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Estimates for the multiple singular integrals via extrapolation

Meng Qu

Correspondence: qumeng@mail.ahnu.edu.cn
School of Mathematical and Computer Sciences, Anhui Normal University, Wuhu, 241002, People's Republic of China

Abstract

This paper is devoted to the study on the L^p estimates for the multiple singular integrals with rough kernels on product spaces $\mathbb{R}^n \times \mathbb{R}^m$ (n, $m \geq 2$). By means of extrapolation method and Fourier transform estimate, we prove that the multiple singular integral operators are bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for the kernel functions: $\Omega \in L(\log L)^2(S^{n-1} \times S^{m-1})$, $h \in \widetilde{\Delta}_\alpha(\mathbb{R}^+ \times \mathbb{R}^+)$ $(\alpha \in (1, 2])$. Furthermore, we prove that when $\Omega \in L(\log L)^2(S^{n-1} \times S^{m-1})$ and h satisfying a 'log' type condition defined on $\mathbb{R}^+ \times \mathbb{R}^+$, the multiple singular integral operators are bounded on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$, which improves the well-known result.

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1 Introduction

Let $\mathbb{R}^n (n \geq 2)$ be n-dimensional Euclidian space and S^{n-1} be the unit sphere in $\mathbb{R}^n$. Suppose that the function $\Omega \in L^1(S^{n-1})$ satisfies the following cancelation condition

$$\int_{S^{n-1}} \Omega(\theta) \mathrm{d}\sigma(\theta) = 0, \tag{1.1}$$

where $\mathrm{d}\sigma$ denotes the usual Lebesgue surface measure on the unit sphere S^{n-1}.

Let $L(\log L)^\alpha$ (S^{n-1}) denotes the functions Ω defined on S^{n-1} satisfying the Zygmund condition: for $\alpha > 0$,

$$\int_{S^{n-1}} |\Omega(\theta)|(\log(2 + |\Omega(\theta)|))^\alpha \mathrm{d}\sigma(\theta) < \infty.$$

It is noted that for any $q > 1$, we have the proper inclusion relations hold:

$$L^q(S^{n-1}) \subset L(\log L)^\alpha(S^{n-1}) \subset L^1(S^{n-1}),$$
$$L(\log L)^\beta(S^{n-1}) \subset L(\log L)^\alpha(S^{n-1}) \text{ if } 0 < \alpha < \beta.$$

For $s \geq 1$, let $\Delta_s(\mathbb{R}^+)$ denote the collection of measurable functions h on $\mathbb{R}^+ = \{t \in \mathbb{R} : t > 0\}$ satisfying

$$||h||_{\Delta_s(\mathbb{R}^+)} = \sup_{j\in\mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} |h(t)|^s dt/t \right)^{1/s} < \infty,$$

where $\mathbb{Z}$ denotes the set of integers. Also by usual modification, $\Delta_\infty(\mathbb{R}^+) = L^\infty(\mathbb{R}^+)$. We note that $\Delta_s \subset \Delta_t$ if $s > t$. We can always assume that $h \in \Delta_1$.

A singular integral operator is defined in the following form:

$$S(f)(x) = p.v. \int_{\mathbb{R}^n} f(x-y)K(y)\mathrm{d}y = \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} f(x-y)K(y)\mathrm{d}y, \tag{1.2}$$

for an appropriate function f on $\mathbb{R}^n$, where $K(y) = |y|^{-n}h(|y|)\Omega(y')$, $y' = |y|^{-1}y$.

It is well known that if $\Omega \in L\log L(S^{n-1})$, $h = 1$, by the method of rotations, Calderón and Zygmund [1] proved that S extends to a bounded operator on L^p for all $p \in (1, \infty)$. In [2], R. Fefferman first introduced the case of rough radial and proved that if $h \in \Delta_\infty(\mathbb{R}^+)$ and Ω satisfy a Lipschitz condition of positive order on S^{n-1}, then S is bounded on L^p for $1 < p < \infty$. Namazi [3] improved this result by replacing the Lipschitz condition by the condition that $\Omega \in L^q(S^{n-1})$ for some $q > 1$. In [4], Duoandikoetxea and Rubio de Francia developed some methods that can be used to study mapping properties of several kinds of operators in harmonic analysis, where they proved that S is bounded on L^p for $1 < p < \infty$ when $h \in \Delta_2(\mathbb{R}^+)$ and $\Omega \in L^q(S^{n-1})$. In [5], Al-Salman and Pan proved that S is bounded on L^p for $1 < p < \infty$ when $h \in \Delta_s(\mathbb{R}^+)$ $(s > 1)$ and $\Omega \in L\log L(S^{n-1})$. Recently, using a method called Yano's extrapolation method [6,7], Sato [8] proved that S extends to be an operator bounded on L^p for $1 < p < \infty$ where $\Omega \in L\log L(S^{n-1})$ and the radial function h satisfying a rougher condition as a log type.

Define the function spaces

$$\mathcal{L}_a(\mathbb{R}^+) = \{h : h \text{ be measurable functions on } \mathbb{R}^+, L_a(h) < \infty\},$$

where

$$L_a(h) = \sup_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} |h(r)|(\log(2 + |h(r)|))^a \frac{dr}{r}.$$

And define the function space

$$\mathcal{N}_a(\mathbb{R}^+) = \{h : h \text{ be measurable functions on } \mathbb{R}^+, N_a(h) < \infty\},$$

where

$$N_a(h) = \sum_{m\geq 1} m^a 2^m d_m(h),$$

with $d_m(h) = \sup_{k\in\mathbb{Z}} 2^{-k}|E(k, m)|$ and $E(k, m) = \{r \in (2^k, 2^{k+1}] : 2^{m-1} < |h(r)| \leq 2^m\}$ for $m \geq 2$, $E(k, 1) = \{r \in (2^k, 2^{k+1}] : |h(r)| \leq 2\}$. Indeed, it is noted that for any $a > 0$, $\mathcal{N}_a(\mathbb{R}^+) \subset \mathcal{L}_a(\mathbb{R}^+)$ and $\mathcal{L}_{a+b}(\mathbb{R}^+) \subset \mathcal{N}_a(\mathbb{R}^+)$ for some $b > 1$.

Sato's main result is the following theorem:

Theorem A. [8] Suppose Ω is a function in $L\log L(S^{n-1})$ satisfying (1.1) and $||S(f)||_{L^p(\mathbb{R}^n)} \leq C||f||_{L^p(\mathbb{R}^n)}$ (or $h \in \mathcal{L}_a(\mathbb{R}^+)$ for some $a > 2$). Let S be as in (1.2). Then, there is a constant C such that

$$||S(f)||_{L^p(\mathbb{R}^n)} \le C||f||_{L^p(\mathbb{R}^n)}$$

for all $p \in (1, \infty)$.

For the one-parameter case, there are also several other papers. Especially, in [9,10], weighted L^p boundedness of singular integrals was discussed. The reader also can refer to [11-13] for more background materials.

In the article, we mainly consider the L^p boundedness for the multiple singular integrals with rough kernels. Suppose that $S^{d-1}(d = n \text{ or } m)$ is the unit sphere of $\mathbb{R}^d$ ($d \ge 2$) equipped with the usual Lebesgue measure dσ. Let $\Omega \in L^1(S^{n-1} \times S^{m-1})$ satisfy the following double cancelation condition:

$$\int_{S^{n-1}} \Omega(u, v)\mathrm{d}\sigma(u) = 0 \text{ and } \int_{S^{m-1}} \Omega(u, v)\mathrm{d}\sigma(v) = 0. \tag{1.3}$$

For $\alpha \ge 1$,

$$\Delta_\alpha(\mathbb{R}^+ \times \mathbb{R}^+) = \{h : h \text{ be measurable functions on } \mathbb{R}^+ \times \mathbb{R}^+, \quad ||h||_{\Delta_\alpha} < \infty\},$$

where

$$||h||_{\Delta_\alpha} = \sup_{k,j\in\mathbb{Z}} \left(\int_{2^k}^{2^{k+1}} \int_{2^j}^{2^{j+1}} |h(r, s)|^\alpha \frac{\mathrm{d}r\mathrm{d}s}{rs} \right)^{\frac{1}{\alpha}}.$$

The multiple singular integral on the product space $\mathbb{R}^n \times \mathbb{R}^m$ is defined by the following form:

$$Tf(x_1, x_2) = \text{p.v.} \int_{\mathbb{R}^n \times \mathbb{R}^m} f(x_1 - y_1, x_2 - y_2)K(y_1, y_2)\mathrm{d}y_1\mathrm{d}y_2 \tag{1.4}$$

for an appropriate function f on $\mathbb{R}^n \times \mathbb{R}^m$, where

$$K(y_1, y_2) = |y_1|^{-n}|y_2|^{-m}\Omega(y_1', y_2')h(|y_1|, |y_2|). \tag{1.5}$$

Let $L(\log L)^\alpha(S^{n-1} \times S^{m-1})$ denote the class of the functions Ω defined on $S^{n-1} \times S^{m-1}$ satisfying the Zygmund condition: for $\alpha > 0$,

$$\int_{S^{n-1} \times S^{m-1}} |\Omega(\theta, \omega)|(\log(2 + |\Omega(\theta, \omega)|))^\alpha \mathrm{d}\sigma(\theta)\mathrm{d}\sigma(\omega) < \infty.$$

Historically, multiple singular integral was introduced by R. Fefferman and Stein's famous work on multiparameter harmonic analysis. Fefferman and Stein [14] proved that when $h \equiv 1$, T is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $1 < p < \infty$ if Ω satisfy certain smooth conditions. Their method mainly relies on so-called square function method. Subsequently, in [15], Duoandikoetxea used the method established in [4] and proved that T is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $1 < p < \infty$ when $\Omega \in L^q(S^{n-1} \times S^{m-1})$ for some $q > 1$ and $h \in \Delta_2(\mathbb{R}^+ \times \mathbb{R}^+)$. In [16], Fan-Guo-Pan proved that T is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $1 < p < \infty$ for the case when Ω belongs to certain block spaces that contain $L^q(S^{n-1} \times S^{m-1})$ (for $p = 2$, it was proved by Jiang and Lu in [17]) and $h = 1$. In [18], Chen proved that T is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $1 < p < \infty$ when $\Omega \in L(\log L)^2 (S^{n-1} \times S^{m-1})$ and $h = 1$ where he mainly relies on the method of rotation. In [19], Al-Salman, Al-Qassem and Pan proved that T is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $1 < p < \infty$ when $\Omega \in L(\log L)^2(S^{n-1} \times S^{m-1})$ and $h \in \Delta_\alpha$ for some $\alpha > 1$, where their technique mostly based on refining the Duoandikoetxea-Rubia's Fourier transform estimates and Littlewood-paley theory. In the same paper, they also pointed out that for any $\varepsilon > 0$, there is a function $\Omega \in L(\log L)^{2-\varepsilon}(S^{n-1} \times S^{m-1})$ such that T may fail to be bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$.

The main purpose of this paper is to improve the above results, especially the rough product radial part. For this reason, we introduce several measurable function spaces defined on $\mathbb{R}^+ \times \mathbb{R}^+$: $\widetilde{\Delta}_\alpha(\mathbb{R}^+ \times \mathbb{R}^+)$, $\mathcal{L}_\alpha(\mathbb{R}^+ \times \mathbb{R}^+)$ and $\mathcal{N}_\alpha(\mathbb{R}^+ \times \mathbb{R}^+)$ for $\alpha > 0$, where these spaces are equipped with the following "norms":

$$\|h\|_{\widetilde{\Delta}_\alpha(\mathbb{R}^+\times\mathbb{R}^+)} = \sup_{k\in\mathbb{Z}} \left(\int_{2^k}^{2^{k+1}} \sup_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} |h(r,s)|^\alpha \frac{\mathrm{d}s}{s}\frac{\mathrm{d}r}{r} \right)^{\frac{1}{\alpha}} + \sup_{j\in\mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} \sup_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} |h(r,s)|^\alpha \frac{\mathrm{d}r}{r}\frac{\mathrm{d}s}{s} \right)^{\frac{1}{\alpha}},$$

$$\|h\|_{\mathcal{L}_\alpha(\mathbb{R}^+\times\mathbb{R}^+)} = \sup_{j,k\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} |h(r,s)| \left(\log(2+|h(r,s)|)\right)^\alpha \frac{\mathrm{d}r\mathrm{d}s}{rs},$$

$$\|h\| \mathcal{N}_{\alpha(\mathbb{R}^+\times\mathbb{R}^+)} = \sum_{m\geq 1} m^a 2^m D_m(h),$$

with $D_m(h) = \sup_{k,j\in\mathbb{Z}} 2^{-k}2^{-j}|E(k,j,m)|$ and $E(k, j, m) = \{(r, s) \in (2^k, 2^{k+1}] \times (2^j, 2^{j+1}] : 2^{m-1} < |h(r, s)| \leq 2^m\}$ for $m \geq 2$, $E(k, j, 1) = \{(r, s) \in (2^k, 2^{k+1}] \times (2^j, 2^{j+1}] : |h(r, s)| \leq 2\}$.

Remark 1.1. Of course by the usual modification, $\Delta_\infty(\mathbb{R}^+ \times \mathbb{R}^+) = \widetilde{\Delta}_\infty(\mathbb{R}^+ \times \mathbb{R}^+) = L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$. For simplicity, we let $\Delta_\alpha = \Delta_\alpha(\mathbb{R}^+ \times \mathbb{R}^+)$, $\widetilde{\Delta}_\alpha = \widetilde{\Delta}_\alpha(\mathbb{R}^+ \times \mathbb{R}^+)$, $\mathcal{L}_\alpha = \mathcal{L}_\alpha(\mathbb{R}^+ \times \mathbb{R}^+)$ and $\mathcal{N}_\alpha = \mathcal{N}_\alpha(\mathbb{R}^+ \times \mathbb{R}^+)$. It is easy to check that (1) $\Delta_\infty \subset \widetilde{\Delta}_\alpha \subset \Delta_\alpha$; (2) $\widetilde{\Delta}_\alpha \subset \widetilde{\Delta}_\beta$ if $1 \leq \beta < \alpha$; (3) for any $\alpha > 0$, $\mathcal{N}_\alpha \subset \mathcal{L}_\alpha$ and $\mathcal{L}_{\alpha+\beta} \subset \mathcal{N}_\alpha$ for any $\beta > 1$; (4) for any $\alpha > 1$ and $\beta > 0$, $\Delta_\alpha \subset \mathcal{L}_\beta \subset \Delta_1$.

Our main results are the following theorems:

Theorem 1.1. Suppose that $\Omega \in L(\log L)^2(S^{n-1} \times S^{m-1})$ satisfying (1.3),

(1) if $h \in \mathcal{N}_2$ or $\mathcal{L}_\alpha$ for some $\alpha > 3$, then there is a constant C such that

$$||Tf||_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} \leq C||f||_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}. \tag{1.6}$$

(2) if $h \in \widetilde{\Delta}_\alpha(\alpha \in (1,2])$, then there is a constant C, which is independent of α, such that

$$||Tf||_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C\frac{1}{(\alpha-1)^2}||h||_{\widetilde{\Delta}_\alpha}||f||_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}, \tag{1.7}$$

for $p \in (1, \infty)$.

Remark 1.2. In [19], it was proved that $h \in \Delta_\alpha$ for some $\alpha > 1$ and $\Omega \in L(\log L)^2(S^{n-1} \times S^{m-1})$ are sufficient for L^p boundedness for the multiple singular integral T. As for $p = 2$, Theorem 1.1 extended this result. For $p \neq 2$, our condition $h \in \widetilde{\Delta}_\alpha(\alpha \in (1,2])$ is strong. However, our result gives a sharp constant estimate, which gives the following corollary when the product radial part is separated (that is, $h(r, s) = h_1(r) \cdot h_2(s)$).

Corollary 1.1. Suppose that $\Omega \in L(\log L)^2(S^{n-1} \times S^{m-1})$ satisfying (1.3) and if $h(r, s) = h_1(r) \cdot \mathrm{h}_2(s)$, where h_1 or h_2 satisfies one of the following case:

(1) $h_1 \in \Delta_\alpha(\mathbb{R}^+)$ (($\alpha > 1$)) and $h_2 \in \mathcal{N}_2(\mathbb{R}^+)$(or $h_2 \in \mathcal{L}_a(\mathbb{R}^+)$ for some $a > 3$);
(2) $h_1 \in \mathcal{N}_2(\mathbb{R}^+)$(or $h_1 \in \mathcal{L}_a(\mathbb{R}^+)$ fsor some $a > 3$) and $h_2 \in \Delta_s(\mathbb{R}^+)$ (($s > 1$)),

then there is a constant C such that

$$||Tf||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C||f||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}, \tag{1.8}$$

for $p \in (1, \infty)$.

Our proof of the above theorem is based on the argument of Sato [8], which mainly relied on Yano's extrapolation method. The following theorem is the key step to prove Theorem 1.1.

Theorem 1.2. Suppose that $\Omega \in L^q(S^{n-1} \times S^{m-1})(q \in (1, 2])$ satisfying (1.3).

(1) If $h \in \Delta_\alpha(\alpha \in (1, 2])$, then there exists a constant C, which is independent of q, α, Ω, h, such that

$$||Tf||_{L^2(\mathbb{R}^n \times \mathbb{R}^m)} \leq C\frac{1}{(\alpha-1)^2}\frac{1}{(q-1)^2}||h||_{\Delta_\alpha}||\Omega||_{L^p(S^{n-1}\times S^{m-1})}||f||_{L^2(\mathbb{R}^n \times \mathbb{R}^m)}. \tag{1.9}$$

(2) If $h \in \widetilde{\Delta}_\alpha(\alpha \in (1, 2])$, then there exists a constant C, which is independent of q, α, Ω, h, such that

$$||Tf||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C\frac{1}{(\alpha-1)^2}\frac{1}{(q-1)^2}||h||_{\widetilde{\Delta}_\alpha}||\Omega||_{L^q(S^{n-1}\times S^{m-1})}||f||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}, \tag{1.10}$$

for $p \in (1, \infty)$.

Remark 1.3. Corollary 1 in [15] asserted that if $h \in \Delta_2$ and $\Omega \in L^q(q > 1)(S^{n-1} \times S^{m-1})$, then T is bounded in $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $p > 1$. After a careful check of its proof, we find that the condition $h \in \Delta_2$ is not sufficient for $p \neq 2$ since the two partial maximal functions are taken supremum both j and k, it seems that if $h \in \Delta_2$, the partial maximal function is not pointwise controlled by the one-parameter maximal function case (line 10-13, [15]). If we substitute $h \in \Delta_2$ with $h \in \widetilde{\Delta}_2$, Corollary 1 in [15] is corrected. This is why we introduce the space $\widetilde{\Delta}_\alpha$. Of course, we remark that our result is mainly influenced by the idea and the technique established in [15]: Littlewood-Paley theory for product theory, Fourier transform estimates, etc.

Remark 1.4. The maximal multiple singular integral is defined as

$$T^*f(x_1, x_2) = \sup_{\varepsilon_1>0, \varepsilon_2>0}\left|\iint_{|y_1|\geq\varepsilon_1, |y_2|\geq\varepsilon_2} f(x_1-y_1, x_2-y_2)K(y_1, y_2)\mathrm{d}y_1\mathrm{d}y_2\right|,$$

where K is as in (1.5). By the estimates we have established and Yano's extrapolation method, combining with [20], we have the same result for the maximal multiple singular integral as in [19]:

Theorem 1.3. Suppose that $\Omega \in L(\log L)^2(S^{n-1} \times S^{m-1})$ satisfies (1.3) and $h \in \Delta_\infty$, then there exists a constant C, such that

$$||T^*f||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C||h||_{\Delta_\infty}||f||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}, \tag{1.11}$$

for $p \in (1, \infty)$.

We leave the proof to the interested reader. But we do not know whether h can be extended to more general case like $\tilde{\Delta}_\alpha(1 < \alpha < \infty)$.

This paper is organized as the following. In Section 2, we give the proof of Theorem 1.2. In Section 3, we give the proof of Theorem 1.1 and Corollary 1.1. Throughout this paper, the letter C will stand for a constant that may vary at each occurrence but that is independent of the essential variables and p' be the conjugation of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$.

2 Proof of Theorem 1.2

Let Ω, h be as in Theorem 1.2. We let $\rho \geq 2$, define

$$E_{k,j} = \{(y_1, y_2) \in \mathbb{R}^{n+m} : \rho^k < |y_1| \leq \rho^{k+1}, \rho^j < |y_2| \leq \rho^{j+1}\}$$

and measures $\sigma_{k,j}$ by

$$\sigma_{k,j} * f(x_1, x_2) = \iint_{E_{k,j}} K(y_1, y_2) f(x_1 - y_1, x_2 - y_2) \mathrm{d}y_1 \mathrm{d}y_2.$$

So

$$Tf = \sum_{k,j} \sigma_{k,j} * f.$$

Define σ^* by $\sigma^* f(x) = \sup_{k,j} ||\sigma_{k,j}| * f(x)|$, where $|\sigma_{k,j}|$ denotes the total variation. Let $\mu_{k,j} = |\sigma_{k,j}|$ and define μ^* by $\mu^* f(x) = \sup_{k,j} |\mu_{k,j} * f(x)|$. Let $\theta \in (0, 1)$, $\delta(p) = |1/p - 1/p'|$, we have the following two lemmas.

Lemma 2.1. For $p > 1 + \theta$, suppose that $\Omega \in L^q(S^{n-1} \times S^{m-1})(q \in (1,2])$ satisfying (1.3) and $h \in \tilde{\Delta}_\alpha(\alpha \in (1, 2])$, we have

$$||\mu^* f||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C \log^2 \rho ||\Omega||_{L^q(S^{n-1} \times S^{m-1})} ||h||_{\tilde{\Delta}_\alpha} \left(1 - 2^{-\frac{\theta}{2}}\right)^{-2.2/p} ||f||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}, \tag{2.1}$$

where the constant C is independent of q, α, Ω, h.

Lemma 2.2. (1). Suppose that $\Omega \in L^q(S^{n-1} \times S^{m-1})(q \in (1,2])$ satisfying (1.3) and $h \in \Delta_\alpha(\alpha \in (1,2])$,

$$||Tf||_{L^2(\mathbb{R}^n \times \mathbb{R}^m)} \leq C \log^2 \rho ||\Omega||_{L^q(S^{n-1} \times S^{m-1})} ||h||_{\Delta_\alpha} ||f||_{L^2(\mathbb{R}^n \times \mathbb{R}^m)}, \tag{2.2}$$

(2). For $p \in (1 + \theta, (1 + \theta)/\theta)$, suppose that $\Omega \in L^q(S^{n-1} \times S^{m-1})(q \in (1, 2])$ satisfying (1.3) and $h \in \tilde{\Delta}_\alpha(\alpha \in (1, 2])$, we have

$$||Tf||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C \log^2 \rho ||\Omega||_{L^q(S^{n-1} \times S^{m-1})} ||h||_{\tilde{\Delta}_\alpha} \left(1 - 2^{-\frac{\theta}{2}}\right)^{-2(1+\delta(p))} ||f||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}, \tag{2.3}$$

where the constant C is independent of q, α, Ω, h.

If Lemma 2.2 is proved, since $\theta \in (0, 1)$ is arbitrary and we choose $\rho = 2^{q'\alpha'}$, then Theorem 1.2 is an immediate consequence of Lemma 2.2 immediately.

Now, we prove part (1) of Lemma 2.2. For simplicity, we let $A = \log^2 \rho ||\Omega||_{L^q(S^{n-1} \times S^{m-1})} ||h||_{\Delta_\alpha}$. Firstly, we have the following estimates for the measures $\sigma_{k,j}$:

$$||\sigma_{k,j}|| \quad \le \quad c_1 A \tag{2.4}$$

$$|\hat{\sigma}_{k,j}(\xi_1, \xi_2) \quad \le \quad c_2 A|\rho^k \xi_1|^{\pm\frac{1}{2q'\alpha'}}|\rho^j \xi_2|^{\pm\frac{1}{2q'\alpha'}} \tag{2.5}$$

for some constants c_i. The equation (2.4) is the consequence of the following result:

$$\begin{aligned}
&||\sigma_{k,j}|| = |\sigma_{k,j}|(\mathbb{R}^n \times \mathbb{R}^m) \\
&\le \int_{\rho^k}^{\rho^{k+1}} \int_{\rho^j}^{\rho^{j+1}} \int_{S^{n-1}\times S^{m-1}} |\Omega(u,v)||h(r,s)| \mathrm{d}\sigma(u)\mathrm{d}\sigma(v)\frac{\mathrm{d}r\mathrm{d}s}{rs} \\
&\le \mathrm{C}\log^2\rho ||\Omega||_{L^1(S^{n-1}\times S^{m-1})}||h||_{\Delta_1}.
\end{aligned}$$

Now, we turn to prove (2.5), note

$$\hat{\sigma}_{k,j}(\xi_1, \xi_2) = \int_{\rho^k}^{\rho^{k+1}} \int_{\rho^j}^{\rho^{j+1}} \int_{S^{n-1}\times S^{m-1}} \Omega(u,v)h(r,s)e^{-2\pi i(\xi_1\cdot ru+\xi_2\cdot sv)}\mathrm{d}\sigma(u)\mathrm{d}\sigma(v)\frac{\mathrm{d}r\mathrm{d}s}{rs}$$

and we define

$$F(r,s,\xi_1,\xi_2) = \int_{S^{n-1}\times S^{m-1}} \Omega(u,v)e^{-2\pi i(\xi_1\cdot ru+\xi_2\cdot sv)}\mathrm{d}\sigma(u)\mathrm{d}\sigma(v).$$

Then, by Hölder's inequality,

$$\begin{aligned}
&|\hat{\sigma}_{k,j}(\xi_1,\xi_2)| = \left| \int_{\rho^k}^{\rho^{k+1}} \int_{\rho^j}^{\rho^{j+1}} F(r,s,\xi_1,\xi_2)h(r,s)\frac{\mathrm{d}r\mathrm{d}s}{rs} \right| \\
&\le \left(\int_{\rho^k}^{\rho^{k+1}} \int_{\rho^j}^{\rho^{j+1}} |h(r,s)|^\alpha \frac{\mathrm{d}r\mathrm{d}s}{rs} \right)^{\frac{1}{\alpha}} \left(\int_{\rho^k}^{\rho^{k+1}} \int_{\rho^j}^{\rho^{j+1}} |F(r,s,\xi_1,\xi_2)|^{\alpha'} \frac{\mathrm{d}r\mathrm{d}s}{rs} \right)^{\frac{1}{\alpha'}} \\
&\le \left(\int_{\rho^k}^{\rho^{k+1}} \int_{\rho^j}^{\rho^{j+1}} |h(r,s)|^\alpha \frac{\mathrm{d}r\mathrm{d}s}{rs} \right)^{\frac{1}{\alpha}} ||\Omega||_{L^1(S^{n-1}\times S^{m-1})}^{\frac{\alpha'-2}{\alpha'}} \left(\int_{\rho^k}^{\rho^{k+1}} \int_{\rho^j}^{\rho^{j+1}} |F(r,s,\xi_1,\xi_2)|^2 \frac{\mathrm{d}r\mathrm{d}s}{rs} \right)^{\frac{1}{\alpha'}}
\end{aligned}$$

while here

$$\begin{aligned}
&\int_{\rho^k}^{\rho^{k+1}} \int_{\rho^j}^{\rho^{j+1}} |F(r,s,\xi_1,\xi_2)|^2 \frac{\mathrm{d}r\mathrm{d}s}{rs} \\
&= \int_{\rho^k}^{\rho^{k+1}} \int_{\rho^j}^{\rho^{j+1}} \iint_{(S^{n-1}\times S^{m-1})^2} \Omega(u,v)\overline{\Omega(u',v')}e^{-2\pi i(\xi_1\cdot r(u-u')+\xi_2\cdot s(v-v'))}\mathrm{d}\sigma(u')\mathrm{d}\sigma(v)\mathrm{d}\sigma(v')\frac{\mathrm{d}r\mathrm{d}s}{rs} \\
&\le \iint_{(S^{n-1}\times S^{m-1})^2} \Omega(u,v)\overline{\Omega(u',v')} \int_{\rho^k}^{\rho^{k+1}} \int_{\rho^j}^{\rho^{j+1}} e^{-2\pi i(\xi_1\cdot r(u-u')+\xi_2\cdot s(v-v'))}\frac{\mathrm{d}r\mathrm{d}s}{rs}\mathrm{d}\sigma(u)\mathrm{d}\sigma(u')\mathrm{d}\sigma(v)\mathrm{d}\sigma(v') \\
&\le \mathrm{C}\log^2\rho ||\Omega||^2_{L^q(S^{n-1}\times S^{m-1})}|\rho^k\xi_1|^{-\varepsilon}|\rho^j\xi_2|^{-\varepsilon} \\
&\cdot \left(\int_{S^{n-1}\times S^{n-1}} \frac{\mathrm{d}\sigma(u)\mathrm{d}\sigma(u')}{|\xi'_1\cdot(u-u')|^{\varepsilon q'}} \right)^{\frac{1}{q'}} \left(\int_{S^{m-1}\times S^{m-1}} \frac{\mathrm{d}\sigma(v)\mathrm{d}\sigma(v')}{|\xi'_2\cdot(v-v')|^{\varepsilon q'}} \right)^{\frac{1}{q'}}.
\end{aligned}$$

When $\varepsilon q' < 1$ (indeed we set $\varepsilon = \frac{1}{2q'}$), the integrals $\left(\int_{S^{n-1}\times S^{n-1}} \frac{\mathrm{d}\sigma(u)\mathrm{d}\sigma(u')}{|\xi'_1\cdot(u-u')|\varepsilon q'}\right)^{\frac{1}{\alpha' q'}}$ and $\left(\int_{S^{m-1}\times S^{m-1}} \frac{\mathrm{d}\sigma(v)\mathrm{d}\sigma(v')}{|\xi'_2\cdot(v-v')|^{\varepsilon q'}}\right)\frac{1}{\alpha' q'}$ are finite and independent of q and α. So we have

$$|\hat{\sigma}_{k,j}(\xi_1,\xi_2)| \le C\log^2\rho||h||_{\Delta_\alpha}||\Omega||_{L^q(S^{n-1}\times S^{m-1})}|\rho^k\xi_1|^{-\frac{1}{2q'\alpha'}}|\rho^j\xi_2|^{-\frac{1}{2q'\alpha'}}. \tag{2.6}$$

Since Ω satisfies the condition (1.3), we have $\hat{\sigma}_{k,j}(0,\xi_2) = 0$ and then $|\hat{\sigma}_{k,j}(\xi_1,\xi_2)|$ equals to

$$\begin{aligned}
&|\hat{\sigma}_{k,j}(\xi_1,\xi_2) - \hat{\sigma}_{k,j}(0,\xi_2)| \\
= &\left|\int_{\rho^k}^{\rho^{k+1}}\int_{\rho^j}^{\rho^{j+1}}\int_{S^{n-1}\times S^{m-1}} \Omega(u,v)h(r,s)[e^{-2\pi i\xi_1\cdot ru}-1]e^{-2\pi i\xi_2\cdot sv}\mathrm{d}\sigma(u)\mathrm{d}\sigma(v)\frac{\mathrm{d}r\mathrm{d}s}{rs}\right| \\
\le &\int_{\rho^k}^{\rho^{k+1}}\int_{S^{n-1}}\left|\int_{\rho^j}^{\rho^{j+1}}\int_{S^{m-1}}\Omega(u,v)h(r,s)e^{-2\pi i\xi_2\cdot sv}dv\frac{\mathrm{d}s}{s}\right||e^{-2\pi i\xi_1\cdot ru}-1|du\frac{\mathrm{d}r}{r} \\
\le &\int_{\rho^k}^{\rho^{k+1}}\int_{S^{n-1}}\left|\int_{\rho^j}^{\rho^{j+1}}\int_{S^{m-1}}\Omega(u,v)h(r,s)e^{-2\pi i\xi_2\cdot sv}dv\frac{\mathrm{d}s}{s}\right|\min(\{2,r|\xi_1|\})du\frac{\mathrm{d}r}{r} \\
\le &C\log^2\rho||h||_{\Delta_\alpha}||\Omega||_{L^q(S^{n-1}\times S^{m-1})}|\rho^k\xi_1|^{\frac{1}{2q'\alpha'}}|\rho^j\xi_2|^{-\frac{1}{2q'\alpha'}}.
\end{aligned} \tag{2.7}$$

The same way as above, we have $|\hat{\sigma}_{k,j}(\xi_1,\xi_2)|$ equals to

$$|\hat{\sigma}_{k,j}(\xi_1,\xi_2) - \hat{\sigma}_{k,j}(\xi_1,0)| \le C\log^2\rho||h||_{\Delta_\alpha}||\Omega||_{L^q(S^{n-1}\times S^{m-1})}|\rho^k\xi_1|^{-\frac{1}{2q'\alpha'}}|\rho^j\xi_2|^{\frac{1}{2q'\alpha'}}. \tag{2.8}$$

Also we have $|\hat{\sigma}_{k,j}(\xi_1,\xi_2)|$ equals to

$$\begin{aligned}
&|\hat{\sigma}_{k,j}(\xi_1,\xi_2) - \hat{\sigma}_{k,j}(\xi_1,0) - \hat{\sigma}_{k,j}(0,\xi_2) + \hat{\sigma}_{k,j}(0,0)| \\
&\le C\log^2\rho||h||_{\Delta_\alpha}||\Omega||_{L^q(S^{n-1}\times S^{m-1})}|\rho^k\xi_1|^{\frac{1}{2q'\alpha'}}|\rho^j\xi_2|^{\frac{1}{2q'\alpha'}}.
\end{aligned} \tag{2.9}$$

Consequently, the inequality (2.5) is just the combination of (2.6), (2.7),(2.8) and (2.9).

Let $\psi^1 \in \mathcal{S}(\mathbb{R}^n)$, $\psi^2 \in \mathcal{S}(\mathbb{R}^m)$, such that

$$\begin{aligned}
&supp(\psi^i(\xi_i)) \subset \{\frac{1}{\rho} \le |\xi_i| < \rho\}, i = 1,2, \\
&\qquad 0 \le \psi^i(\xi_i) \le 1, i = 1,2,
\end{aligned}$$

and

$$\sum_{k=-\infty}^{\infty}|(\psi^1)(\rho^k\xi_1)|^2 = \sum_{j=-\infty}^{\infty}|(\psi^2)(\rho^j\xi_2)|^2 = 1.$$

Let ψ_k^1, ψ_j^2 as $(\psi_k^1)^\wedge(\xi_1) = \psi^1(\rho^k\xi_1)$, $(\psi_j^2)^\wedge(\xi_2) = \psi^2(\rho^j\xi_2)$, respectively. Then, we have

$$\begin{aligned} Tf &= \sum_{k,j} \sigma_{k,j} * f \\ &= \sum_{k,j} \sum_{l,m} \sigma_{k,j} * (\psi^1_{k+1} \otimes \psi^2_{j+m}) * (\psi^1_{k+l} \otimes \psi^2_{j+m}) * f \\ &\triangleq \sum_{l,m} T_{l,m} f, \end{aligned}$$

where

$$T_{l,m} f = \sum_{k,j} \sigma_{k,j} * (\psi^1_{k+1} \otimes \psi^2_{j+m}) * (\psi^1_{k+l} \otimes \psi^2_{j+m}) * f. \tag{2.10}$$

Then, by Plancherel's theorem and (2.5), we have

$$\begin{aligned} ||T_{l,m} f||^2_{L^2(\mathbb{R}^n \times \mathbb{R}^m)} &\leq \sum_{k,j} C \iint_{D(k+l,j+m)} |\hat{\sigma}_{k,j}(\xi_1, \xi_2)|^2 |\hat{f}(\xi_1, \xi_2)|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\ &\leq CA^2 \min\{1, \rho^{-2\frac{1}{2q'\alpha'}(|l|-1)}\} \min\{1, \rho^{-2\frac{1}{2q'\alpha'}(|m|-1)}\} \sum_{k,j \in \mathbb{Z}} \iint_{D(k+l,j+m)} |\hat{f}(\xi_1, \xi_2)|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\ &\leq CA^2 \min\{1, \rho^{-2\frac{1}{2q'\alpha'}(|l|-1)}\} \min\{1, \rho^{-2\frac{1}{2q'\alpha'}(|m|-1)}\} ||f||^2_{L^2(\mathbb{R}^n \times \mathbb{R}^m)} \end{aligned} \tag{2.11}$$

where $D(k, j) = \{(\xi, \eta) : \rho^{-k-1} \leq |\xi| \leq \rho^{-k+1}, \rho^{-j-1} \leq |\eta| \leq \rho^{-j+1}\}$. By above estimates and Minkowski's inequality, we give the proof of part (1).

Now, we turn to prove part (2) of Lemma 2.2, take for Lemma 2.1 is granted. We let $A' = \log^2 \rho ||\Omega||_{L^q(S^{n-1} \times S^{m-1})} ||h||_{\tilde{\Delta}_\alpha}$ and $B = \left(1 - 2^{-\frac{\theta}{2}}\right)^{-2}$ for simplicity. We have

$$||\sigma_{k,j}|| \quad \leq \quad c_1 A' \tag{2.12}$$

$$|\hat{\sigma}_{k,j}(\xi_1, \xi_2)| \quad \leq \quad c_2 A' |\rho^k \xi_1|^{\pm \frac{1}{2q'\alpha'}} |\rho^j \xi_2|^{\pm \frac{1}{2q'\alpha'}} \tag{2.13}$$

$$||\sigma^*(f)||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \quad \leq \quad C_p A' B^{\frac{2}{p}} ||f||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \quad \text{for} \quad p > 1 + \theta, \tag{2.14}$$

for some constants c_i and C_p. where Eqs. (2.12) and (2.13) follow (2.4) and (2.5), respectively, (2.14) is just (2.1).

Lemma 2.3. Let $u \in (1 + \theta, 2]$, define a number v by $\frac{1}{v} - \frac{1}{2} = \frac{1}{2u}$. Then, we have the vector-valued inequality

$$\left\| \left(\sum_{k,j} |\sigma_{k,j} * g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{L^v(\mathbb{R}^n \times \mathbb{R}^m)} \leq (c_1 C_u)^{\frac{1}{2}} A' B^{\frac{1}{u}} \left\| \left(\sum_{k,j} |g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{L^v(\mathbb{R}^n \times \mathbb{R}^m)},$$

where c_1 and C_u are as in (2.4) and (2.14), respectively.

Proof. The proof is the same way as in one parameter case, and we prove it here for completeness.

Since

$$||\sum_{k,j}|\sigma_{k,j}*g_{k,j}|||_{L^1(\mathbb{R}^n\times\mathbb{R}^m)}\le c_1A'||\sum_{k,j}|g_{k,j}|||_{L^1(\mathbb{R}^n\times\mathbb{R}^m)}$$

and

$$||\sup_{k,j}|\sigma_{k,j}*g_{k,j}|||_{L^u(\mathbb{R}^n\times\mathbb{R}^m)}\le||\sigma*(\sup_{k,j}|g_{k,j}|)||_{L^u(\mathbb{R}^n\times\mathbb{R}^m)}\le C_uA'B^{\frac{2}{u}}||\sup_{k,j}|g_{k,j}|||_{L^u(\mathbb{R}^n\times\mathbb{R}^m)}$$

Interpolation between the above two inequalities completed the proof of the lemma.

By the Littlewood-Paley theory, we have

$$\left\|T_{l,m}f\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\le C_p\left\|\Big(\sum_{k,j}\left|\sigma_{k,j}*(\psi^1_{k+l}\otimes\psi^2_{j+m})*f\right|\Big)^{1/2}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)},\tag{2.15}$$

$$||(\sum_{k,j}|(\psi^1_{k+1}\otimes\psi^2_{j+m})*f|)^{1/2}||_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\le C_p||f||_{L^p(\mathbb{R}^n\times\mathbb{R}^m)},\tag{2.16}$$

where $p\in(1,\infty)$ and C_p is independent of ρ. Suppose that $1+\theta\le p\le\frac{4}{3-\theta}$. Then, we can find $u\in(1+\theta,2]$ such that $\frac{1}{p}=\frac{1}{2}+\frac{1-\theta}{2u}$. Let $v:\frac{1}{v}=\frac{1}{2}+\frac{1}{2u}$, by Lemma 2.3, (2.15) and (2.16), we have

$$||T_{l,m}f||_v\le CA'B^{\frac{1}{u}}||f||_v.$$

Since $\frac{1}{p}=\frac{1-\theta}{v}+\frac{\theta}{2}$, by interpolation, we have

$$||T_{l,m}f||_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\le CA'B^{\frac{1-\theta}{u}}\min\{1,\rho^{-\frac{\theta}{2q'\alpha'}(|m|-1)}\}||f||_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}$$

Then

$$||Tf||_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\le\sum_{l,m}||T_{l,m}f||_p\le CA'B^{\frac{1-\theta}{u}}\left(1-\rho^{-\frac{\theta}{2q'\alpha'}}\right)^{-2}||f||_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}.$$

Since $\rho=2^{q'\alpha'}$, $B=\left(1-2^{-\frac{\theta}{2}}\right)^{-2}$ and $\frac{1-\theta}{u}+1=\frac{2}{p}$, then we have

$$||Tf||_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\le CA'B^{\frac{2}{p}}||f||_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}.\tag{2.17}$$

When $p=2$, by Eq. (2.11) and $B>\left(1-2^{-\frac{1}{2}}\right)^{-2}$, we have

$$||Tf||_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}\le\sum_{l,m}||T_{l,m}f||_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}\le CA'B||f||_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}.$$

By duality and interpolation, we can now finish the proof of Lemma 2.2.

Now, we give a proof of Lemma 2.1. Since $||\mu^*f||_\infty\le c_1A||f||_\infty$, by taking into account an interpolation, it suffices to prove (2.1) for $p\in(1+\theta,2]$. We recall that $\mu_{k,j}=|\sigma_{k,j}|$ and $\mu^*f(x)=\sup_{k,j}|\mu_{k,j}*f(x)|$. The following four estimates for $\mu_{k,j}$ are similar with the equations (2.4) and (2.5):

$$||\mu_{k,j}|| \le A', \tag{2.18}$$

$$|\hat{\mu}_{k,j}(\xi_1, \xi_2) - \hat{\mu}_{k,j}(0, \xi_2)| \le CA'|\rho^k \xi_1|^{\frac{1}{2q'\alpha'}} |\rho^j \xi_2|^{-\frac{1}{2q'\alpha'}}, \tag{2.19}$$

$$|\hat{\mu}_{k,j}(\xi_1, \xi_2) - \hat{\mu}_{k,j}(\xi_1, 0)| \le CA'|\rho^k \xi_1|^{-\frac{1}{2q'\alpha'}} |\rho^j \xi_2|^{\frac{1}{2q'\alpha'}}, \tag{2.20}$$

$$|\hat{\mu}_{k,j}(\xi_1, \xi_2) - \hat{\mu}_{k,j}(\xi_1, 0) - \hat{\mu}_{k,j}(0, \xi_2) + \hat{\mu}_{k,j}(0, 0)| \le CA'|\rho^k \xi_1|^{\frac{1}{2q'\alpha'}} |\rho^j \xi_2|^{\frac{1}{2q'\alpha'}}, \tag{2.21}$$

where C is independent of q, Ω, h, α. Choose positive real value functions $\phi_j \in C_0^\infty(\mathbb{R})$ $(j = 1, 2)$ satisfying $supp(\varphi_j) \subset \{|r| < 1\}$ and $\varphi_j = 1$, when $|r| < \frac{1}{2}$. Define

$$(\Phi_k^1)^\Lambda(\xi_1) = \phi_1(|\rho^k \xi_1|),$$
$$(\Phi_j^2)^\Lambda(\xi_2) = \phi_2(|\rho^j \xi_2|),$$

and measures

$$\begin{aligned} \hat{\tau}_{k,j}(\xi) &= \hat{\mu}_{k,j}(\xi) - (\Phi_k^1)^\Lambda(\xi_1)\hat{\mu}_{k,j}(0, \xi_2) \\ &-(\Phi_j^2)^\Lambda(\xi_2)\hat{\mu}_{k,j}(\xi_1, 0) + (\Phi_k^1)^\Lambda(\xi_1)(\Phi_j^2)^\Lambda(\xi_2)\hat{\mu}_{k,j}(0, 0). \end{aligned} \tag{2.22}$$

So by the definition of $\tau_{k,j}$ and estimates (2.18)-(2.21), it is easy to check that $\tau_{k,j}$ satisfies the same estimates as $\sigma_{k,j}$, i.e.,

$$|\hat{\tau}_{k,j}(\xi_1, \xi_2)| \le CA'|\rho^k \xi_1|^{\pm\frac{1}{2q'\alpha'}} |\rho^j \xi_2|^{\pm\frac{1}{2q'\alpha'}}, \tag{2.23}$$

where C is independent of q, α and Ω, h. Also we have

$$\begin{aligned} \mu^* f(x_1, x_2) \le \sup_{k,j}(\Phi_k^1 \otimes \mu_{k,j}^{(1)}) * f(x_1, x_2) + \sup_{k,j}(\mu_{k,j}^{(2)} \otimes \Phi_j^2) * f(x_1, x_2) \\ + \sup_{k,j}(\mu_{k,j}^{(1,2)} \otimes \Phi_k^1 \otimes \Phi_j^2) * f(x_1, x_2) + g(f)(x_1, x_2), \end{aligned} \tag{2.24}$$

where

$$g(f)(x_1, x_2) = (\sum_{k,j} |\tau_{k,j} * f(x_1, x_2)|^2)^{\frac{1}{2}}$$

and $\mu_{k,j}^{(1)}$, $\mu_{k,j}^{(2)}$ and $\mu_{k,j}^{(1,2)}$ defined as follows:

$$\hat{\mu}_{k,j}^{(1)}(\xi_2) = \hat{\mu}_{k,j}(0, \xi_2), \hat{\mu}_{k,j}^{(2)}(\xi_1) = \hat{\mu}_{k,j}(\xi_1, 0), \hat{\mu}_{k,j}^{(1,2)}(\xi_1, \xi_2) = \hat{\mu}_{k,j}(0, 0).$$

Then, we have

$$\begin{aligned} \sup_{k,j}(\Phi_j^1 \otimes \mu_{k,j}^{(1)} * f(x_1, x_2) &\le CM_1 M^{(1)} f(x_1, x_2) \\ \sup_{k,j}(\mu_{k,j}^{(2)} \otimes \Phi_j^2) * f(x_1, x_2) &\le CM_2 M^{(2)} f(x_1, x_2) \\ \sup_{k,j}(\mu_{k,j}^{(1,2)} \otimes \Phi_k^1 \otimes \Phi_j^2) * f(x_1, x_2) &\le CM_1 M_2 f(x_1, x_2)\hat{\mu}_{k,j}(0, 0) \end{aligned} \tag{2.25}$$

where M_i is the Hardy-Littlewood maximal function acting on the x_i-variable and $M^{(i)}$ is the partial maximal function, defined as the following

$$M^{(i)}g_i = \sup_{k,j} |\mu_{k,j}^{(i)} * g_i|, i = 1, 2. \tag{2.26}$$

Since

$$\begin{aligned} M^{(1)}g_1(x_2) &\le \sup_{k,j} \int_{\rho^k}^{\rho^{k+1}} \int_{\rho^j}^{\rho^{j+1}} \int_{S^{n-1}\times S^{m-1}} |\Omega(u,v)||h(r,s)||g(x_2 - sv)| \mathrm{d}\sigma(u)\mathrm{d}\sigma(v) \frac{\mathrm{d}r\mathrm{d}s}{rs} \\ &\le \sup_{k,j} C \log\rho \int_{\rho^j}^{\rho^{j+1}} \int_{S^{m-1}} \int_{S^{n-1}} |\Omega(u,v)|\mathrm{d}\sigma(u) \int_{2^k}^{2^{k+1}} |h(r,s)| \frac{\mathrm{d}r}{r} |g(x_2 - sv)| \mathrm{d}\sigma(v) \frac{\mathrm{d}s}{s}. \end{aligned} \tag{2.27}$$

We let $\bar{h}(s) = \sup_k \int_{2^k}^{2^{k+1}} |h(r,s)| \frac{\mathrm{d}r}{r}$ and $\bar{\Omega}(v) = \int_{S^{n-1}} |\Omega(u,v)|\mathrm{d}\sigma(u)$. Since $h \in \widetilde{\Delta}_\alpha$ and $\Omega \in L^q(S^{n-1} \times S^{m-1})$, then $\bar{h} \in \Delta_\alpha(\mathbb{R}^+)$ and $\bar{\Omega} \in L^q(S^{m-1})$. By Lemma 1 of [8], the one-parameter case, we have for $p > 1 + \theta$,

$$\begin{aligned} ||M^{(1)}g_1||_{L^p(\mathbb{R}^m)} &\le C\log^2\rho ||\Omega||_{L^q(S^{n-1}\times S^{m-1})} ||h||_{\widetilde{\Delta}_\alpha} (1 - 2^{-\frac{\theta}{2}})^{-\frac{2}{p}} ||g_1||_{L^p(\mathbb{R}^m)} \\ &\le CA'B^{\frac{2}{p}} ||g_1||_{L^p(\mathbb{R}^m)}, \end{aligned} \tag{2.28}$$

and the same way we have

$$\begin{aligned} \| M^{(2)}g_2\|_{L^p(\mathbb{R}^n)} &\le C\log^2\rho \ \| \Omega\|_{L^q(S^{n-1}\times S^{m-1})} \ \| h\|_{\widetilde{\Delta}_\alpha} (1 - 2^{-\frac{\theta}{p}})^{-\frac{2}{p}} \ \| g_2\|_{L^p(\mathbb{R}^n)} \\ &\le CA'B^{\frac{2}{p}} ||g_2||_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{2.29}$$

On the other hand, it is easy to check,

$$\sup_{k,j} \mu_{k,j}^{(1,2)} * f(x_1, x_2) \le C\log^2\rho \ \| \Omega\|_{L^q(S^{n-1}\times S^{m-1})} \ \| h\|_{\Delta_\alpha} |f(x_1, x_2)|. \tag{2.30}$$

So with (2.28)-(2.30) and (2.25), we concluded that for $p \in (1 + \theta, 2]$,

$$\begin{aligned} &\| \sup_{k.j} \left(\Phi_j^1 \otimes \mu_{k.j}^{(1)}\right) * f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \le CA'B^{\frac{2}{p}} \ \| f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}, \\ &\| \sup_{k,j} \left(\mu_{k,j}^{(2)} \otimes \Phi_j^2\right) * f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \le CA'B^{\frac{2}{p}} \ \| f\|_{L^p(R^n\times R^m)}, \\ &\| \sup_{k,j} \left(\mu_{k,j}^{(1,2)} \otimes \Phi_k^1 \otimes \Phi_j^2\right) * f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \le CA'B^{\frac{2}{p}} \ \| f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}. \end{aligned} \tag{2.31}$$

To prove Lemma 2.1, it suffices to prove $\| g(f)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \le CAB^{\frac{2}{p}} \ \| f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}$ for $p \in (1 + \theta, 2]$. By a well-known property of Rademacher's function, this follows from

$$\| U(f)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \le CA'B^{\frac{2}{p}} \ \| f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}, \tag{2.32}$$

for $p \in (1 + \theta, 2]$, where $U(f) = \sum_{k,j} \varepsilon_{k,j}\tau_{k,j} * f$ with $\varepsilon_{k,j} = 1$ or -1, and the constant C is independent of $\varepsilon_{k,j}$. The estimate (2.32) is a consequence of the following lemma:

Lemma 2.4. We define a sequence $\{p_j\}_{j=1}^{\infty}$ by $p_1 = 2$ and $\frac{1}{p_{j+1}} = \frac{1}{2} + \frac{1-\theta}{2p_j}$ for $j \ge 1$. (We note that $\frac{1}{p_j} = \frac{1-a^j}{1+\theta}$, where $a = \frac{1-\theta}{2}$, so $\{p_j\}$ is decreasing and converges to $1 + \theta$.) Then, for $j \ge 1$ we have

$$\| U(f)\|_{L^{p_j}(\mathbb{R}^n\times\mathbb{R}^m)} \le CA'B^{2/p_j} \| f\|_{L^{p_j}(\mathbb{R}^n\times\mathbb{R}^m)}. \tag{2.33}$$

Proof. Let

$$U_{k,m}(f) = \sum_{k,j} \varepsilon_{k,j}\tau_{k,j} * (\not{\psi}^1_{k+l}\otimes \not{\psi}^2_{j+m}) * (\not{\psi}^1_{k+l}\otimes \not{\psi}^2_{j+m}) * f$$

By Plancherel's theorem and the estimates (2.23), the same way as in (2.11), we have that

$$\| U_{l,m}(f) \|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} \le CA'^2 \min\{1, \rho^{-2\frac{1}{2q'\alpha'}(|l|-1)}\} \min\{1, \rho^{-2\frac{1}{2q'\alpha'}(|m|-1)}\} \| f \|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}. \tag{2.34}$$

It follows that $\| U(f)\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} \le \sum_{l,m} \| U_{k,m}(f)\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} \le CAB \| f\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}$. If we denote by $A(s)$ the claim of Lemma 2.4 for $j = s$, this proves $A(1)$.

Now, we derive $A(s + 1)$ from $A(s)$ assuming that $A(s)$ holds, which will complete the proof of Lemma 2.4 by induction. By (2.22) and (2.24), we have that

$$\begin{aligned}
\tau^*(f)(x) &\le \mu^*(|f|)(x) + \sup_{k,j}(\Phi^1_k \otimes \mu^{(1)}_{k,j}) * f(x) + \sup_{k,j}(\mu^{(2)}_{k,j} \otimes \Phi^2_j) * f(x) \\
&\quad + \sup_{k,j}(\mu^{(1,2)}_{k,j} \otimes \Phi^1_k \otimes \Phi^2_j) * f(x) \\
&\le g(f)(x) + 2(\sup_{k,j}(\Phi^1_k \otimes \mu^{(1)}_{k,j}) * f(x) + \sup_{k,j}(\mu^{(2)}_{k,j} \otimes \Phi^2_j) * f(x) \\
&\quad + \sup_{k,j}(\mu^{(1,2)}_{k,j} \otimes \Phi^1_k \otimes \Phi^2_j) * f(x))
\end{aligned} \tag{2.35}$$

Note that A_s means that $\| g(f)\|_{p_s} \le CA'B^{\frac{2}{p_s}} \| f\|_{p_s}$. By (2.35) and (2.31) we have

$$\begin{aligned}
\| \tau^*(f)\|_{L^{p_s}(\mathbb{R}^n\times\mathbb{R}^m)} &\le \| g(f)\|_{L^{p_s}(\mathbb{R}^n\times\mathbb{R}^m)} + 2(\| \sup_{k,j}(\Phi^1_j \otimes \mu^{(1)}_{k,j}) * f\|_{L^{p_s}(\mathbb{R}^n\times\mathbb{R}^m)} \\
&\quad + \| \sup_{k,j}(\mu^{(2)}_{k,j} \otimes \Phi^2_j) * f\|_{L^{p_s}(\mathbb{R}^n\times\mathbb{R}^m)} + \| \sup_{k,j}(\mu^{(1,2)}_{k,j} \otimes \Phi^1_k \otimes \Phi^2_j) * f\|_{L^{p_s}(\mathbb{R}^n\times\mathbb{R}^m)} \\
&\le CA'B^{\frac{2}{p_s}} \| f\|_{L^{p_s}(\mathbb{R}^n\times\mathbb{R}^m)}.
\end{aligned} \tag{2.36}$$

By (2.36) and (2.34), we can now apply the arguments used in the proof of (2.17) to get $A(s + 1)$. This completes the proof of Lemma 2.4.

Now, we prove the inequality (2.32) for $p \in (1 + \theta, 2]$. Let $\{p_j\}_{j=1}^{\infty}$ be as in Lemma 2.4. Then, we have $p_{N+1} \le p \le p_N$ for some N. Thus, interpolation between the estimates of Lemma 2.4 for $j = N$ and $j = N + 1$, we have (2.36). This completes the proof of Lemma 2.1.

3 Proofs of Theorem 1.1 and Corollary 1.1

Proof of Theorem 1.1: We first need to establish a suitable decomposition for Ω defined on $S^{n-1} \times S^{m-1}$. The main technique is mainly based on Chen [18]. Define a sequence of sets $\{F_k\}(k \in \mathbb{N})$ on $S^{n-1} \times S^{m-1}$ as:

$$F_\kappa = \{(\theta, w) \in S^{n-1} \times S^{m-1} : 2^{\kappa-1} \le |\Omega(\theta, w)| < 2^\kappa\} \text{ for } \kappa = 2, 3, \ldots$$

and

$$F_1 = \{(\theta, w) \in S^{n-1} \times S^{m-1} : |\Omega(\theta, w)| < 2\} \text{ for } \kappa = 1.$$

We define

$$\Omega_\kappa(\theta,w)=\Omega\chi_{F_\kappa}(\theta,w)-\frac{1}{\sigma(S^{n-1})}\int_{S^{n-1}}\Omega\chi_{F_\kappa}(u,w)\mathrm{d}\sigma(u)-\frac{1}{\sigma(S^{m-1})}\int_{S^{m-1}}\Omega\chi_{F_\kappa}(\theta,v)\mathrm{d}\sigma(v)+\frac{1}{\sigma(S^{n-1})\sigma(S^{m-1})}\int_{S^{n-1}\times S^{m-1}}\Omega\chi_{F_\kappa}(u,v)\mathrm{d}\sigma(u)\sigma(v).$$

Then, it is easy to check that

$$\Omega(\theta,w)=\sum_{\kappa=1}^{\infty}\Omega_\kappa(\theta,w) \tag{3.1}$$

and all Ω_κ satisfies the condition (1.3), i.e.,

$$\int_{S^{n-1}}\Omega_\kappa(\theta,w)\mathrm{d}\sigma(\theta)=\int_{S^{m-1}}\Omega_\kappa(\theta,w)\mathrm{d}\sigma(w)=0 \tag{3.2}$$

Furthermore, if we set $e_\kappa=\sigma(F_\kappa)=\int_{F_\kappa}\mathrm{d}\sigma(u)d\sigma(v)$, then for $r\in(1,\infty)$, we have

$$\|\Omega_\kappa\|_{L^r(S^{n-1}\times S^{m-1})}\le C2^\kappa e_\kappa^{\frac{1}{r}}\ \text{for}\ \kappa\in\mathbb{N}. \tag{3.3}$$

Now, fix $h\in\tilde{\Delta}_\alpha(\alpha\in(1,2]), p\in(1,\infty)$ and a function f with $\|f\|_{L^p(\mathbb{R}^{n+m})}\le 1$, we denote $R(Tf,\Omega)=\|T_{\Omega,h}f\|_{L^p(\mathbb{R}^{n+m})}$. Then, by Theorem 1.2, Eqs. (3.1), (3.2) and (3.3), we have

$$\begin{aligned}R(Tf,\Omega)&\le\sum_{\kappa=1}^{\infty}R(Tf,\Omega_\kappa)\le C\frac{1}{(\alpha-1)^2}\|h\|_{\tilde{\Delta}_\alpha}\sum_{\kappa=1}^{\infty}\kappa^2\|\Omega_\kappa\|_{L^{1+\frac{1}{\kappa}}(S^{n-1}\times S^{m-1})}\\&\le C\frac{1}{(\alpha-1)^2}\|h\|_{\tilde{\Delta}_\alpha}\sum_{\kappa=1}^{\infty}\kappa^2 2^\kappa e_\kappa^{\frac{\kappa}{\kappa+1}}\\&\le C\frac{1}{(\alpha-1)^2}\|h\|_{\tilde{\Delta}_\alpha}\left(\sum_{e_\kappa<3^{-\kappa}}+\sum_{e_\kappa\ge3^{-\kappa}}\right)\kappa^2 2^\kappa e_\kappa^{\frac{\kappa}{\kappa+1}}\\&\le C\frac{1}{(\alpha-1)^2}\|h\|_{\tilde{\Delta}_\alpha}\left(\sum_{\kappa\ge1}\kappa^2 2^\kappa 3^{-\frac{\kappa^2}{\kappa+1}}+\sum_{\kappa\ge1}\kappa^2 2^\kappa e_\kappa 3^{\frac{\kappa}{\kappa+1}}\right)\\&\le C\frac{1}{(\alpha-1)^2}\|h\|_{\tilde{\Delta}_\alpha}\left(1+\int_{S^{n-1}\times S^{m-1}}|\Omega(\theta,\omega)|\log^2(2+|\Omega(\theta,\omega)|)\mathrm{d}\sigma(\theta)\mathrm{d}\sigma(\omega)\right).\end{aligned} \tag{3.4}$$

For $p=2$ and a function f with $\|f\|_{L^2(\mathbb{R}^{n+m})}\le 1$. Denote $O(h)=\|T_hf\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}$. Suppose that $h\in\Delta_\alpha(\alpha\in(1,2])$ and $\Omega\in L(\log L)^2(S^{n-1}\times S^{m-1})$, with the same estimate as in (3.4), we have that

$$O(h)\le C\frac{1}{(\alpha-1)^2}\|h\|_{\Delta_\alpha(\mathbb{R}^+\times\mathbb{R}^+)} \tag{3.5}$$

Put $E_1=\{(r,s)\in\mathbb{R}^+\times\mathbb{R}^+:|h(r,s)|\le 2\}$ and $E_m=\{(r,s)\in\mathbb{R}^+\times\mathbb{R}^+:2^{m-1}<|h(r,s)|\le 2^m\}$ for $m=2,3,\ldots$. Then, by (3.5), we have

$$O(h_{\chi E_m})=\|T_{h_{\chi E_m}}\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}\le\frac{C}{(\alpha-1)^2}\|h_{\chi E_m}\|_{\Delta_\alpha(\mathbb{R}^+\times\mathbb{R}^+)}.$$

We follow the extrapolation argument of Zygmund [7]. First we note that

$$\|h_{\chi E_m}\|_{\Delta_{1+\frac{1}{m}}(\mathbb{R}^+\times\mathbb{R}^+)}\le 2^m D_m^{\frac{m}{m+1}}(h) \tag{3.6}$$

for $m \geq 1$, where $D_m(h)$ be as in the definition of N_α in Section 1. By (3.5) and (3.6), we have

$$\begin{aligned} O(h) &\leq \sum_{m\geq 1} O(h\chi_{E_m}) \\ &\leq C\sum_{m\geq 1} m^2 \| h\chi_{E_m}\|_{\Delta_{1+\frac{1}{m}}(\mathbb{R}^+\times\mathbb{R}^+)} \\ &\leq C\sum_{m\geq 1} m^2 2^m D_m^{\frac{m}{m+1}}(h) \\ &= C\left(\sum_{D_m(h)<3^{-m}} + \sum_{D_m(h)\geq 3^{-m}}\right) m^2 2^m D_m^{\frac{m}{m+1}}(h) \\ &\leq C\left(1+ \| h\|_{N_2(\mathbb{R}^+\times\mathbb{R}^+)}\right) \end{aligned}$$

This ends the proof of Theorem 1.1.

Proof of Corollary 1.1: Since h can be written as separate case $h_1(r) \cdot h_2(s)$, we deal (1) and (2) by the same procession. We only need to prove part (1) of the corollary. Suppose that $\Omega \in L(\log L)^2(S^{n-1} \times S^{m-1})$ and $h_1, h_2 \in \Delta_\alpha(\mathbb{R}^+)$ for $\alpha \in (1, 2]$, then $h = h_1 h_2 \in \tilde{\Delta}_\alpha$. By part (2) of Theorem 1.1, for $p \in (1, \infty)$ we have

$$\begin{aligned} \| Tf\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} &\leq C\frac{1}{(\alpha-1)^2} \| h\|_{\tilde{\Delta}_\alpha} \| f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \\ &\leq C\frac{1}{(\alpha-1)^2} \| h_1\|_{\Delta_\alpha(\mathbb{R}^+)} \| h_2\|_{\Delta_\alpha(\mathbb{R}^+)} \| f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \end{aligned} \tag{3.7}$$

Suppose that $\| \Omega\|_{L(\log L)^2(S^{n-1}\times S^{m-1})} \leq 1$, $\| f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq 1$ and $h_1 \in \Delta_\alpha(\mathbb{R}^+)$ with $\| h_1\|_{\Delta_\alpha(\mathbb{R}^+)} \leq 1$. We define $U(h_2) =\| T_{h_2}f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}$. Set $E_1 = \{r \in \mathbb{R}^+ : |h_2(r)| \leq 2\}$ and $E_m = \{r \in \mathbb{R}^+ : 2^{m-1} < |h_2(r)| \leq 2^m\}$ for $m \geq 2$. Then, by (3.7), there exists a constant C, which is independent of α such that

$$U(h_2\chi_{E_m}) \leq \frac{C}{(\alpha-1)^2} \| h_2\chi_{E_m}\|_{\Delta_\alpha(\mathbb{R}^+)} \tag{3.8}$$

for $\alpha \in (1, 2]$. We note

$$\| h_2\chi_{E_m}\|_{\Delta_{1+\frac{1}{m}}(\mathbb{R}^+)} \leq 2^m d_m^{\frac{m}{m+1}}(h_2) \tag{3.9}$$

for $m \geq 1$, where $d_m(h)$ is as in Section 1. By (3.8) and (3.9), we have

$$\begin{aligned} U(h) &\leq \sum_{m\geq 1} O(h_2\chi_{E_m}) \\ &\leq C\sum_{m\geq 1} m^2 \| h_2\chi_{E_m}\|_{\Delta_{1+\frac{1}{m}}(\mathbb{R}^+)} \\ &\leq C\sum_{m\geq 1} m^2 2^m d_m^{\frac{m}{m+1}}(h_2) \\ &= C\left(\sum_{D_m(h_1)<3^{-m}} + \sum_{d_m(h_2)\geq 3^{-m}}\right) m^2 2^m d_m^{\frac{m}{m+1}}(h_2) \\ &\leq C(1 + N_2(h_2)), \end{aligned}$$

which finishes the proof of the corollary.

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Competing interests
The authors declare that they have no competing interests.

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