $|y| \le M$. *Then*

(a) *there exists* $\kappa_j\ (\varepsilon, x_{k,\ n}, n) > 0$ *such that*

$$\left|(\partial/\partial y)^s P_{kn}^{[j]}(y)\right| \le C\kappa_j(\varepsilon, x_{k,n}, n)\left(\frac{n}{\sqrt{a_n}}\right)^{2j}x_{k,n}^{-j}, \quad s = 0, 1 \tag{3.13}$$

and $\kappa_j\ (\varepsilon, x_{k,\ n}, n) \to 0$ *as* $n \to \infty$ *and* $\varepsilon \to 0$.

(b) *there exists* $\gamma_j\ (\varepsilon, n) > 0$ *such that*

$$\left|(\partial/\partial x)^{2j+1}F_{kn}(x_{k,n}, y)\right| \le C\gamma_j(\varepsilon, n)\left(\frac{n}{\sqrt{a_n}}\right)^{2j+1}x_{k,n}^{-\frac{2j+1}{2}} \tag{3.14}$$

and $\gamma_j\ (\varepsilon, n) \to 0$ *as* $n \to \infty$ *and* $\varepsilon \to 0$.

Lemma 3.8. *[4, Lemma 13] If* $y < 0$, *then for* $j = 0, 1, 2 \ldots$,

$$(-1)^j\Psi_j(y) > 0.$$

Lemma 3.9. *For positive integers* s *and* m *with* $1 \le m \le v$,

$$\sum_{r=0}^{s}\binom{2s}{2r}\Psi_r(-m)\varphi_{s-r}(m) = 0.$$

Proof. If we let $C_s(y) = \sum_{r=0}^{s}\binom{2s}{2r}\Psi_r(-y)\Psi_{s-r}(y)$, then it suffices to show that $C_s(m) = 0$. For every s,

$$\begin{aligned}
0 = (l_{k,n}^{-m+m})^{2s}(x_{k,n}) &= \sum_{i=0}^{2s}\binom{2s}{i}(l_{k,n}^{-m})^{(i)}(x_{k,n})(l_{k,n}^{m})^{(2s-i)}(x_{k,n}) \\
&= \sum_{r=0}^{s}\binom{2s}{2r}(\partial/\partial x)^{2r}F_{kn}(x_{k,n}, -m)(l_{k,n}^{m})^{(2s-2r)}(x_{k,n}) \\
&\quad + \sum_{r=0}^{s-1}\binom{2s}{2r+1}(\partial/\partial x)^{2r+1}F_{kn}(x_{k,n}, -m)(l_{k,n}^{m})^{(2s-2r-1)}(x_{k,n}).
\end{aligned}$$

By (1.11), (3.12) and (3.13), we see that the first sum $\sum_{r=0}^{s}$ has the form of

$$\sum_{r=0}^{s} = (-1)^s \beta^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s} x_{k,n}^{-s}\left(\sum_{r=0}^{s}\binom{2s}{2r}\Psi_r(-m)\phi_{s-r}(m) + \tilde{\eta}_s(-m,\varepsilon,x_{k,n},n)\right).$$

Then, since

$$\begin{aligned}\tilde{\eta}_s(-m,\varepsilon,x_{k,n},n) &= \sum_{r=0}^{s}\binom{2s}{2r}\Psi_r(-m)\phi_{s-r}(m)\xi_{s-r}(m,\varepsilon,x_{k,n},n)\\ &+ \sum_{r=0}^{s}\binom{2s}{2r}(-1)^{-r}\beta^{-r}(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{-2r} x_{k,n}^{r}\\ &\times\ \phi_{s-r}(m)P_{kn}^{[r]}(-m)(1+\xi_{s-r}(m,\varepsilon,x_{k,n},n)),\end{aligned}$$

we know that $|\tilde{\eta}_s(-m,\varepsilon,x_{k,n},n)| \to 0$ as $n\to\infty$ and $\varepsilon\to 0$ (see (3.12)). By (3.14) and (3.8), the second sum $\sum_{r=0}^{s-1}$ is bounded by $C\left(\frac{n}{\sqrt{a_n}}\right)^{2s+1} x_{k,n}^{-\frac{2s+1}{2}}\sum_{r=0}^{2s-1}\gamma_r(\varepsilon,n)$, and we know that $\sum_{r=0}^{2s-1}\gamma_r(\varepsilon,n)\to 0$ as $n\to\infty$ and $\varepsilon\to 0$. Therefore, we obtain the following result: for every s,

$$0 = \sum_{r=0}^{s}\binom{2s}{2r}\Psi_r(-m)\Psi_{s-r}(m).$$

Theorem 1.9 is important to show a divergence theorem with respect to L_n $(m, f; x)$, where m is an odd integer.

Proof of Theorem 1.9. We prove (1.12) by induction on s. Since $e_0(m, k, n) = 1$ and $\Psi_0(y) = 1$, (1.12) holds for $s = 0$. From (3.6) we write e_{2s} (m, k, n) in the form of

$$\begin{aligned}e_{2s}(m,k,n) &= -\sum_{r=0}^{s-1}\frac{1}{(2s-2r)!}e_{2r}(m,k,n)(l_{k,n}^{m})^{(2s-2r)}(x_{k,n})\\ &\quad -\sum_{r=1}^{s}\frac{1}{(2s-2r+1)!}e_{2r-1}(m,k,n)(l_{k,n}^{m})^{(2s-2r+1)}(x_{k,n})\\ &=: I + II.\end{aligned}$$

From (1.6), we know that for $x_{k,\ n} \le a_n/4$,

$$|\, e_j(l,m,k,n)\,| \le C\left(\frac{n}{\sqrt{a_n}}\right)^{j} x_{k,n}^{-\frac{j}{2}}. \tag{3.15}$$

Then, by (1.10) and (3.15), $|II|$ is bounded by $C\sum_{r=1}^{s}\delta(\varepsilon,n)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}x_{k,n}^{-s}$. For $0 \le i < s$, we suppose (1.12). Then, we have for I,

$$\begin{aligned}-\sum_{r=0}^{s-1} &= \frac{(-1)^{s+1}}{(2s)!}\beta^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}x_{k,n}^{-s}\\ &\times\sum_{r=0}^{s-1}\binom{2s}{2r}\Psi_r(-m)\phi_{s-r}(m)(1+\eta_r)(1+\xi_{s-r}),\end{aligned}$$

where $\xi_{s-r} := \xi_{s-r}(m, \varepsilon, x_{k,n}, n)$ and $\eta_r := \eta_r(m, \varepsilon, x_{k,n}, n)$ which are defined in (1.11) and (1.12). Then, using Lemma 3.9 and $\varphi_0(m) = 1$, we have the following form:

$$e_{2s}(m,k,n) = \frac{(-1)^s}{(2s)!}\Psi_s(-m)\beta^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s} x_{k,n}^{-s}(1+\eta_s(m,\varepsilon,x_{k,n},n)).$$

Here, since

$$\eta_s(m,\varepsilon,x_{k,n},n) = \sum_{r=0}^{s-1}\binom{2s}{2r}\Psi_r(-m)\phi_{s-r}(m)(\eta_r+\xi_{s-r}+\eta_r\xi_{s-r})$$
$$+\ (-1)^s\beta^{-s}(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{-2s} x_{k,n}^{s}\frac{(2s)!}{\Psi_s(-m)}II,$$

we see that $|\eta_s(m, \varepsilon, x_{k,n}, n)| \to 0$ as $n \to \infty$ and $\varepsilon \to 0$ (recall above estimation of $|II|$). Therefore, we proved the result.

Lemma 3.10. *[5, Theorem 1.3] Let* $\rho > -\frac{1}{2}$*and* $w(x)$ (C^2+). *There exists* n_0 *such that uniformly for* $n \geq n_0$, *we have the following:*

(a) *For* $1 \leq j \leq n$,

$$|\, p'_{n,\rho}(x_{j,n})\,|\, w_\rho(x_{j,n})\tilde{\varphi}_n(x_{j,n})^{-1}[x_{j,n}(a_n - x_{j,n})]^{-1/4}. \tag{3.16}$$

(b) *For* $j \leq n - 1$ *and* $x \in [x_{j+1,n}, x_{j,n}]$,

$$|\, p_{n,\rho}(x)\,|\, w(x)\left(x+\frac{a_n}{n^2}\right)^\rho$$
$$\sim \min\{|\, x - x_{j,n}\,|, |\, x - x_{j+1,n}\,|\}\varphi_n(x_{j,n})^{-1}[x_{j,n}(a_n - x_{j,n})]^{-1/4}. \tag{3.17}$$

(c) *For* $0 < a \leq x_{k,n} \leq b < \infty$,

$$|\, p'_{n,\rho}(x_{k,n})\,|\, w_\rho(x_{k,n}) \sim \frac{n}{a_n^{3/4}}. \tag{3.18}$$

(d) *For* $0 < a \leq x_{k+1,n}, x_{k,n} \leq b < \infty$ *and* $x \in [(x_{k+1,n} + x_{k,n})/2\ x_{k,n} + x_{k-1,n})/2]$,

$$|\, p_{n,\rho}(x)\,|\, w(x)\left(x+\frac{a_n}{n^2}\right)^\rho \sim \frac{1}{a_n^{1/4}}. \tag{3.19}$$

Moreover, for $0 < a \leq x \leq b < \infty$, *there exists a constant* $C > 0$ *such that*

$$|\, p_{n,\rho}(x)\,|\, w(x)\left(x+\frac{a_n}{n^2}\right)^\rho \leq C\frac{1}{a_n^{1/4}}. \tag{3.20}$$

(e) *Uniformly for* $n \geq 1$ *and* $1 \leq j < n$,

$$x_{j,n} - x_{j+1,n} \sim \varphi_n(x_{j,n}). \tag{3.21}$$

(f) *Let* Λ *be defined in Definition 1.2 (d). There exists* $C > 0$ *such that for* $n \geq 1$,

$$a_n \leq Cn^{1/\Lambda}.$$

Proof. (a) and (b) follow from [5, Theorem 1.3]. (e) follows from [5, Theorem 1.4]. We need to prove (c), (d), and (f).

(c) For $0 < a \leq x_{k,n} \leq b < \infty$, we have (2.11);

$$\varphi_n(x_{k,n}) \sim \frac{\sqrt{a_n}}{n},$$

so applying (a), we have the result.

(d) Let $0 < a \leq x_{k+1,n} < x_{k,n} \leq b < \infty$. We take a constant $\delta > 0$ as

$$\min\{|x - x_{k,n}|, |x - x_{k+1,n}|\} = \delta\frac{\sqrt{a_n}}{n}.$$

Then, by (b) we have

$$|p_{n,\rho}(x)|w(x)\left(x + \frac{a_n}{n^2}\right)^{\rho} \sim \delta\frac{\sqrt{a_n}}{n}\frac{n}{a_n^{3/4}} = \delta\frac{1}{a_n^{1/4}}.$$

Moreover, by [6, Theorem 1.2] the second inequality holds.

(f) We see

$$\frac{R'(x)}{R(x)} = \frac{T(x)}{x} \geq \frac{\Lambda}{x},$$

so that by an integration, $R(x) \geq R(1)\, x^{\Lambda}$ for $x \geq 1$, and hence we have

$$R'(x) \geq \Lambda R(1)x^{\Lambda-1} \quad (x \geq 1).$$

Since $\lim_{n\to\infty} a_n = \infty$, we can choose n_0 such that $a_n \geq 2$ for all $n \geq n_0$. Then for some $C_1 > 0$,

$$\begin{aligned} n &= \frac{2}{\pi}\int_0^1 \frac{a_n t R'(a_n t)}{\sqrt{t(1-t)}}\mathrm{d}t \geq \frac{2}{\pi}\int_0^1 \frac{a_n t \Lambda R(1)(a_n t)^{\Lambda-1}}{\sqrt{t(1-t)}}\mathrm{d}t \\ &\geq a_n^{\Lambda}\frac{2\Lambda R(1)}{\pi}\int_{1/2}^1 \frac{t^{\Lambda}}{\sqrt{t(1-t)}}\mathrm{d}t =: C_1^{\Lambda}a_n^{\Lambda}. \end{aligned}$$

Hence, we have the result.

Lemma 3.11. *Let the function* $h_{kn}(m; x)$ *be defined by (1.4) and let* $0 < c < a < b < d < \infty$. *Then we have*

$$\max_{a\leq x\leq b}\sum_{c\leq x_{k,n}\leq d}\left|l_{k,n}^m(x)\sum_{i=0}^{m-2}e_i(m,k,n)(x - x_{k,n})^i\right| \leq C.$$

Proof. Let $c \leq x_{k+1,n} < x_{k,n} \leq d$. Then by (3.21), we see $|x_{k,n} - x_{k+1,n}| \sim \sqrt{a_n}/n$.

Now, choose $\alpha, \beta > 0$ satisfying for all $x_{k+1,n}, x_{k,n} \in [c, d]$,

$$\alpha \frac{\sqrt{a_n}}{n} \leq \left|x_{k,n} - x_{k+1,n}\right| \leq \beta \frac{\sqrt{a_n}}{n}. \tag{3.22}$$

Let $x \in [a, b]$ and $|x - {}_{xj(x),n}| = \min\{|x - x_{k,n}\}|; x_{k,n} \in [a, b]\}$, ${}_{xj(c)+1,n} < c \leq {}_{xj(c),n}$, and ${}_{xj(d),n} \leq d <_{xj(d)-1,n}$. Moreover, we take a non-negative integer j_k satisfying for each $x_{k,n} \in [a, b]$ and $k \neq j(x)$,

$$\left(j_k + \frac{1}{2}\right) \alpha \frac{\sqrt{a_n}}{n} \leq \left|x - x_{k,n}\right| \leq (j_k + 1)\beta \frac{\sqrt{a_n}}{n}. \tag{3.23}$$

Then we have

$$\begin{aligned}
&\max_{a \leq x \leq b} \sum_{c \leq x_{k,n} \leq d} \left| l_{k,n}^m(x) \sum_{i=0}^{m-2} e_i(m,k,n)(x - x_{k,n})^i \right| \\
&\leq \max_{a \leq x \leq b} \sum_{i=0}^{m-2} \sum_{c \leq x_{k,n} \leq d} \left| \left(\frac{p_n(w_\rho^2; x)}{(x - x_{k,n}) p_n'(w_\rho^2; x_{k,n})} \right)^m e_i(m,k,n)(x - x_{k,n})^i \right| \\
&\leq \max_{a \leq x \leq b} \sum_{i=0}^{m-2} \left[\left| \frac{p_n(w_\rho^2; x)}{(x - x_{j(x),n}) p_n'(w_\rho^2; x_{j(x),n})} \right|^m \mid e_i(m,k,n)(x - x_{j(x),n})^i \mid \right. \\
&\quad \left. + \sum_{\substack{c \leq x_{k,n} \leq d \\ x_{k,n} \neq x_{j(x),n}}} \left| \left(\frac{p_n(w_\rho^2; x)}{(x - x_{k,n}) p_n'(w_\rho^2; x_{k,n})} \right)^m e_i(m,k,n)(x - x_{k,n})^i \right| \right].
\end{aligned}$$

Here, by (3.16) and (3.17) we see

$$\left| \frac{p_n(w_\rho^2; x)}{(x - x_{j(x),n}) p_n'(w_\rho^2; x_{j(x),n})} \right|^m \leq C$$

and from (3.20), we have $|{}_{xj(x),n}| \leq b + 1$, and so by (1.6) we have

$$\mid e_i(m, j(x), n)(x - x_{j(x),n})^i \mid \leq C \left(\frac{n}{\sqrt{a_{2n} - x_{j(x),n}}} \right)^i x_{j(x),n}^{-\frac{i}{2}} \left(\frac{\sqrt{a_n}}{n} \right)^i \leq C.$$

Consequently, we have

$$\left| \frac{p_n(w_\rho^2; x)}{(x - x_{j(x),n}) p_n'(w_\rho^2; x_{j(x),n})} \right|^m \mid e_i(m, j(x), n)(x - x_{j(x)n})^i \mid \leq C.$$

Similarly, for $c \leq x_{k,n} \leq d$ with $x_{k,n} \neq {}_{xj(x),n}$, we have by (3.18) and (3.20),

$$\left| \frac{p_n(w_\rho^2; x)}{p_n'(w_\rho^2; x_{k,n})} \right|^m \leq \left(\frac{\sqrt{a_n}}{n} \right)^m$$

and by (1.6) and (3.23),

$$\mid e_i(m,k,n)(x - x_{k,n})^{i-m} \mid \leq C \left(\frac{n}{\sqrt{a_n}} \right)^i \left(\frac{n}{\sqrt{a_n}} \frac{1}{(j_k + 1/2)\alpha} \right)^{m-i}.$$

Therefore, we have for $0 \le i \le m - 2$,

$$\sum_{\substack{c\le x_{k,n}\le d\\ x_{k,n}\ne x_{j(x),n}}} \left| \left(\frac{p_n(w_\rho^2; x)}{(x - x_{k,n}) p_n'(w_\rho^2; x_{k,n})} \right)^m e_i(m,k,n)(x - x_{k,n})^i \right|$$
$$\le C \sum_{\substack{c\le x_{k,n}\le d\\ x_{k,n}\ne x_{j(x),n}}} \left(\frac{1}{(j_k + 1/2)\alpha} \right)^2 \le C.$$

Therefore,

$$\max_{a\le x\le b} \sum_{c\le x_{k,n}\le d} \left| l_{k,n}^m(x) \sum_{i=0}^{m-2} e_i(m,k,n)(x - x_{k,n})^i \right| \le C.$$

Proof of Theorem 1.10. We use Theorem 1.9 and Lemma 3.11. We find a lower bound for the Lebesgue constants $\lambda_n(m, [a, b]) = \max_{a\le x\le b} \sum_{k=1}^n |h_{kn}(m; x)|$ with a positive odd order m and a given interval $[a, b]$, $0 < a < b < \infty$. By the expression (1.4) we have

$$|h_{kn}(m; x)| \ge | l_{k,n}^m(x) e_{m-1}(m,k,n)(x - x_{k,n})^{m-1} |$$
$$- \left| l_{k,n}^m(x) \sum_{i=0}^{m-2} e_i(m,k,n)(x - x_{k,n})^i \right|.$$

Let $c = a/2$, $d = b + (b - a)$ and

$$\lambda_n(m, [a,b]) \ge \max_{a\le x\le b} \sum_{c\le x_{k,n}\le d} \left| l_{k,n}^m(x) e_{m-1}(m,k,n)(x - x_{k,n})^{m-1} \right|$$
$$- \max_{a\le x\le b} \sum_{c\le x_{k,n}\le d} \left| l_{k,n}^m(x) \sum_{i=0}^{m-2} e_i(m,k,n)(x - x_{k,n})^i \right|$$
$$= \max_{a\le x\le b} F_n(x) - \max_{a\le x\le b} G_n(x).$$

It follows from Lemma 3.11 that $\max_{a \le x \le b} G_n(x) \le C$ with C independent of n. Therefore, it is enough to show that $\max_{a \le x \le b} F_n(x) \ge C \log(1 + n)$. We consider α, β and j_k defined in (3.22) and (3.23). Let $K(x; [c, d])$ be the set of numbers defined as

$$K(x;[c,d]) = \left\{ j_k;\ (j_k + 1/2)\alpha \frac{\sqrt{a_n}}{n} \le |x - x_{k,n}| \le (j_k + 1)\beta \frac{\sqrt{a_n}}{n}, x_{k,n} \in [c,d], k \ne j(x) \right\},$$

where j_k is a non-negative integer. Then, there exist $\gamma > 0$ and $C > 0$ such that

$$Cn^\gamma \le max\{j_k \in K(x;[c,d])\},$$

that is, we see

$$\{0, 1, 2, \ldots, [Cn^\gamma]\} \subset K(x;[c,d]). \tag{3.24}$$

In fact, from Lemma 3.10 (f), we see $a_n \le c_1 n^{1/\Lambda}$, $\Lambda > 1/2$. By (3.22) and (3.23), we see $0 \in K(x; [c, d])$ and

$$(\max\{j_k \in K(x;[c,d])\} + 1)\beta \frac{\sqrt{a_n}}{n} \ge b - a.$$

Hence, we have

$$\max\{j_k \in K(x;[c,d])\} + 1 \geq \frac{b-a}{\beta}\frac{n}{\sqrt{a_n}} \geq \frac{b-a}{\beta}\frac{n}{\sqrt{a_n}}\frac{1}{\sqrt{c_1}}n^{1-1/(2\Lambda)} =: Cn^{\gamma}.$$

So, we have (3.24). Now, we take an interval $[x_{l+1,n}, x_{l,\ n}] \subset [a, b]$, and we put

$$x^* := (x_{\ell+1,n} + x_{\ell,n})/2.$$

By Lemma 3.10 (d), we have

$$\left|(p_n w_\rho)(x^*)\right| \sim \frac{1}{a_n^{1/4}}.$$

From Lemma 3.10 (e), we know for $c \leq x_{k,\ n} \leq d$

$$\left|x^* - x_{k,n}\right| \geq \frac{\alpha}{2}\frac{\sqrt{a_n}}{n}. \tag{3.25}$$

By Lemma 3.10 (c), we have for $c \leq x_{k,\ n} \leq d$

$$\left|p_n'(x_{k,n})\right| w_\rho(x_{k,n}) \sim \frac{n}{a_n^{3/4}}.$$

Then, we have

$$\begin{aligned} |\ l_{k,n}(x^*)\ | &= \frac{|\ p_n(x^*)\ |\ w(x^*)\left(x^* + \frac{a_n}{n^2}\right)^\rho}{|\ (x^* - x_{k,n})p_n'(x_{k,n})\ |\ w_\rho(x_{k,n})}\frac{w_\rho(x_{k,n})}{w(x^*)(x^* + \frac{a_n}{n^2})^\rho} \\ &\geq C\frac{\sqrt{a_n}}{n}\frac{1}{\left|x^* - x_{k,n}\right|}. \end{aligned}$$

Here, we used the following facts:

$$\left(x^* + \frac{a_n}{n}\right)^\rho \sim x^{*\rho}$$

and

$$\frac{w_\rho(x_{k,n})}{w_\rho(x^*)} \sim 1, \quad x^*, x_{k,n} \in [a,b].$$

Now we use Theorem 1.9, that is, for $c \leq x_{k,\ n} \leq d$ we have

$$e_{m-1}(m,k,n) \sim \left(\frac{n}{\sqrt{a_n}}\right)^{m-1}.$$

Therefore, with (3.25) and (3.26), we have

$$\begin{aligned} F_n\left(x^*\right) &\geq \sum_{j_k \in K(x^*;[c,d])} \left|l_{k,n}^m\left(x^*\right) e_{m-1}\left(m,k,n\right)\left(x^* - x_{k,n}\right)^{m-1}\right| \\ &\geq C \sum_{j_k \in K(x^*;[c,d])} \left(\frac{\sqrt{a_n}}{n}\frac{1}{\left|x^* - x_{k,n}\right|}\right)^m \left(\frac{n}{\sqrt{a_n}}\right)^{m-1} \left|x^* - x_{k,n}\right|^{m-1} \\ &= C \sum_{j_k \in K(x^*;[c,d])} \left(\frac{\sqrt{a_n}}{n}\right)^m \frac{1}{\left|x^* - x_{k,n}\right|}\left(\frac{n}{\sqrt{a_n}}\right)^{m-1} \\ &\geq C \sum_{j_k \in K(x^*;[c,d])} \left(\frac{\sqrt{a_n}}{n}\right)^m \frac{1}{(j_k+1)\beta\left(\sqrt{a_n}/n\right)}\left(\frac{n}{\sqrt{a_n}}\right)^{m-1} \\ &\geq C\left(\beta\right) \sum_{j_k \in K(x^*;[c,d])} \frac{1}{j_k+1} \geq C \sum_{0 \leq j \leq n^\gamma} \frac{1}{j+1} \geq C\ \log\ n. \end{aligned}$$

Consequently, the theorem is complete. □

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Author details
[1]Department of Mathematics Education, Sungkyunkwan University Seoul 110-745, Republic of Korea [2]Department of Mathematics, Meijo University Nagoya 468-8502, Japan

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment. All authors read and approved the final manuscript.

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