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A note on the Frobenius conditional number with positive definite matrices

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Abstract

In this article, we focus on the lower bounds of the Frobenius condition number. Using the generalized Schwarz inequality, we present some lower bounds for the Frobenius condition number of a positive definite matrix depending on its trace, determinant, and Frobenius norm. Also, we give some results on a kind of matrices with special structure, the positive definite matrices with centrosymmetric structure.

1 Introduction and preliminaries

In this article, we use the following notations. Let $\mathbb{C}^{n\times n}$ and $\mathbb{R}^{n\times n}$ be the space of $n\times n$ complex and real matrices, respectively. The identity matrix in $\mathbb{C}^{n\times n}$ is denoted by $I = I_n$. Let A^T, $\bar{A}$, A^H, and tr(A) denote the transpose, the conjugate, the conjugate transpose, and the trace of a matrix A, respectively. Re(a) stands for the real part of a number a. The Frobenius inner product $< \cdot, \cdot >_F$ in $\mathbb{C}^{m\times n}$ is defined as $<A,B >_F = \mathrm{Re}(\mathrm{tr}(B^H A))$, for $A,B \in \mathbb{C}^{m\times n}$, i.e., $<A,B>$ is the real part of the trace of $B^H A$. The induced matrix norm is $||A||_F = \sqrt{< A, A>_F} = \sqrt{\mathrm{Re}\ (\mathrm{tr}(A^H A))} = \sqrt{\mathrm{tr}(A^H A)}$, which is called the Frobenius (Euclidean) norm. The Frobenius inner product allows us to define the consine of the angle between two given real $n \times n$ matrices as

$$\cos(\mathrm{A, B}) = \frac{< A, B>_F}{||A||_F ||B||_F}. \tag{1.1}$$

The cosine of the angle between two real $n \times n$ depends on the Frobenius inner product and the Frobenius norms of given matrices. A matrix $A \in \mathbb{C}^{n\times n}$ is Hermitian if $A^H = A$. An Hermitian matrix A is said to be positive semidefinite or nonnegative definite, written as $A \geq 0$, if (see, e.g., [[1], p. 159])

$$x^H Ax \geq 0, \quad \forall x \in \mathbb{C}^n, \tag{1.2}$$

A is further called positive definite, symbolized $A > 0$, if the strict inequality in (1.2) holds for all nonzero $x \in \mathbb{C}^n$. An equivalent condition for $A \in \mathbb{C}^n$ to be positive definite is that A is Hermitian and all eigenvalues of A are positive real numbers.

The quanity

$$\mu(A) = \begin{cases} ||A|| \cdot ||A^{-1}||, & \text{if } A \text{ is nonsingular;} \\ \infty & \text{if } A \text{ is singular.} \end{cases} \tag{1.3}$$

is called the condition number of matrix $\mu(A)$ with respect to the matrix norm $|| \cdot ||$. Notice that $\mu(A) = ||A^{-1}|| \cdot ||A|| \geq ||A^{-1}A|| = ||I|| \geq 1$ for any matrix norm (see, e.g., [[2], p. 336]). The condition number $\mu(A)$ of a nonsingular matrix A plays an important role in the numerical solution of linear systems since it measures the sensitivity of the solution of linear system $Ax = b$ to the perturbations on A and b. There are several methods that allow to find good approximations of the condition number of a general square matrix.

We first introduce some inequalities. Buzano [3] obtained the following extension of the celebrated Schwarz inequality in a real or complex inner product space $(H, < \cdot, \cdot >)$.

Lemma 1.1 ([3]). *For any* $a, b, x \in H$, there is

$$| < a, x >< x, b > | \leq \frac{1}{2}(||a|| \cdot ||b|| + | < a.b > |)||x||^2. \tag{1.4}$$

It is clear that for $a = b$, (1.4) becomes the standard Schwarz inequality

$$| < a, x >^2 | \leq ||a||^2 ||x||^2, \quad a, x \in H \tag{1.5}$$

with equality if and only if there exists a scalar λ such that $x = \lambda a$.

Also Dragomir [4] has stated the following inequality.

Lemma 1.2 [4]. *For any* $a,b,x \in H$, *and* $x \neq 0$, *there is the following*

$$\left| \frac{< a, x >< x, b >}{||x||^2} - \frac{< a, b >}{2} \right| \leq \frac{||a|| \cdot ||b||}{2} \tag{1.6}$$

Dannan [5] showed the following inequality by using arithmetic-geometric inequality.

Lemma 1.3 [5]. *For n-square positive definite matrices A and B,*

$$n(\mathrm{detA} \cdot \det \mathrm{B})^{m/n} \leq \mathrm{tr}(\mathrm{A^m B^m}), \tag{1.7}$$

where m is a positive integer.

By taking $A = I$, $B = A^{-1}$, and $m = 1$ in (1.7), we obtain

$$n(\mathrm{detI} \cdot \det \mathrm{A}^{-1})^{1/\mathrm{n}} \leq \mathrm{tr}(\mathrm{I} \cdot \mathrm{A}^{-1}), \tag{1.8}$$

$$n\left(\frac{1}{\mathrm{detA}}\right)^{1/n} \leq \mathrm{tr}(\mathrm{A}^{-1}) \tag{1.9}$$

In [6], Türkmen and Ulukök proposed the following,

Lemma 1.4 [6]. *Let both A and B be n-square positive definite matrices, then*

$$\cos(A, I)\cos(B, I) \leq \frac{1}{2}\cos(A, B) + 1], \tag{1.10}$$

$$\cos(A, A^{-1}) \leq \cos(A, I)\cos(A^{-1}, I) \leq \frac{1}{2}[\cos(A, A^{-1}) + 1] \leq 1, \tag{1.11}$$

$$\cos(A, I) \leq 1, \cos(A^{-1}, I) \leq 1, \tag{1.12}$$

As a consequence, in the following section, we give some bounds for the Frobenius condition numbers by considering inequalities given in this section.

2 Main results

Theorem 2.1. *Let A be a positive definite real matrix, α be any real number. Then,*

$$2n\frac{\mathrm{trA}^{1+\alpha}}{\mathrm{trA}^{2\alpha}(\mathrm{detA})^{(1-\alpha)/n}} - n \le \mu_F(A); \tag{2.1}$$

$$2\frac{\mathrm{trA}^{1+\alpha}\mathrm{trA}^{\alpha-1}}{\mathrm{trA}^{2\alpha}} - n \le \mu_F(A). \tag{2.2}$$

where $\mu_F(A)$ is the Frobenius conditional number of A.

Proof. Let X, A, B be positive definite real matrices. From Lemma 1.2, we have the following

$$\left|\frac{< A, X>_F\ < X, B>_F}{||X||_F^2}\frac{< A, B>_F}{2}\right| \le \frac{||A||_F||B||_F}{2}, \tag{2.3}$$

i.e.,

$$\left|\frac{\mathrm{tr}(\mathrm{A^TX})\mathrm{tr}(\mathrm{X^TB})}{||X||_F^2} - \frac{\mathrm{tr}(\mathrm{A^TB})}{2}\right| \le \frac{||A||_F||B||_F}{2}. \tag{2.4}$$

Let $B = A^{-1}$, then (2.4) turns into

$$\left|\frac{\mathrm{tr}(\mathrm{A^TX})\mathrm{tr}(\mathrm{X^TA^{-1}})}{||X||_F^2} - \frac{\mathrm{tr}(\mathrm{A^TA^{-1}})}{2}\right| \le \frac{||A||_F||A^{-1}||_F}{2}. \tag{2.5}$$

Since both X and A are positive definite, we have

$$\left|\frac{\mathrm{tr}(\mathrm{AX})\mathrm{tr}(\mathrm{XA^{-1}})}{||X||_F^2} - \frac{n}{2}\right| \le \frac{||A||_F||A^{-1}||_F}{2} = \frac{\mu_F(A)}{2}, \tag{2.6}$$

where $\mu_F(A)$ is the Frobenius condition number of A.

By taking $X = A^\alpha$ (α is an arbitrary real number) into (2.6), there exists

$$\left|\frac{\mathrm{trA}^{1+\alpha}\mathrm{trA}^{-(1-\alpha)}}{\mathrm{trA}^{2\alpha}} - \frac{n}{2}\right| \le \frac{\mu_F(A)}{2}. \tag{2.7}$$

Thus, it follows that

$$\frac{\mathrm{trA}^{1+\alpha}\mathrm{trA}^{-(1-\alpha)}}{\mathrm{trA}^{2\alpha}} - \frac{n}{2} \le \frac{\mu_F(A)}{2}, \tag{2.8}$$

i.e.,

$$2\frac{\mathrm{trA}^{1+\alpha}\mathrm{trA}^{\alpha-1}}{\mathrm{trA}^{2\alpha}} - \frac{n}{\le \mu_F(A)}. \tag{2.9}$$

From (1.9), by replacing A with $A^{1-\alpha}$, we get

$$0 < n \cdot \frac{1}{(\det A)^{(1-\alpha)/n}} \le \mathrm{tr}(\mathrm{A}^{-(1-\alpha)}). \tag{2.10}$$

Taking (2.10) into (2.8), we can write

$$n \cdot \frac{\mathrm{trA}^{1+\alpha}}{\mathrm{trA}^{2\alpha}} \cdot \frac{1}{(\det A)^{(1-\alpha)/n}} - \frac{n}{2} \leq \frac{\mu_F(A)}{2}, \tag{2.11}$$

i.e.,

$$2n \cdot \frac{\mathrm{trA}^{1+\alpha}}{\mathrm{trA}^{2\alpha}(\mathrm{detA})^{(1-\alpha)/n}} - n \leq \mu_F(A). \tag{2.12}$$

In particular, let $\alpha = 1$, and by taking it into (2.7), we have the following

$$\left| \frac{\mathrm{trA}^2 \mathrm{trI}}{\mathrm{trA}^2} - \frac{n}{2} \right| \leq \frac{\mu_F(A)}{2}, \tag{2.13}$$

i.e.,

$$n \leq \mu_F(A). \tag{2.14}$$

Note, when $\alpha = \frac{1}{2}$, (2.12) becomes

$$2n \cdot \frac{\mathrm{trA}^{3/2}}{trA \cdot (\det A)^{1/2n}} - n \leq \mu_F(A), \tag{2.15}$$

Taking $\alpha = 1$ into (2.12), we obtain that

$$2\frac{\mathrm{tr}A}{(\det A)^{1/n}} - n \leq \mu_F(A). \tag{2.16}$$

(2.15), (2.16) can be found in [6].

Example 2.2.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2^{-1} \end{bmatrix}.$$

Here $\mathrm{tr}A = 2.5$, $\det A = 1$ and $n = 2$. Then, from (2.15) and (2.16), we obtain two lower bounds of $\mu_F(A)$:

$$\mu_F(A) \geq 2n \cdot \frac{\mathrm{trA}^{3/2}}{\mathrm{trA} \cdot (\mathrm{detA})^{1/2\mathrm{n}}} = 3.091168, \text{and } \mu_F(A) \geq 2 \cdot \frac{\mathrm{tr}A}{(\det A)^{1/n}} = 3.$$

Taking $\alpha = 1/4$ into (2.1) and (2.2) from Theorem 2.1, another two lower bounds are obtained as follows:

$$\mu_F(A) \geq 2n\frac{\mathrm{trA}^{1+\alpha}}{\mathrm{trA}^{2\alpha}(\det A)^{(1-\alpha)/n}} - n = 3.277585, \text{ and } \mu_F(A) \geq 2\frac{\mathrm{trA}^{1+\alpha}\mathrm{trA}^{\alpha-1}}{\mathrm{trA}^{2\alpha}} - n = 4.006938.$$

In fact, $\mu_F(A) = 4.25$. Thus, Theorem 2.1 is indeed a generalization of (2.15) and (2.16) given in [6].

Lemma 2.3. *Let $a_1, a_2, \ldots, a_n$ be positive numbers, and*

$$f(x) = a_1^x + a_2^x + \cdots + a_n^x + a_1^{-x} + \cdots + a_n^{-x}.$$

Then, $f(x)$ is monotonously increasing for $x \in [0, +\infty)$.

Proof. It is obvious that

$$\begin{aligned} f'(x) &= a_1^x \ln a_1 + a_2^x \ln a_2 + \cdots + a_n^x \ln a_n - a_1^{-x} \ln a_1 - \cdots - a_n^x \ln a_n \\ &= \ln a_1(a_1^x - a_1^{-x}) + \ln a_2(a_2^x - a_1^{-x}) + \ln a_n(a_n^x - a_n^{-x}). \end{aligned}$$

When $a_i \geq 1$, and $x \in [0, +\infty)$, $\ln a_i \geq 0, a_i^x \geq 1$, and $a_i^{-x} \leq 1$. Thus, $\ln a_i(a_i^x - a_i^{-x}) \geq 0$.

When $0 < a_i < 1$, and $x \in [0, +\infty)$, we have $\ln a_i < 0, 0 < a_i^x \leq 1$, and $a_i^{-x} \geq 1$. Thus, $\ln a_i(a_i^x - a_i^{-x}) \geq 0$. Therefore, $f'(x) \geq 0$, and $f(x)$ is increasing for $x \in [0, \infty)$.

Theorem 2.4. *Let A be a positive definite real matrix. Then*

$$\frac{n\sqrt{n} \cdot \left\| A^{1/2} \right\|_F}{\mathrm{tr} A^{1/2}} \leq \mu_F(A). \tag{2.17}$$

Proof. Since A is positive definite, from Lemma 1.4, we have

$$\cos(A^{1/2}, A^{-1/2} \leq \cos(A^{1/2}, I). \tag{2.18}$$

That is

$$\frac{n}{\mu_F(A^{1/2})} = \frac{\mathrm{tr}(A^{1/2}A^{-1/2})}{||A^{1/2}||_F \cdot ||(A^{1/2})^{-1}||_F} \leq \frac{\mathrm{tr}A^{1/2}}{\sqrt{n} \cdot ||A^{1/2}||_F}. \tag{2.19}$$

Let all the positive real eigenvalues of A be $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$. Then, the eigenvalues of A^{-1} are $1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_n > 0$, the eigenvalues of A^2 are $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2 > 0$, and the eigenvalue of A^{-2} are $\lambda_1^{-2}, \lambda_2^{-2}, \ldots, \lambda_n^{-2} > 0$. Next, we will prove the following by induction.

$$\left(\sum_{i=1}^{n} \lambda_i^2\right)\left(\sum_{i=1}^{n} 1/\lambda_i^2\right) \geq \left(\sum_{i=1}^{n} \lambda_i\right)\left(\sum_{i=1}^{n} 1/\lambda_i\right). \tag{2.20}$$

In case $n = 1$, it is obvious that (2.20) holds.

In case $n = 2$, $\left(\sum_{i=1}^n \lambda_i^2\right)\left(\sum_{i=1}^n 1/\lambda_i^2\right) = 2 + (\lambda_1/\lambda_2)^2 + (\lambda_2/\lambda_1)^2$. From Lemma 2.3, $2 + (\lambda_1/\lambda_2)^2 + (\lambda_2/\lambda_1)^2 \geq 2 + \lambda_1/\lambda_2 + \lambda_2/\lambda_1 = (\lambda_1 + \lambda_2)(1/\lambda_1 + 1/\lambda_2)$. Thus, (2.20) holds. Suppose that (2.20) holds, when $n = k$, i.e.,

$$\left(\sum_{i=1}^{k} \lambda_i^2\right)\left(\sum_{i=1}^{k} 1/\lambda_i^2\right) \geq \left(\sum_{i=1}^{k} \lambda_i\right)\left(\sum_{i=1}^{k} 1/\lambda_i\right). \tag{2.21}$$

In case $n = k + 1$,

$$\begin{aligned} \left(\sum_{i=1}^{k+1} \lambda_i^2\right)\left(\sum_{i=1}^{k+1} 1/\lambda_i^2\right) &= \left(\sum_{i=1}^{k} \lambda_i^2 + \lambda_{k+1}^2\right)\left(\sum_{i=1}^{k} 1/\lambda_i^2 + 1/\lambda_{k+1}^2\right) \\ &= \left(\sum_{i=1}^{k} \lambda_i^2\right)\left(\sum_{i=1}^{k} 1/\lambda_i^2\right) + \sum_{i=1}^{k} (\lambda_i/\lambda_{k+1})^2 + \sum_{i=1}^{k} (\lambda_{k+1}/\lambda_i)^2 + 1. \end{aligned} \tag{2.22}$$

By Lemma 2.3, we get

$$\begin{aligned}\left(\sum_{i=1}^{k+1}\lambda_i^2\right)\left(\sum_{i=1}^{k+1}1/\lambda_i^2\right) &\geq \left(\sum_{i=1}^{k}\lambda_i\right)\left(\sum_{i=1}^{k}1/\lambda_i\right)+\sum_{i=1}^{k}(\lambda_i/\lambda_{k+1})+\sum_{i=1}^{k}(\lambda_{k+1}/\lambda_i)+1\\ &= \left(\sum_{i=1}^{k+1}\lambda_i\right)\left(\sum_{i=1}^{k+1}1/\lambda_i\right).\end{aligned} \tag{2.23}$$

Thus, when $n = k + 1$, (2.20) holds. On the other hand, $||A||_F = \sqrt{\mathrm{tr}(\mathrm{A}^2)} = \sqrt{\sum_{i=1}^n \lambda_i^2}$, $||A^{-1}||_F = \sqrt{\mathrm{trA}^{-2}} = \sqrt{\sum_{i=1}^n \lambda_i^{-2}}$, $||A^{1/2}||_F = \sqrt{\mathrm{trA}} = \sqrt{\sum_{i=1}^n \lambda_i}$ and $||A^{-1/2}||_F = \sqrt{\mathrm{trA}^{-1}} = \sqrt{\sum_{i=1}^n \lambda_i^{-1}}$. Therefore,

$$||A||_F^2||A^{-1}||_F^2 \geq ||A^{1/2}||_F^2||A^{-1/2}||_F^2, \tag{35}$$

i.e.,

$$\mu_F(A) \geq \mu_F(A^{1/2}). \tag{2.25}$$

Taking (2.25) into (2.19), we obtain

$$\frac{n}{\mu_F(A)} \leq \frac{n}{\mu_F(A^{1/2})} \leq \frac{\mathrm{trA}^{1/2}}{\sqrt{n}||A^{1/2}||_F}, \tag{2.26}$$

i.e.,

$$\frac{n\sqrt{n}||A^{1/2}||_F}{\mathrm{trA}^{1/2}} \leq \mu_F(A). \tag{2.27}$$

Remark 2.5. (2.17) can be extended to any $0 < \alpha \leq 1$, i.e., $\frac{n\sqrt{n}||A^{\alpha}||_F}{\mathrm{trA}^{\alpha}} \leq \mu_F(A)$.

3 The Frobenius condition number of a centrosymmetric positive definite matrix

Definition 3.1 (see [7]). *Let* $A = (a_{ij})_{pxq} \in \mathbb{R}^{p\times q}$. *A is a centrosymmetric matrix, if*

$$a_{ij} = a_{p-i+1,q-j+1},\ 1 \leq i \leq p,\ 1 \leq j \leq q,\ \text{ or } J_pAJ_q = A,$$

where $J_n = (e_n, e_{n-1},..., e_1)$, e_i *denotes the unit vector with the i-th entry 1.*

Using the partition of matrix, the central symmetric character of a square centrosymmetric matrix can be described as follows (see [7]):

Lemma 3.2. *Let* $A = (a_{ij})_{n\times n}$ $(n = 2m)$ *be centrosymmetric. Then, A has the form,*

$$A = \begin{bmatrix} B & J_mCJ_m \\ C & J_mBJ_m \end{bmatrix},\ P^TAP = \begin{bmatrix} B - J_mC & 0 \\ 0 & B + J_mC \end{bmatrix}, \tag{3.1}$$

Where $B,C \in \mathbb{C}^{m\times m}$, $P = \frac{1}{2}\begin{bmatrix} I_m & I_m \\ -J_m & J_m \end{bmatrix}$.

Lemma 3.3. *Let Abeann* $\times$ $n(n = 2m)$ *centrosymmetric positive definite matrix with the following form*

$$A = \begin{bmatrix} B & J_m C J_m \\ C & J_m B J_m \end{bmatrix},$$

where $B, C \in \mathbb{R}^{m\times m}$*. Then there exists an orthogonal matrix P such that*

$$P^T A P = \begin{bmatrix} M & \\ & N \end{bmatrix},$$

where $M = B - J_m C$, $N = B + J_m C$ *and* M^{-1}, N^{-1} *are positive definite matrices.*

Proof. From Lemma 3.2, there exists an orthogonal matrix P such that

$$H = P^T A P = \begin{bmatrix} M & \\ & N \end{bmatrix}.$$

Since A is positive definite, then any eigenvalue of A is positive real number. Thus, the eigenvalues of H are all positive real numbers. That is to say, all eigenvalues of M and N are positive real numbers. Thus, M and N are both positive definite. It is obvious that M^{-1} and N^{-1} are positive definite matrices.

Lemma 3.4. *Let A, B are positive definite real matrices. Then,*

$$\frac{\mathrm{tr}A}{(\mathrm{detB})^{1/n}} \le ||A||_F||B^{-1}||_F. \tag{3.2}$$

Proof. Let X be positive definite. By Lemma 1.2, we have

$$\left|\frac{< A, X>_F < X, B>_F}{||X||_F^2} - \frac{< A, B>_F}{2}\right| \le \frac{||A||_F B||_F}{2}. \tag{3.3}$$

Thus,

$$\left|\frac{\mathrm{tr(A^TX)tr(X^TB)}}{||X||_F^2} - \frac{\mathrm{tr(A^TB)}}{2}\right| \le \frac{||A||_F B||_F}{2}. \tag{3.4}$$

By replacing X, B with I_n and B^{-1}, respectively, in inequality (3.4), we can obtain

$$\left|\frac{\mathrm{tr(A)tr(B^{-1})}}{n} - \frac{\mathrm{tr(A^TB^{-1})}}{2}\right| \le \frac{||A||_F||B^{-1}||_F}{2}. \tag{3.5}$$

That is,

$$\frac{\mathrm{tr(A)tr(B^{-1})}}{n} - \frac{\mathrm{tr(A^TB^{-1})}}{2} \le \frac{||A||_F||B^{-1}||_F}{2}. \tag{3.6}$$

From Schwarz inequality (1.5),

$$\mathrm{tr}(A^TB^{-1}) =< A, B^{-1}>_F \le ||A||_F||B^{-1}||_F. \tag{3.7}$$

By taking (3.7) into (3.6), we have

$$\frac{\mathrm{tr(A)tr}(B^{-1})}{n} \le ||A||_F||B^{-1}||_F \le \frac{||A||_F||B^{-1}||_F}{2} + \frac{\mathrm{tr(A^TB^{-1})}}{2}. \tag{3.8}$$

From (1.9),

$$n(1/\mathrm{detB})^{1/n} \le \mathrm{trB^{-1}}. \tag{3.9}$$

Therefore,

$$\frac{\mathrm{tr}A}{(\det B)^{1/n}} \le ||A||_F||B^{-1}||_F. \tag{3.10}$$

Theorem 3.5. *Let A be a centrosymmetric positive definite matrix with the form*

$$A = \begin{bmatrix} B & J_mCJ_m \\ C & J_mBJ_m \end{bmatrix}, B, C \in \mathbb{R}^{m\times m}.$$

Let $M = B - J_mC$, $N = B + J_mC$. *Then,*

$$\mu_F(A) \ge \sqrt{\left(2\frac{\mathrm{tr}M}{(\det M)^{1/m}} - m\right)^2 + \left(2\frac{\mathrm{tr}N}{(\det N)^{1/m}} - m\right)^2 + \left(\frac{\mathrm{tr}M}{(\det N)^{1/m}}\right)^2 + \left(\frac{\mathrm{tr}N}{(\det M)^{1/m}}\right)^2}. \tag{3.11}$$

Proof. By Lemma 3.2, there exists an orthogonal matrix P such that

$$H = P^TAP = \begin{bmatrix} B - J_mC & \\ & B + J_mC \end{bmatrix} = \begin{bmatrix} M & \\ & N \end{bmatrix}.$$

Thus,

$$\mu_F(A) = ||A||_F||A^{-1}||_F = \mu_F(H) = ||H||_F||H^{-1}||_F = \sqrt{||M||_F^2 + ||N||_F^2}\sqrt{||M^{-1}||_F^2 + ||N^{-1}||_F^2}. \tag{3.12}$$

Therefore,

$$\begin{aligned} \mu_F^2(A) &= (||M||_F^2 + ||N||_F^2)(||M^{-1}||_F^2 + ||N^{-1}||_F^2) \\ &= ||M||_F^2||M^{-1}||_F^2 + ||N||_F^2||N^{-1}||_F^2 + ||M||_F^2||N^{-1}||_F^2 + ||N||_F^2||M^{-1}||_F^2 \\ &= \mu_F^2(M) + \mu_F^2(N) + (||M||_F||N^{-1}||_F)^2 + (||N||_F||M^{-1}||_F)^2. \end{aligned}$$

From Lemma 3.4,

$$\frac{\mathrm{tr}M}{(\det N)^{1/m}} \le ||M||_F||N^{-1}||_F, \quad \frac{\mathrm{tr}N}{(\det M)^{1/m}} \le ||N||_F||M^{-1}||_F. \tag{3.14}$$

From (2.18),

$$2\frac{\mathrm{tr}M}{(\det M)^{1/m}} - m \le \mu_F(M), \text{ and } 2\frac{\mathrm{tr}N}{(\det N)^{1/m}} - m \le \mu_F(N). \tag{3.15}$$

Thus,

$$\mu_F(A) \ge \sqrt{\left(2\frac{\mathrm{tr}M}{(\det M)^{1/m}} - m\right)^2 + \left(2\frac{\mathrm{tr}N}{(\det N)^{1/m}} - m\right)^2 + \left(\frac{\mathrm{tr}M}{(\det N)^{1/m}}\right)^2 + \left(\frac{\mathrm{tr}N}{(\det M)^{1/m}}\right)^2}.$$

Example 3.6.

$$A = \begin{bmatrix} 5 & 0 & 0 & 3 \\ 0 & 5 & 3 & 0 \\ 0 & 3 & 5 & 0 \\ 3 & 0 & 0 & 5 \end{bmatrix}, \text{ and } P^TAP = \begin{bmatrix} M & \\ & N \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} & \\ & \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} \end{bmatrix}.$$

Here $n = 4$, $m = 2$, det $(A) = 256$, tr $A = 20$, tr $M = 4$, tr $N = 16$, $\det(M) = 4$, $\det(N) = 64$. From Theorem 3.5, a lower bound of $\mu_F(A)$ is as follows:

$$\mu_F(A) \geq \sqrt{\left(2\frac{\text{trM}}{(\text{detM})^{1/m}} - m\right)^2 + \left(2\frac{\text{trN}}{(\text{detN})^{1/m}} - m\right)^2 + \left(\frac{\text{trM}}{(\text{detN})^{1/m}}\right)^2 + \left(\frac{\text{trN}}{(\text{detM})^{1/m}}\right)^2} = 8.5.$$

In fact

$$\mu_F(A) = \sqrt{\mu_F^2(M) + \mu_F^2(N) + (||M||_F||N^{-1}||_F)^2 + (||N||_F||M^{-1}||_F)^2} = 8.5.$$

On the other hand, the lower bounds of $\mu_F(A)$ in (2.15) and (2.16) provided by [6] are

$$\mu_F(A) \geq 2n \cdot \frac{\text{trA}^{3/2}}{\text{tr}A \cdot (\text{detA})^{1/2\text{n}}} - n = 6.1823376, \quad \text{and } \mu_F(A) \geq 2\frac{\text{tr}A}{(\det A)^{1/n}} - n = 6.$$

It can easily be seen that, in this example, the best lower bound is the first one given by Theorem 3.5.

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Authors' contributions
Hongyi Li carried out studies on the linear algebra and matrix theory with applications, and drafted the manuscript. Zongsheng Gao read the manuscript carefully and gave valuable suggestions and comments. Di Zhao provided numerical examples, which demonstrated the main results of this article. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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References
1. Zhang, F: Matrix Theory: Basic Results and Techniques. Springer, New York (1999)
2. Horn, RA, Johnson, CR: Matrix Analysis. Cambridge University Press, Cambridge (1985)
3. Buzano, ML: Gemeralizzazione della diseguaglianza di Cauchy-Schwarz. Rendiconti del Seminario Matematico Universita e Politecnico di Torino **31**, 405–409 (1971). (1974) (Italian)
4. Dragomir, SS: Refinemens of Buzano's and Kurepa's inequalities in inner product spaces. Facta Universitatis. **20**, 65–73 (2005)
5. Danna, FM: Matrix and operator inequalites. J Inequal Pure Appl Math **2**(3) (2001). article 34
6. Ramazan, T, Zübeyde, U: On the Frobenius condition number of positive definite matrices. J Inequal Appl (2010). Article ID 897279 (2010)
7. Liu, ZY: Some properties of centrosymmetric matrices. Appl Math Comput. **141**, 17–26 (2002)

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