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On the maximum modulus of a polynomial and its polar derivative

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Abstract

For a polynomial $p(z)$ of degree n, having all zeros in $|z| \leq 1$, Jain is shown that

$$\max_{|z|=1} \left|D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} p(z)\right| \geq \frac{n(n-1)\cdots(n-t+1)}{2^t} \times \Bigg[\{(|\alpha_1|-1)\cdots(|\alpha_t|-1)\} \max_{|z|=1} |p(z)| + \left\{2^t\left(|\alpha_1|\cdots|\alpha_t|\right) - \{(|\alpha_1|-1)\cdots(|\alpha_t|-1)\}\right\} \min_{|z|=1} |p(z)| \Bigg],$$

$$|\alpha_1| \geq 1, \quad |\alpha_2| \geq 1, \cdots |\alpha_t| \geq 1, \quad (t < n).$$

In this paper, the above inequality is extended for the polynomials having all zeros in $|z| \leq k$, where $k \leq 1$. Our result generalizes certain well-known polynomial inequalities.

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1. Introduction and statement of results

Let $p(z)$ be a polynomial of degree n, then according to the well-known Bernstein's inequality [1] on the derivative of a polynomial, we have

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}$$

This result is best possible and equality holding for a polynomial that has all zeros at the origin.

If we restrict to the class of polynomials which have all zeros in $|z| \leq 1$, then it has been proved by Turan [2] that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

The inequality (1.2) is sharp and equality holds for a polynomial that has all zeros on $|z| = 1$.

As an extension to (1.2), Malik [3] proved that if $p(z)$ has all zeros in $|z| \leq k$, where $k \leq 1$, then

$$\max_{|z|=1}\left|p'(z)\right| \geq \frac{n}{1+k}\max_{|z|=1}\left|p(z)\right|. \tag{1.3}$$

This result is best possible and equality holds for $p(z) = (z - k)^n$.

Aziz and Dawood [4] obtained the following refinement of the inequality (1.2) and proved that if $p(z)$ has all zeros in $|z| \leq 1$, then

$$\max_{|z|=1}\left|p'(z)\right| \geq \frac{n}{2}\left\{\max_{|z|=1}\left|p(z)\right| + \min_{|z|=1}\left|p(z)\right|\right\}. \tag{1.4}$$

This result is best possible and equality attains for a polynomial that has all zeros on $|z| = 1$.

Let $D_\alpha p(z)$ denote the polar differentiation of the polynomial $p(z)$ of degree n with respect to $\alpha \in \mathbb{C}$. Then, $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$. The polynomial $D_\alpha p(z)$ is of degree at most $n - 1$, and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left[\frac{D_\alpha p(z)}{\alpha}\right] = p'(z).$$

Shah [5] extended (1.2) to the polar derivative of $p(z)$ and proved that if all zeros of the polynomial $p(z)$ lie in $|z| \leq 1$, then for every α with $|\alpha| \geq 1$, we have

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \geq \frac{n}{2}\left(|\alpha| - 1\right)\max_{|z|=1}\left|p(z)\right|. \tag{1.5}$$

This result is best possible and equality holds *as* $p(z) = (z - 1)^n$ with $\alpha \geq 1$.

Aziz and Rather [6] generalized (1.5) by extending (1.3) to the polar derivative of a polynomial. In fact, they proved that if all zeros of $p(z)$ lie in $|z| \leq k$, where $k \leq 1$, then for every α with $|\alpha| \geq k$, we get

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \geq \frac{n}{1+k}\left(|\alpha| - k\right)\max_{|z|=1}\left|p(z)\right|. \tag{1.6}$$

This result is best possible and equality holds for $p(z) = (z - k)^n$ with $\alpha \geq k$.

In the same paper, Aziz and Rather [6] sharpened the inequality (1.5) by proving that if all the zeros of $p(z)$ lie in $|z| \leq 1$, then for every α with $|\alpha| \geq 1$, we would obtain

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \geq \frac{n}{2}\left\{\left(|\alpha| - 1\right)\max_{|z|=1}\left|p(z)\right| + \left(|\alpha| - 1\right)\min_{|z|=1}\left|p(z)\right|\right\}. \tag{1.7}$$

This result is best possible and equality attains for $p(z) = (z - 1)^n$ with $\alpha \geq 1$.

As an extension to the inequality (1.7), Jain [7] proved that if $p(z)$ has all zeros in $|z| \leq 1$, then for all $\alpha_1,\ldots \alpha_t \in \mathbb{C}$ with $|\alpha_1| \geq 1$, $|\alpha_2| \geq 1$, ..., $|\alpha_t| \geq 1$, $(1 \leq t < n)$, we have

$$\begin{aligned}\max_{|z|=1}\left|D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}p(z)\right| \geq \frac{n(n-1)\cdots(n-t+1)}{2^t}\Big[\\ \left\{\left(|\alpha_1| - 1\right)\cdots\left(|\alpha_t| - 1\right)\right\}\max_{|z|=1}\left|p(z)\right| + \\ \left\{2^t\left(|\alpha_1|\cdots|\alpha_t|\right) - \left\{\left(|\alpha_1| - 1\right)\cdots\left(|\alpha_t| - 1\right)\right\}\right\}\min_{|z|=1}\left|p(z)\right|\Big],\end{aligned} \tag{1.8}$$

where

$$D_{\alpha_j} D_{\alpha_{j-1}} \cdots D_{\alpha_1} p(z) = p_j(z) = \\ (n-j+1) p_{j-1}(z) + (\alpha_j - z) p_{j-1'}(z), \quad j = 1, 2, \cdots, t, \\ p_0(z) = p(z).$$

This result is best possible and equality holds as $p(z) = (z - 1)^n$ with $\alpha_1 \geq 1$, $\alpha_2 \geq 1$,..., $\alpha_t \geq 1$.

The following result proposes an extension to (1.8). In a precise set up, we have

Theorem 1.1. *Let $p(z)$ be a polynomial of degree n having all zeros in $|z| \leq k$, where $k \leq 1$, then for all $\alpha_1, \ldots \alpha_t \in \mathbb{C}$ with $|\alpha_1| \geq k$, $|\alpha_2| \geq k$,..., $|\alpha_t| \geq k$, $(1 \leq t < n)$,*

$$\max_{|z|=1} \left| D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} p(z) \right| \geq \frac{n(n-1)\cdots(n-t+1)}{(1+k)^t} \Big[\\ \{(|\alpha_1| - k) \cdots (|\alpha_t| - k)\} \max_{|z|=1} |p(z)| + \\ \left\{ (1+k)^t (|\alpha_1| \cdots |\alpha_t|) - \{(|\alpha_1| - k) \cdots (|\alpha_t| - k)\} \right\} k^{-n} \min_{|z|=k} |p(z)| \Big]. \tag{1.9}$$

This result is best possible and equality holds for $p(z) = (z - k)^n$ with $\alpha_1 \geq k$, $\alpha_2 \geq k$,..., $\alpha_t \geq k$.

If we take $k = 1$ in Theorem 1.1, then inequality (1.9) reduces to inequality (1.8).

If we take $t = 1$ in Theorem 1.1, the following refinement of inequality (1.6) can be obtained.

Corollary 1.2. Let $p(z)$ be a polynomial of degree n, having all zeros in $|z| \leq k$, where $k \leq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,

$$\max_{|z|=1} \left| D_\alpha p(z) \right| \geq \frac{n}{1+k} \left\{ (|\alpha| - k) \max_{|z|=1} |p(z)| + (|\alpha| + 1) k^{-(n-1)} \min_{|z|=k} |p(z)| \right\}. \tag{1.10}$$

This result is best possible and equality occurs if $p(z) = (z - k)^n$ with $\alpha \geq k$.

If we divide both sides of the above inequality in (1.10) by $|\alpha|$ and make $|\alpha| \to \infty$, we obtain a result proved by Govil [8].

2. Lemmas

For proof of the theorem, the following lemmas are needed. The first lemma is due to Laguerre [9].

Lemma 2.1. *If all the zeros of an nth degree polynomial $p(z)$ lie in a circular region C and w is any zero of $D_\alpha p(z)$, then at most one of the points w and α may lie outside C.*

Lemma 2.2. *If $p(z)$ is a polynomial of degree n, having all zeros in the closed disk $|z| \leq k$, $k \leq 1$, then on $|z| = 1$,*

$$\left| p'(z) \right| \geq \frac{n}{1+k} \left| p(z) \right|. \tag{2.1}$$

This lemma is due to Govil [10].

Lemma 2.3. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < k$, $k \geq 1$, then on $|z| = 1$,*

$$k \left| p'(z) \right| \leq \left| q'(z) \right|, \tag{2.2}$$

where $q(z) = z^n \overline{p(1/\bar{z})}$.

The above lemma is due to Chan and Malik [11].

Lemma 2.4. *If $p(z)$ is a polynomial of degree n, having all zeros in the closed disk $|z| \le k$, $k \le 1$, then on $|z| = 1$,*

$$\left|q'(z)\right| \le k\left|p'(z)\right|, \tag{2.3}$$

where $q(z) = z^n\overline{p(1/\bar{z})}$.

Proof. Since $p(z)$ has all its zeros in $|z| \le k$, $k \le 1$, therefore $q(z)$ has no zero in $|z| < 1/k$, $1/k \ge 1$. Now applying Lemma 2.3 to the polynomial $q(z)$ and the result follows.

Lemma 2.5. *If $p(z)$ is a polynomial of degree n, having all zeros in the closed disk $|z| \le k$, $k \le 1$, then for every real or complex number α with $|\alpha| \ge k$ and $|z| = 1$, we have*

$$\left|D_\alpha p(z)\right| \ge \frac{n}{1+k}\left(|\alpha| - k\right)\left|p(z)\right|. \tag{2.4}$$

Proof. Let $q(z) = z^n\overline{p(1/\bar{z})}$, then $|q'(z)| = |np(z) - zp'(z)|$ on $|z| = 1$. Thus, on $|z| = 1$, we get

$$\begin{aligned}\left|D_\alpha p(z)\right| = \left|np(z) + (\alpha - z)p'(z)\right| = \left|\alpha p'(z) + np(z) - zp'(z)\right| \ge \\ \left|\alpha p'(z) - |np(z) - zp'(z)\right|,\end{aligned}$$

that implies

$$\left|D_\alpha p(z)\right| \ge |\alpha|\left|p'(z)\right| - \left|q'(z)\right|. \tag{2.5}$$

By combining (2.3) and (2.5), we obtain

$$\left|D_\alpha p(z)\right| \ge \left(|\alpha| - k\right)\left|p'(z)\right|.$$

that along Lemma 2.2, yields

$$\left|D_\alpha p(z)\right| \ge \frac{n}{1+k}\left(|\alpha| - k\right)\left|p(z)\right|.$$

Lemma 2.6. *If* $p(z) = a_0 + a_1 z + \sum_{i=2}^{n} a_i z^i$ *is a polynomial of degree n, having no zeros in $|z| < k$, $k \ge 1$, then*

$$\frac{k\,|a_1|}{|a_0|} \le n. \tag{2.6}$$

The above lemma is due to Gardner et al. [12].

Lemma 2.7. *If* $p(z) = \sum_{i=0}^{n} a_i z^i$ *is a polynomial of degree n, having all zeros in $|z| \le k$, $k \le 1$, then*

$$\frac{|a_{n-1}|}{|a_n|} \le nk. \tag{2.7}$$

Proof. Since $p(z)$ has all zeros in $|z| \le k$, $k \le 1$, therefore

$$q(z) = z^n\overline{p(1/\bar{z})} = \overline{a_n} + \overline{a_{n-1}}z + \cdots + \overline{a_1}z^{n-1} + \overline{a_0}z^n,$$

is a polynomial of degree at most n, which does not vanish in $|z| < 1/k$, $1/k \ge 1$. By applying Lemma 2.6 for $q(z)$, we get

$$\frac{\frac{1}{k}|a_{n-1}|}{|a_n|} \leq \text{degree}\{q(z)\} \leq n,$$

which completes the proof.

Lemma 2.8. *If* $p(z)$ *is a polynomial of degree* n *having all zeros in* $|z| \leq k$, $k \leq 1$, *then for all* $\alpha_1, \ldots \alpha_t \in \mathbb{C}$ *with* $|\alpha_1| \geq k$, $|\alpha_2| \geq k, \ldots, |\alpha_t| \geq k$, $(1 \leq t < n)$, *and* $|z| = 1$ *we have*

$$\left|D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} p(z)\right| \geq \frac{n(n-1)\cdots(n-t+1)}{(1+k)^t} \times \{(|\alpha_1| - k) \cdots (|\alpha_t| - k)\} \left|p(z)\right|. \tag{2.8}$$

Proof. If $|\alpha_j| = k$ for at least one j; $1 \leq j \leq t$, then inequality (2.8) is trivial. Therefore, we assume that $|\alpha_j| > k$ for all j; $1 \leq j \leq t$.

In the rest, we proceed by mathematical induction. The result is true for $t = 1$, by Lemma 2.5, that means if $|\alpha_1| > k$ then

$$\left|D_{\alpha_1} p(z)\right| \geq \frac{n}{1+k}(|\alpha_1| - k)\left|p(z)\right|. \tag{2.9}$$

Now for $t = 2$, since $D_{\alpha_1} p(z) = (na_n\alpha_1 + a_{n-1}) z^{n-1} + \cdots + (na_0 + \alpha_1 a_1)$, and $|\alpha_1| > k$, then $D_{\alpha_1} p(z)$ will be a polynomial of degree $(n - 1)$. If it is not true, then the coefficient of z^{n-1} must be equal to zero, which implies

$$na_n\alpha_1 + a_{n-1} = 0,$$

i.e,

$$|\alpha_1| = \frac{|a_{n-1}|}{n\,|a_n|}.$$

Applying Lemma 2.7, we get

$$|\alpha_1| = \frac{|a_{n-1}|}{n\,|a_n|} \leq k.$$

But this result contradicts the fact that $|\alpha_1| > k$. Hence, the polynomial $D_{\alpha_1} p(z)$ must be of degree $(n - 1)$.

On the other hand, since all the zeros of $p(z)$ lie in $|z| \leq k$, therefore by applying Lemma 2.1, all the zeros of $D_{\alpha_1} p(z)$ lie in $|z| \leq k$, then using Lemma 2.5 for the polynomial $D_{\alpha_1} p(z)$ of degree $n - 1$, and $|\alpha_2| > k$, it concludes that

$$\left|D_{\alpha_2}\left\{D_{\alpha_1} p(z)\right\}\right| \geq \frac{(n-1)}{1+k}(|\alpha_2| - k)\left|D_{\alpha_1} p(z)\right|.$$

Substituting the term $D_{\alpha_1} p(z)$ from (2.9) in the above inequality, we obtain

$$\left|D_{\alpha_2} D_{\alpha_1} p(z)\right| \geq \frac{n(n-1)}{(1+k)^2}(|\alpha_1| - k)(|\alpha_2| - k)\left|p(z)\right|.$$

This implies result is true for $t = 2$.

At this stage, we assume that the result is true for $t = s < n$; it means that for $|z| = 1$, we have

$$\left|D_{\alpha_s}\cdots D_{\alpha_2}D_{\alpha_1}p(z)\right| \geq \frac{n(n-1)\cdots(n-s+1)}{(1+k)^s}\times \\ \{(|\alpha_1|-k)\cdots(|\alpha_s|-k)\}\left|p(z)\right|, \tag{2.10}$$

and we will prove that the result is true for $t = s + 1 < n$.

According to the above procedure, using Lemmas 2.7 and 2.1, the polynomial $D_{\alpha_2}D_{\alpha_1}p(z)$ must be of degree $(n - 2)$ for $|\alpha_1| > k$, $|\alpha_2| > k$, and has all zeros in $|z| \leq k$. One can continue that $D_{\alpha_s}\cdots D_{\alpha_2}D_{\alpha_1}p(z)$ will be a polynomial of degree $(n - s)$ for all $\alpha_1,\ldots \alpha_s \in \mathbb{C}$ with $|\alpha_1| \geq k$, $|\alpha_2| \geq k,\ldots, |\alpha_s| \geq k$, $(s < n)$, and has all zeros in $|z| \leq k$. Therefore, for $|\alpha_{s+1}| > k$, by applying Lemma 2.5 to $D_{\alpha_s}\cdots D_{\alpha_2}D_{\alpha_1}p(z)$, we get

$$\left|D_{\alpha_{s+1}}\left\{D_{\alpha_s}\cdots D_{\alpha_2}D_{\alpha_1}p(z)\right\}\right| \geq \frac{(n-s)}{1+k}\left(|\alpha_{s+1}|-k\right)\left|D_{\alpha_s}\cdots D_{\alpha_2}D_{\alpha_1}p(z)\right|. \tag{2.11}$$

By combining the terms (2.10) and (2.11), we obtain

$$\left|D_{\alpha_{s+1}}D_{\alpha_s}\cdots D_{\alpha_2}D_{\alpha_1}p(z)\right| \geq \frac{n(n-1)\cdots(n-s)}{(1+k)^{s+1}}\times \\ \{(|\alpha_1|-k)\cdots(|\alpha_{s+1}|-k)\}\left|p(z)\right|.$$

This implies that the result is true for $t = s + 1$. The proof is complete.

Lemma 2.9. *If* $p(z) = \sum_{i=0}^{n} a_i z^i$ *is a polynomial of degree n,* $p(z) \neq 0$ *in* $|z| < k$, *then* $m < |p(z)|$ *for* $|z| < k$, *and in particular* $m < |a_0|$, *where* $m = \min_{|z|=k} |p(z)|$.

The above lemma is due to Gardner et al. [13].

Lemma 2.10. *If* $p(z) = \sum_{i=0}^{n} a_i z^i$ *is a polynomial of degree n having all zeros in* $|z| \leq k$, *then*

$$m \leq k^n |a_n|, \tag{2.12}$$

where $m = \min_{|z|=k} |p(z)|$.

Proof. If $k = 0$, then inequality (2.12) is trivial. Now we suppose that $k > 0$. Since the polynomial $p(z) = \sum_{i=0}^{n} a_i z^i$ has all zeros in $|z| \leq k$, the polynomial $q(z) = z^n p(1/z) = a_n + \ldots + a_0 z^n$ has no zero in $|z| < \frac{1}{k}$. Thus, by applying Lemma 2.9 for the polynomial $q(z)$, we get

$$\min_{|z|=\frac{1}{k}} \left|q(z)\right| < |a_n|. \tag{2.13}$$

Since $\min_{|z|=\frac{1}{k}} \left|q(z)\right| = \frac{1}{k^n}\min_{|z|=k}\left|p(z)\right|$, (2.13) implies that $\frac{m}{k^n} < |a_n|$.

3. Proof of the theorem

Proof of Theorem 1.1. Let $m = \min_{|z|=k} |p(z)|$. If $p(z)$ has a zero on $|z| = k$, then $m = 0$ and the result follows from Lemma 2.8. Henceforth, we suppose that all the zeros of $p(z)$ lie in $|z| < k$, so that $m > 0$. Now $m \leq |p(z)|$ for $|z| = k$, therefore if λ is any real or complex number such that $|\lambda| < 1$, then $\left|\lambda m\left(\frac{z}{k}\right)^n\right| < \left|p(z)\right|$ for $|z| = k$. Since all zeros of $p(z)$ lie in $|z| < k$, by Rouche's theorem we can deduce that all zeros of the polynomial $G(z) = p(z) - \lambda m\left(\frac{z}{k}\right)^n$ lie in $|z| < k$. Also it follows from Lemma 2.10, that

$G(z) = p(z) - \lambda(\frac{m}{k^n})z^n$, hence the polynomial $G(z) = p(z) - \lambda(\frac{m}{k^n})z^n$ is of degree n. Now we can apply Lemma 2.8 for the polynomial $G(z)$ of degree n which has all zeros in $|z| \le k$. This implies that for all $\alpha_1, \ldots \alpha_t \in \mathbb{C}$ with $|\alpha_1| \ge k$, $|\alpha_2| \ge k$, ..., $|\alpha_t| \ge k$, ($t < n$), on $|z| = 1$,

$$\left|D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}G(z)\right| \ge \frac{n(n-1)\cdots(n-t+1)}{(1+k)^t}\times \{(|\alpha_1|-k)\cdots(|\alpha_t|-k)\}\left|G(z)\right|.$$

Equivalently

$$\left|D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}p(z) - \lambda\frac{m}{k^n}\left\{n(n-1)\cdots(n-t+1)\alpha_1\alpha_2\cdots\alpha_1\right\}z^{n-t}\right| \ge \frac{n(n-1)\cdots(n-t+1)}{(1+k)^t}\{(|\alpha_1|-k)\cdots(|\alpha_t|-k)\}\left|p(z) - \lambda m\left(\frac{z}{k}\right)^n\right|. \tag{3.1}$$

But by Lemma 2.1, the polynomial $T(z) = D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}G(z)$ has all zeros in $|z| \le k$. That is,

$$T(z) = D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}G(z) \ne 0, \quad \textit{for} \ \ |z| > k.$$

Then, substituting $G(z)$ in the above, we conclude that for every λ with $|\lambda| < 1$, and $|z| > k$,

$$T(z) = D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}p(z) - \lambda\frac{m}{k^n}\left\{n(n-1)\cdots(n-t+1)\alpha_1\alpha_2\cdots\alpha_t\right\}z^{n-t} \ne 0. \tag{3.2}$$

Thus, for $|z| > k$,

$$\left|D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}p(z)\right| \ge \frac{m}{k^n}\left\{n(n-1)\cdots(n-t+1)\,|\alpha_1|\,|\alpha_2|\cdots|\alpha_t|\right\}\left|z^{n-t}\right|. \tag{3.3}$$

If the inequality (3.3) is not true, then there is a point $z = z_0$ with $|z_0| > k$ such that

$$\left|D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}p(z_0)\right| < \frac{m}{k^n}\left\{n(n-1)\cdots(n-t+1)\,|\alpha_1|\,|\alpha_2|\cdots|\alpha_t|\right\}\left|z_0^{n-t}\right|.$$

Now take

$$\lambda = \frac{D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}p(z_0)}{\frac{m}{k^n}\left\{n(n-1)\cdots(n-t+1)\alpha_1\alpha_2\cdots\alpha_t\right\}z_0^{n-t}},$$

then $|\lambda| < 1$ and with this choice of λ, we have, $T(z_0) = 0$ for $|z_0| > k$, from (3.2). But it contradicts the fact that $T(z) \ne 0$ for $|z| > k$. Hence, for $|z| > k$, we have

$$\left|D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}p(z)\right| \ge \frac{m}{k^n}\left\{n(n-1)\cdots(n-t+1)\,|\alpha_1|\,|\alpha_2|\cdots|\alpha_t|\right\}\left|z^{n-t}\right|.$$

Taking a relevant choice of argument of λ, $\arg\lambda = \arg\left\{D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}p(z)\right\} - \arg\left\{\alpha_1\alpha_2\cdots\alpha_t z^{n-t}\right\}$, we have

$$|D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}p(z)- \lambda\frac{m}{k^n}\left\{n(n-1)\cdots(n-t+1)\alpha_1\alpha_2\cdots\alpha_t\right\}z^{n-t}| = |D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}p(z)- |\lambda|\frac{m}{k^n}\left\{n(n-1)\cdots(n-t+1)|\alpha_1||\alpha_2|\cdots|\alpha_t|\right\}|z^{n-t}|,$$

where $|z| = 1$.

Therefore, we can rewrite (3.1) as

$$\left|D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}p(z)\right| - |\lambda|\frac{m}{k^n}\left\{n(n-1)\cdots(n-t+1)|\alpha_1||\alpha_2|\cdots|\alpha_t|\right\}z^{n-t}\Big| \geq \frac{n(n-1)\cdots(n-t+1)}{(1+k)^t}\{(|\alpha_1|-k)\cdots(|\alpha_t|-k)\}\left(p(z)-|\lambda|\frac{m}{k^n}|z|^n\right),$$

where $|z| = 1$.

In an equivalent way

$$\left|D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}p(z)\right| \geq \frac{n(n-1)\cdots(n-t+1)}{(1+k)^t}[\left\{(|\alpha_1|-k)\cdots(|\alpha_t|-k)\left|p(z)\right|\right\} + |\lambda|\left\{(1+k)^t(|\alpha_1||\alpha_2|\cdots|\alpha_t|) - \{(|\alpha_1|-k)\cdots(|\alpha_t|-k)\}\right\}\frac{m}{k^n}].$$

Making $|\lambda| \to 1$, Theorem 1.1 follows.

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Competing interests
The author declares that they have no competing interests.

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