Uyanik and Owa Journal of Inequalities and Applications 2011, 2011:107
http://www.journalofinequalitiesandapplications.com/content/2011/1/107

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Radius properties for analytic and p-valently starlike functions

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Abstract

Let $\mathcal{A}_p$ be the class of functions *f(z)* which are analytic in the open unit disk $\mathbb{U}$ and satisfy $\frac{z^p}{f(z)} \neq 0 (z \in \mathbb{U})$. Also, let $\mathcal{S}_p^*(\alpha)$ denotes the subclass of $\mathcal{A}_p$ consisting of *f(z)* which are *p*-valently starlike of order $\alpha(0 \leqq \alpha < p)$. A new subclass $\mathcal{U}_p(\lambda)$ of $\mathcal{A}_p$ is introduced by

$$\left| z^2 \left(\frac{z^{p-1}}{f(z)} - \frac{1}{z} \right)' \right| \leqq \lambda \quad (z \in \mathbb{U})$$

for some real $\lambda > 0$. The object of the present paper is to consider some radius properties for $f(z) \in \mathcal{S}_p^*(\alpha)$ such that $\delta^{-p} f(\delta z) \in \mathcal{U}_p(\lambda)$.

2010 Mathematics Subject Classification: Primary 30C45.

Keywords: Analytic, radius problem, Cauchy-Schwarz inequality, *p*-valently starlike of order *α*.

1 Introduction

Let $\mathcal{A}_p$ be the class of functions *f*(*z*) of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p = 1, 2, 3, \ldots) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy

$$\frac{z^p}{f(z)} = 1 + \sum_{n=p+1}^{\infty} b_n z^{n-p} \neq 0 \quad (z \in \mathbb{U}). \tag{1.2}$$

For $f(z) \in \mathcal{A}_p$, we say that *f*(*z*) belongs to the class $\mathcal{U}_p(\lambda)$ if it satisfies

$$\left| z^2 \left(\frac{z^{p-1}}{f(z)} - \frac{1}{z} \right)' \right| \leqq \lambda \quad (z \in \mathbb{U}) \tag{1.3}$$

for some real number $\lambda > 0$.

Let us consider a function $f_\delta(z)$ given by

$$f_\delta(z) = \frac{z^p}{(1-z)^\delta} \quad (\delta \in \mathbb{R}). \tag{1.4}$$

Then, we can write that

$$f_\delta(z) = \frac{z^p}{1 + \sum_{n=1}^{\infty} a_n z^n}$$

with

$$a_n = (-1)^n \binom{\delta}{n}$$

and

$$\left| z^2 \left(\frac{z^{p-1}}{f_\delta(z)} - \frac{1}{z} \right)' \right| = \left| \sum_{n=1}^{\infty} (n-1) a_n z^n \right| < \sum_{n=1}^{\infty} (n-1)|a_n|.$$

Thus, if $\delta = 2$, then

$$\left| z^2 \left(\frac{z^{p-1}}{f_2(z)} - \frac{1}{z} \right)' \right| < 1.$$

This shows that $f_2(z) \in \mathcal{U}_p(\lambda)$ for $\lambda \geqq 1$.
If $\delta = 3$, then we have that

$$\left| z^2 \left(\frac{z^{p-1}}{f_3(z)} - \frac{1}{z} \right)' \right| < 5$$

Which shows that $f(z) \in \mathcal{U}_p(\lambda)$ for $\lambda \geqq 5$.
Further, if $\delta = 4$, then

$$\left| z^2 \left(\frac{z^{p-1}}{f_4(z)} - \frac{1}{z} \right)' \right| < 11$$

which shows that $f(z) \in \mathcal{U}_p(\lambda)$ for $\lambda \geqq 11$.
If $p = 1$, then $f(z) \in \mathcal{U}_1(\lambda)$ is defined by

$$\left| z^2 \left(\frac{1}{f(z)} - \frac{1}{z} \right)' \right| \leqq \lambda \quad (z \in \mathbb{U}) \tag{1.5}$$

for some real number $\lambda > 0$. Note that (1.5) is equivalent to

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leqq \lambda \quad (z \in \mathbb{U}).$$

Therefore, this class $\mathcal{U}_1(\lambda)$ was considered by Obradović and Ponnusamy [1]. Further-more, this class was extended as the class $\mathcal{U}(\beta_1, \beta_2; \lambda)$ by Shimoda et al. [2].

Let $\mathcal{S}_p^*(\alpha)$ denotes the subclass of $\mathcal{A}_p$ consisting of $f(z)$ which satisfy

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}) \tag{1.6}$$

for some real α ($0 \leqq \alpha < p$).

A function $f(z) \in \mathcal{S}_p^*(\alpha)$ is said to be p-valently starlike of order α in $\mathbb{U}$ (cf. Robertson [3]).

2 Coefficient inequalities

For $f(z) \in \mathcal{A}_p$, we consider the sufficient condition for $f(z)$ to be in the class $\mathcal{U}_p(\lambda)$.

Lemma 1 *If* $f(z) \in \mathcal{A}_p$ *satisfies*

$$\sum_{n=p+2}^{\infty} (n-p-1)|b_n| \leqq \lambda, \tag{2.1}$$

then $f(z) \in \mathcal{U}_1(\lambda)$.

Proof We note that

$$\left|z^2\left(\frac{z^{p-1}}{f(z)} - \frac{1}{z}\right)'\right| = \left|\sum_{n=p+1}^{\infty} (n-p-1)b_n z^{n-p}\right|$$
$$< \sum_{n=p+1}^{\infty} (n-p-1)|b_n|.$$

Therefore, if

$$\sum_{n=p+1}^{\infty} (n-p-1)|b_n| = \sum_{n=p+2}^{\infty} (n-p-1)|b_n| \leqq \lambda,$$

then $f(z) \in \mathcal{U}_p(\lambda)$.

Example 1 If we consider a function $f(z) \in \mathcal{A}_p$ given by

$$\frac{z^p}{f(z)} = 1 + b_{p+1}z + \sum_{n=p+2}^{\infty} \frac{\lambda e^{i\varphi}}{(n-p)(n-p-1)^2} z^{n-p} \neq 0 \quad (z \in \mathbb{U})$$

with

$$b_n = \frac{\lambda e^{i\varphi}}{(n-p)(n-p-1)^2} (\lambda > 0, \varphi \in \mathbb{R})$$

for $n \geqq p + 2$, then we see that

$$\sum_{n=p+2}^{\infty} (n-p-1)|b_n| = \sum_{n=p+2}^{\infty} \frac{\lambda e^{i\varphi}}{(n-p)(n-p-1)}$$
$$< \lambda \sum_{n=p+2}^{\infty} \left(\frac{1}{n-p-1} - \frac{1}{n-p}\right) = \lambda.$$

Thus, this function $f(z)$ satisfies the inequality (2.1). Also, we see that

$$\left| z^2 \left(\frac{z^{p-1}}{f(x)} - \frac{1}{z} \right)' \right| = \left| \sum_{n=p+2}^{\infty} \frac{\lambda e^{i\varphi}}{n-p-1)(n-p)} z^{n-p} \right|$$
$$< \lambda \sum_{n=p+2}^{\infty} \left(\frac{1}{n-p-1} - \frac{1}{n-p} \right) = \lambda.$$

Therefore, we say that $f(z) \in \mathcal{U}_p(\lambda)$.

Next, we discuss the necessary condition for the class $\mathcal{S}_p^*(\alpha)$.

Lemma 2 *If* $f(z) \in \mathcal{S}_p^*(\alpha)$ *satisfies*

$$\frac{z^p}{f(z)} = 1 + \sum_{n=p+1}^{\infty} b_n z^{n-p} \neq 0 \quad (z \in \mathbb{U})$$

with $b_n = |b_n| e^{i(n-p)\theta}$ $(n = p + 1, p + 2, p + 3, \ldots)$, *then*

$$\sum_{n=p+1}^{\infty} (n + \alpha - 2p)|b_n| \leqq p - \alpha.$$

Proof Let us define the function $F(z)$ by

$$F(z) = \frac{z^p}{f(z)} = 1 + \sum_{n=p+1}^{\infty} b_n z^{n-p}.$$

It follows that

$$Re\left(\frac{zf'(z)}{f(z)} \right) = Re\left(p - \frac{zF'(z)}{F(z)} \right)$$
$$= Re\left(\frac{p - \sum_{n=p+1}^{\infty} (n-2p) b_n z^{n-p}}{1 + \sum_{n=p+1}^{\infty} b_n z^{n-p}} \right)$$
$$= Re\left(\frac{p - \sum_{n=p+1}^{\infty} (n-2p)|b_n| e^{i(n-p)\theta} z^{n-p}}{1 + \sum_{n=p+1}^{\infty} |b_n| e^{i(n-p)\theta} z^{n-p}} \right) > \alpha$$

for $z \in \mathbb{U}$. Letting $z = |z| e^{-i\theta}$, we have that

$$\frac{p - \sum_{n=p+1}^{\infty} (n-2p)|b_n||z|^{n-p}}{1 + \sum_{n=p+1}^{\infty} |b_n||z|^{n-p}} > \alpha \quad (z \in \mathbb{U}).$$

If we take $|z| \to 1^-$, we obtain that

$$\frac{p - \sum_{n=p+1}^{\infty} (n-2p)|b_n|}{1 + \sum_{n=p+1}^{\infty} |b_n|} \geqq \alpha$$

which implies that

$$\sum_{n=p+1}^{\infty} (n + \alpha - 2p)|b_n| \leqq p - \alpha.$$

Remark 1 If we take $p = 1$ in Lemmas 1 and 2, then we have that

$$(i)\ f(z) \in \mathcal{A}_1, \sum_{n=2}^{\infty} (n-2)|b_n| \leqq \lambda \Rightarrow f(z) \in \mathcal{U}_1(\lambda)$$

and

$$(ii)\ f(z) \in \mathcal{S}^*(\alpha), \quad |b_n| = |b_n| e^{i(n-1)\theta} \Rightarrow \sum_{n=2}^{\infty} (n+\alpha-2)|b_n| \leqq 1-\alpha.$$

3 Radius problems

Our main result for the radius problem is contained in

Theorem 1 *Let* $f(z) \in \mathcal{S}_p^*(\alpha)$ $(p - 1 \leqq \alpha < p)$ *with*

$$\frac{z^p}{f(z)} = 1 + \sum_{n=p+1}^{\infty} b_n z^{n-p} \neq 0 \quad (z \in \mathbb{U}).$$

and $b_n = |b_n| e^{i(n-p)\theta}$ $(n = p + 1, p + 2, p + 3, \ldots)$. *If* $\delta \in \mathbb{C}(|\delta| < 1)$, *then* $\frac{1}{\delta^p} f(\delta z)$ *belongs to the class* $\mathcal{U}_p(\lambda)$ *for* $0 < |\delta| \leqq |\delta_0(\lambda)|$, *where* $|\delta_0(\lambda)|$ *is the smallest positive root of the equation*

$$|\delta|^2 \sqrt{1-\alpha} - (1 - |\delta|^2)\lambda = 0, \tag{3.1}$$

that is,

$$|\delta_0(\lambda)| = \sqrt{\frac{\lambda}{\lambda + \sqrt{1-\alpha}}}. \tag{3.2}$$

Proof Since

$$f(\delta z) = \delta^p z^p + \sum_{n=p+1}^{\infty} a_n \delta^n z^n,$$

we have that

$$\frac{z^p}{\frac{1}{\delta^p} f(\delta z)} = 1 + \sum_{n=p+1}^{\infty} b_n \delta^{n-p} z^{n-p}.$$

In view of Lemma 1, we have to show that

$$\sum_{n=p+2}^{\infty} (n-p-1)|b_n||\delta|^{n-p} \leqq \lambda.$$

Note that $f(z) \in \mathcal{S}_p^*(\alpha)$ satisfies

$$|b_n| \leqq \frac{p-\alpha}{n+\alpha-2p} < 1 \quad (p-1 \leqq \alpha < p).$$

Applying Cauchy-Schwarz inequality, we obtain that

$$\sum_{n=p+2}^{\infty}(n-p-1)|b_n||\delta|^{n-p} \leqq \left(\sum_{n=p+2}^{\infty}(n-p-1)|b_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=p+2}^{\infty}(n-p-1)|\delta|^{2(n-p)}\right)^{\frac{1}{2}}$$
$$\leqq \left(\sum_{n=p+2}^{\infty}(n-p-1)|\delta|^{2(n-p)}\right)^{\frac{1}{2}}\sqrt{p-\alpha}.$$

Let $|\delta|^2 = x$. Then, we have that

$$\begin{aligned}\sum_{n=p+2}^{\infty}(n-p-1)x^{n-p} &= x^2\left(\sum_{n=p+2}^{\infty}(n-p-1)x^{n-p-2}\right)\\ &= x^2\left(\sum_{n=p+2}^{\infty}x^{n-p-1}\right)'\\ &= x^2\left(\sum_{n=1}^{\infty}x^{n-1}\right)\\ &= \frac{x^2}{(1-x)^2}.\end{aligned}$$

This gives us that

$$\sum_{n=p+2}^{\infty}(n-p-1)|b_n||\delta|^{n-p} \leqq \frac{|\delta|^2\sqrt{p-\alpha}}{1-|\delta|^2}.$$

Let us define the function $h(|\delta|)$ by

$$h(|\delta|) = |\delta|^2\sqrt{p-\alpha} - (1-|\delta|^2)\lambda.$$

Then, $h(|\delta|)$ satisfies $h(0) = -\lambda < 0$ and $h(1) = \sqrt{p-\alpha} > 0$. Indeed, we have that $h(|\delta_0(\lambda)|) = 0$ for

$$0 < |\delta_0(\lambda)| = \sqrt{\frac{\lambda}{\lambda + \sqrt{p-\alpha}}} < 1.$$

This completes the proof of the theorem.

Corollary 1 *Let* $f(z) \in \mathcal{S}_1^*(\alpha) \quad (0 \leqq \alpha < 1)$ *with*

$$\frac{z}{f(z)} = 1 + \sum_{n=2}^{\infty}b_n z^{n-1} \neq 0 \quad (z \in \mathbb{U})$$

and $b_n = |b_n| e^{i(n-1)\theta}$ $(n = 2, 3, 4, \ldots)$. *If* $\delta \in \mathbb{C}$ $(|\delta| < 1)$, *then* $\frac{1}{\delta}f(\delta z)$ *belongs to the class* $\mathcal{U}_1(\lambda)$ *for* $0 < |\delta| \leqq |\delta_0(\lambda)|$, *where* $|\delta_0(\lambda)|$ *is the smallest positive root of the equation*

$$|\delta|^2\sqrt{1-\alpha} - (1-|\delta|^2)\lambda = 0,$$

that is,

$$|\delta_0(\lambda)| = \sqrt{\frac{\lambda}{\lambda + \sqrt{1-\alpha}}}.$$

Remark 2 In view of (3.2), we define the function $g(\lambda)$ by

$$g(\lambda) = |\delta_0(\lambda)| = \sqrt{\frac{\lambda}{\lambda + \sqrt{p-\alpha}}}.$$

Then, we have that

$$g'(\lambda) = \frac{1}{2}\sqrt{\frac{p-\alpha}{\lambda(\lambda + \sqrt{p-\alpha})^3}} > 0$$

for $\lambda > 0$. Therefore, $|\delta_0(\lambda)|$ given by (3.2) is increasing for $\lambda > 0$.

Remark 3 If we put $\alpha = p - \frac{1}{2}$ in Theorem 1, then

$$|\delta_0(\lambda)| = \sqrt{\frac{2\lambda}{2\lambda + \sqrt{2}}}.$$

Therefore, if we consider $\lambda = \frac{1}{2}$, then we see that

$$\left|\delta_0\left(\frac{1}{2}\right)\right| = \sqrt{\frac{1}{1+\sqrt{2}}} = 0.64359\ldots$$

and if we make $\lambda = 5$, then we have that

$$|\delta_0(5)| = \sqrt{\frac{10}{10+\sqrt{2}}} = 0.93600\ldots$$

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Authors' contributions
QF carried out the main part of this article. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 28 June 2011 Accepted: 7 November 2011 Published: 7 November 2011

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doi:10.1186/1029-242X-2011-107
Cite this article as: Uyanik and Owa: **Radius properties for analytic and p-valently starlike functions.** *Journal of Inequalities and Applications* 2011 **2011**:107.