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Approximate Cauchy functional inequality in quasi-Banach spaces

Hark-Mahn Kim and Eunyoung Son[*]

* Correspondence:
sey8405@hanmail.net
Department of Mathematics, Chungnam National University, 79 Daehangno, Yuseong-gu, Daejeon 305-764, Korea

Abstract

In this article, we prove the generalized Hyers-Ulam stability of the following Cauchy functional inequality:

$$||f(x) + f(y) + nf(z)|| \le nf \left\| \left(\frac{x+y}{n} + x \right) \right\|$$

in the class of mappings from n-divisible abelian groups to p-Banach spaces for any fixed positive integer $n \ge 2$.

1 Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms.

We are given a group G_1 and a metric group G_2 with metric $\rho\,(\cdot,\cdot)$. *Given* $\epsilon > 0$, *does there exist a* $\delta > 0$ *such that if* $f : G_1 \to G_2$ *satisfies* $\rho(f(xy), f(x)f(y)) < \delta$ *for all* $x, y \in G_1$, *then a homomorphism* $h : G_1 \to G_2$ *exists with* $\rho(f(x), h(x)) < \epsilon$ *for all* $x \in G_1$*?*

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it.

In 1941, Hyers [2] considered the case of approximately additive mappings between Banach spaces and proved the following result. Suppose that E_1 and E_2 are Banach spaces and $f : E_1 \to E_2$ satisfies the following condition: there is a constant $\epsilon \ge 0$ such that

$$|f(x+y) - f(x) - (y)|| \le \varepsilon$$

for all $x,y \in E_1$. Then, the limit $h(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E_1$, and it is a unique additive mapping $h : E_1 \to E_2$ such that $||f(x) - h(x)|| \le \epsilon$.

The method which was provided by Hyers, and which produces the additive mapping h, was called a direct method. This method is the most important and most powerful tool for studying the stability of various functional equations. Hyers' theorem was generalized by Aoki [3] and Bourgin [4] for additive mappings by considering an unbounded Cauchy difference. In 1978, Rassias [5] also provided a generalization of Hyers' theorem for linear mappings which allows the Cauchy difference to be unbounded like this $||x||^p + ||y||^p$. Let E_1 and E_2 be two Banach spaces and $f : E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in $t \in \mathbf{R}$ for each fixed x. Assume that there exists $\epsilon > 0$ and $0 \le p < 1$ such that

$$||f(x+y) - f(x) - f(y)|| \le \varepsilon(||x||^p + ||y||^p), \quad \forall x, y \in E_1.$$

Then, there exists a unique **R**-linear mapping $T : E_1 \to E_2$ such that

$$||f(x) - T(x)|| \le \frac{2}{2-2^p}||x||^p$$

for all $x \in E_1$. A generalized result of Rassias' theorem was obtained by Găvruta in [6] and Jung in [7]. In 1990, Rassias [8] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \ge 1$. In 1991, Gajda [9], following the same approach as in [5], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [9], as well as by Rassias and [001]emrl [10], that one cannot prove a Rassias' type theorem when $p = 1$. The counterexamples of Gajda [9], as well as of Rassias and [001]emrl [10], have stimulated several mathematicians to invent new approximately additive or approximately linear mappings. In particular, Rassias [11,12] proved a similar stability theorem in which he replaced the unbounded Cauchy difference by this factor $||x||^p||y||^q$ for $p,q \in \mathbf{R}$ with $p + q \neq 1$.

Let G be an n-divisible abelian group $n \in \mathbf{N}$ (i.e., $a \mapsto na : G \to G$ is a surjection) and X be a normed space with norm $|| \cdot ||$. Now, for a mapping $f : G \to X$, we consider the following generalized Cauchy-Jensen equation

$$f(x) + f(y) + nf(z) = nf\left(\frac{x+y}{n} + z\right), \quad n \ge 2$$

for all $x,y, z \in G$, which has been introduced in [13].

Proposition 1.1. For a mapping $f : G \to X$, the following statements are equivalent.

(a) *f is additive*,

(b) $f(x) + f(y) + nf(z) = nf\left(\frac{x+y}{n} + z\right)$,

(c) $||f(x) + f(y) + nf(z)|| \le \left\|nf\left(\frac{x+y}{n} + z\right)\right\|$

for all $x, y, z \in G$.

As a special case for $n = 2$, the generalized Hyers-Ulam stability of functional equation (b) and functional inequality (c) has been presented in [13]. We remark that there are some interesting papers concerning the stability of functional inequalities and the stability of functional equations in quasi-Banach spaces [14-18]. In this article, we are going to improve the theorems given in [13] without using the oddness of approximate additive functions concerning the functional inequality (c) for a more general case.

2 Generalized Hyers-Ulam stability of (c)

We recall some basic facts concerning quasi-Banach spaces and some preliminary results. Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following:

(1) $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$.
(2) $||\lambda x|| = |\lambda|||x||$ for all $\lambda \in \mathbf{R}$ and all $x \in X$.
(3) There is a constant $M \geq 1$ such that $||x + y|| \leq M(||x|| + ||y||)$ for all $x,y \in X$.

The pair $(X, || \cdot ||)$ is called a quasi-normed space if $|| \cdot ||$ is a quasi-norm on X [19,20]. The smallest possible M is called the modulus of concavity of $|| \cdot ||$. A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $|| \cdot ||$ is called a p-norm ($0 < p \leq 1$) if

$$||x + y||^p \leq ||x||^p + ||y||^p$$

for all $x,y \in X$. In this case, a quasi-Banach space is called a p-Banach space.

Given a p-norm, the formula $d(x,y) := ||x - y||^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem [20], each quasi-norm is equivalent to some p-norm (see also [19]). Since it is much easier to work with p-norms, henceforth, we restrict our attention mainly to p-norms. We observe that if $x_1, x_2,..., x_n$ are non-negative real numbers, then

$$\left(\sum_{i=1}^{n} x_i\right)^p \leq \sum_{i=1}^{n} {x_i}^p,$$

where $0 < p \leq 1$ [21].

From now on, let G be an n-divisible abelian group for some positive integer $n \geq 2$, and let Y be a p-Banach space with the modulus of concavity M.

Theorem 2.1. Suppose that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the functional inequality

$$||f(x) + f(y) + nf(z)|| \leq \left\| nf\left(\frac{x+y}{n} + z\right)\right\| + \varphi(x, y, z) \tag{1}$$

for all $x, y, z \in G$, and the perturbing function $\phi : G^3 \to \mathrm{R}^+$ satisfies

$$\Phi(x, y, z) := \sum_{i=0}^{\infty} \frac{\varphi(n^i x, n^i y, n^i z)^p}{n^{ip}} < \infty$$

for all $x,y,z \in G$. Then, there exists a unique additive mapping $h : G \to Y$, defined as $h(x) = \lim_{k\to\infty} \frac{f(n^k x) - f(-n^k x)}{2n^k}$, such that

$$||f(x) - h(x)|| \leq \frac{M^2}{2n}[\Phi(nx, 0, -x) + \Phi(-nx, 0, x)]^{\frac{1}{p}} + \frac{M}{2}\varphi(x, -x, 0) \tag{2}$$

for all $x \in G$.

Proof. Let $y = -x$, $z = 0$ in (1) and dividing both sides by 2, we have

$$\left\|\frac{f(x) + f(-x)}{2}\right\| \leq \frac{\varphi(x, -x, 0)}{2} \tag{3}$$

for all $x \in G$. Replacing x by nx and letting $y = 0$ and $z = -x$ in (1), we get

$$||f(nx) + nf(-x)|| \leq \varphi(nx, 0, -x) \tag{4}$$

for all $x \in G$. Replacing x by $-x$ in (4), one has

$$||f(-nx) + nf(x)|| \leq \varphi(-nx, 0, x) \tag{5}$$

for all $x \in G$. Put $g(x) = \dfrac{f(x) - f(-x)}{2}$. Combining (4) and (5) yields

$$||ng(x) - g(nx)|| \leq \frac{M}{2}(\varphi(nx, 0, -x) + \varphi(-nx, 0, x))$$

that is,

$$\left\| g(x) - \frac{1}{n}g(nx) \right\| \leq \frac{M}{2n}(\varphi(nx, 0, -x) + \varphi(-nx, 0, x)) \tag{6}$$

for all $x \in G$. It follows from (6) that

$$\begin{aligned}
&\left\| \frac{g(n^l x}{n^l} - \frac{g(n^m x)}{n^m} \right\|^p \\
&\leq \sum_{k=l}^{m-1} \left\| \frac{1}{n^k} g(n^k x) - \frac{1}{n^{k+1}} g(n^{k+1} x) \right\|^p \\
&= \sum_{k=1}^{m-1} \frac{1}{n^{kp}} \left\| g(n^k x) - \frac{1}{n} g(n^{k+1} x) \right\|^p \\
&\leq \sum_{k=1}^{m-1} \frac{M^p}{2^p n^{(k+1)_p}} [\varphi(n^{k+1} x, 0, -n^k x)^p + \varphi(-n^{k+1} x, 0, n^k x)^p]
\end{aligned} \tag{7}$$

for all nonnegative integers m and l with $m > l \geq 0$ and $x \in G$. Since the right-hand side of (7) tends to zero as $l \to \infty$, we obtain the sequence $\left\{ \dfrac{g(n^m x}{n^m} \right\}$ is Cauchy for all $x \in G$. Because of the fact that Y is complete, it follows that the sequence $\left\{ \dfrac{g(n^m x}{n^m} \right\}$ converges in Y. Therefore, we can define a function $h : G \to Y$ by

$$h(x) = \lim_{m \to \infty} \frac{g(n^m x)}{n^m} = \lim_{m \to \infty} \frac{f(n^m x) - f(-n^m x)}{2n^m}, \quad x \in G.$$

Moreover, letting $l = 0$ and taking $m \to \infty$ in (7), we get

$$\left\| \frac{f(x) - f(-x)}{2} - h(x) \right\| \leq ||g(x) - h(x)|| \leq \frac{M}{2n} [\Phi(nx, 0 - x) + \Phi(-nx, 0, x)]^{\frac{1}{p}} \tag{8}$$

for all $x \in G$. It follows from (3) and (8) that

$$||f(x) - h(x)|| \leq \frac{M^2}{2n} [\Phi(nx, 0, -x) + \Phi(-nx, 0, x)]^{\frac{1}{p}} + \frac{M}{2} \varphi(x, -x, 0)$$

for all $x \in G$.

It follows from (1) and (4) that

$$\begin{aligned}
||h(x)+h(y)-h(x+y)||^p &= ||h(x)+h(y)+h(-x-y)||^p \\
&= \lim_{k\to\infty}\frac{1}{n^{kp}}||g(n^kx)+g(n^ky)+g(-n^k(x+y))||^p \\
&\le \lim_{k\to\infty}\frac{1}{2^pn^{kp}}(||f(n^kx)+f(n^ky)+nf(-n^{k-1}(x+y))||^p \\
&\quad +||-f(-n^kx)-f(-n^ky)-nf(n^{k-1}(x+y))||^p \\
&\quad +||nf(n^{k-1}(x+y))+f(-n^k(x+y))||^p \\
&\quad +||-nf(-n^{k-1}(x+y))+f(n^k(x+y))||^p \\
&\le \lim_{k\to\infty}\frac{1}{2^pn^{kp}}(\varphi(n^kx,n^ky,-n^{k-1}(x+y))^p+\varphi(-n^kx,-n^ky,n^{k-1}(x+y))^p \\
&\quad +\varphi(-n^k(x+y),0,n^{k-1}(x+y))^p+\varphi(n^k(x+y),0,-n^{k-1}(x+y))^p) \\
&= 0
\end{aligned}$$

for all $x,y \in G$. This implies that the mapping h is additive.

Next, let $h' : G \to Y$ be another additive mapping satisfying

$$||f(x)-h'(x)|| \le \frac{M^2}{2n}[\Phi(nx,0,-x)+\Phi(-nx,0,x)]^{\frac{1}{p}}+\frac{M}{2}\varphi(x,-x,0)$$

for all $x \in G$. Then, we have

$$\begin{aligned}
||h(x)-h'(x)||^p &= \left\|\frac{1}{n^k}h(n^kx)-\frac{1}{n^k}h'(n^kx)\right\|^p \\
&\le \frac{1}{n^{kp}}(||h(n^kx)-f(n^kx)||^p+||f(n^kx)-h'(n^kx)||^p) \\
&\le \frac{2M^{2p}}{2^pn^{(k+1)_p}}[\Phi(n^{k+1}x,0,-n^kx)+\Phi(-n^{k+1}x,0,n^kx)]+\frac{2M^p}{2^pn^{kp}}\varphi(n^kx,-n^kx,0)^p \\
&= \sum_{i=k}^{\infty}\frac{2M^{2p}}{2^pn^{(i+1)p}}[\varphi(n^{i+1}x,0,-n^ix)^p+\varphi(-n^{i+1}x,0,n^ix)^p]+\frac{2M^p\varphi(n^kx,-n^kx,0)^p}{2^pn^{kp}}
\end{aligned}$$

for all $k \in$ N and all $x \in G$. Taking the limit as $k \to \infty$, we conclude that

$$h(x) = h'(x)$$

for all $x \in G$. This completes the proof.

Suppose that X is a normed space in the following corollaries. If we put $\phi(x,y,z) := \theta(||x||^q||y||^r||z||^s)$ and $\phi(x,y,z) := \theta(||x||^q+||y||^r+||z||^s)$ in Theorem 2.1, respectively, then we get the following Corollaries 2.2 and 2.3.

Corollary 2.2. Let $q + r + s < 1$, $q, r, s > 0$, $\theta > 0$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following functional inequality:

$$||f(x)+f(y)+nf(z)|| \le \left\|nf\left(\frac{x+y}{n}+x\right)\right\|+\theta(||x||^q||y||^r||z||^s$$

for all $x, y, z \in X$, then f is additive.

Corollary 2.3. Let $0 < q,r,s < 1$, $\theta_1,\theta_2 > 0$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following functional inequality:

$$||f(x)+f(y)+nf(z)|| \le \left\|nf\left(\frac{x+y}{n}+z\right)\right\|+\theta_1(||x||^q+||y||^r+||z||^s)+\theta_2$$

for all $x,y,z \in X$, then there exists a unique additive mapping $h : X \in Y$, defined as $h(x) = \lim_{k\to\infty}\frac{f(n^kx)-f(-n^kx)}{2n^k}$, such that

$$||f(x)-h(x)|| \le \frac{M^2\sqrt[p]{2}}{2}\left(\frac{n^{pq}\theta_1^p||x||^{pq}}{n^p-n^{pq}}+\frac{\theta_1^p||x||^{ps}}{n^p-n^{ps}}+\frac{\theta_2^p}{n^p-1}\right)^{\frac{1}{p}}$$
$$+\frac{M}{2}(\theta_1||x||^q+\theta_1||x||^r+\theta_2)$$

for all $x \in X$.

Noting the inequality

$$||f(nx)-nf(x)|| \le M[\varphi(nx,0,-x)+n\varphi(x,-x,0)]$$

according to the inequalities (3) and (4), then we can similarly prove another stability theorem under the same condition as in Theorem 2.1:

Remark 2.4. Let $\phi : G^3 \to \mathbf{R}+$ and $f : G \to Y$ satisfy the assumptions of Theorem 2.1. Then, there exists a unique additive mapping $h : G \to Y$, defined by $h(x)=\lim_{k\to\infty}\frac{f(n^kx)}{n^k}$, such that

$$||f(x)-h(x)|| \le \frac{M}{n}[\Phi(nx,0,-x)+n^p\Phi(x,-x,0)]^{\frac{1}{p}}$$

for all $x \in G$ using the similar argument to Theorem 2.1.

In particular, if a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following functional inequality:

$$||f(x)+f(y)+nf(z)|| \le \left\|nf\left(\frac{x+y}{n}+z\right)\right\|+\theta_1(||x||^q+||y||^r+||z||^s)+\theta_2$$

for all x,y,z in a normed space X, where $0 < q,r,s < 1$, $\theta_1,\theta_2 > 0$, then there exists a unique additive mapping $h : X \to Y$ such that

$$||f(x)-h(x)|| \le M\left(\frac{(n^{pq}+n^p)\theta_1^p||x||^{pq}}{n^p-n^{pq}}+\frac{n^p\theta_1^p||x||^{pr}}{n^p-n^{pr}}+\frac{\theta_1^p||x||^{ps}}{n^p-n^{ps}}+\frac{(1+n^p)\theta_2^2}{n^p-1}\right)^{\frac{1}{p}}$$

for all $x \in X$.

We may obtain more simple and sharp approximation than that of Theorem 2.1 for the stability result under the oddness condition.

Remark 2.5. Let $\phi : G^3 \to \mathrm{R}^+$ and $f : G \to Y$ satisfy the assumptions of Theorem 2.1. Moreover, if the mapping f is odd, then there exists a unique additive mapping $h : G \to Y$, defined by $h(x)=\lim_{k\to\infty}\frac{f(n^kx)}{n^k}$, such that

$$||f(x)-h(x)|| \le \frac{1}{n}\Phi(nx,0,-x)^{\frac{1}{p}}$$

for all $x \in G$.

Now, we consider another stability result of functional inequality (c) in the followings.

Theorem 2.6. Suppose that a mapping $f: G \to Y$ satisfies

$$||f(x) + f(y) + nf(z)|| \leq \left\| nf\left(\frac{x+y}{n} + z\right)\right\| + \varphi(x, y, z) \tag{9}$$

and the perturbing function $\phi : G^3 \to \mathrm{R}^+$ is such that

$$\Psi(x, y, z) := \sum_{i=1}^{\infty} n^{ip} \varphi\left(\frac{x}{n^i}, \frac{y}{n^i}, \frac{z}{n^i}\right)^p < \infty$$

for all $x,y,z \in G$. Then, there exists a unique additive mapping $h : G \to Y$, defined $h(x)\lim_{k\to\infty} \frac{n^k}{2}\left(f(\frac{x}{n^k}) - f(-\frac{x}{n^k})\right)$, such that

$$||f(x) - h(x)|| \leq \frac{M^2}{2n}[\Psi(nx, 0, -x) + \Psi(-nx, 0, x)]^{\frac{1}{p}} + \frac{M}{2}\varphi(x, -x, 0) \tag{10}$$

for all $x \in G$.

Proof. We observe that $f(0) = 0$ because of $\phi(0,0,0) = 0$ by the convergence of $\Psi(0,0,0) < \infty$. Now, combining (4) and (5) yields the functional inequality

$$||g(x) - ng\left(\frac{x}{n}\right)|| \leq \frac{M}{2}\left(\varphi\left(x, 0, -\frac{x}{n}\right) + \varphi\left(-x, 0, \frac{x}{n}\right)\right),$$

where $g(x) = \dfrac{f(x) - f(-x)}{2}$, $x \in G$. It follows from the last inequality that

$$\left\|g(x) - n^m g\left(\frac{x}{n^m}\right)\right\|^p \leq \frac{M^p}{2^p}\sum_{i=0}^{m-1} n^{ip}\left[\varphi\left(\frac{x}{n^i}, 0, -\frac{x}{n^{i+1}}\right)^p + \varphi\left(-\frac{x}{n^i}, 0, \frac{x}{n^{i+1}}\right)^p\right] \tag{11}$$

for all $x \angle G$.

The remaining proof is similar to the corresponding proof of Theorem 2.1. This completes the proof.

Suppose that X is a normed space in the following corollaries. If we put $\phi(x,y,z) := \theta(||x||^q||y||^r||z||^s)$ and $\phi(x,y,z) := \theta(||x||^q + ||y||^r + ||z||^s)$ in Theorem 2.6, respectively, then we get the following Corollaries 2.7 and 2.8.

Corollary 2.7. Let $q + r + s > 1$, $q,r, s > 0$, $\theta > 0$. If a mapping $f: X \to Y$ satisfies the following functional inequality:

$$||f(x) + f(y) + nf(z)|| \leq \left\| nf\left(\frac{x+y}{n} + x\right)\right\| + \theta(||x||^q||y||^r||z||^s$$

for all $x, y, z \in$ X, then f is additive.

Corollary 2.8. Let $q,r,s > 1$, $\theta_1 > 0$. If a mapping $f: X \to Y$ satisfies the following functional inequality:

$$||f(x) + f(y) + nf(z)|| \leq \left\| nf\left(\frac{x+y}{n} + z\right)\right\| + \theta_1(||x||^q + ||y||^r + ||z||^s)$$

for all $x,y,z \in X$, then there exists a unique additive mapping $h : X \to Y$, defined as $h(x)\lim_{k\to\infty} \frac{n^k}{2}\left(f(\frac{x}{n^k}) - f(-\frac{x}{n^k})\right)$, such that

$$||f(x) - h(x)|| \le \frac{M^2\sqrt[p]{2\theta_1}}{2}\left(\frac{n^{pq}||x||^{pq}}{n^{pq}-n^p} + \frac{||x||^{ps}}{n^{ps}-n^p}\right)^{\frac{1}{p}} + \frac{M\theta_1}{2}(||x||^q + ||x||^r)$$

for all $x \in X$.

We can similarly prove another stability theorem under somewhat different conditions as follows:

Remark 2.9. Let $\phi : G^3 \to \mathrm{R}^+$ and $f : G \to Y$ satisfy the assumptions of Theorem 2.6. Then, there exists a unique additive mapping $h : G \to Y$, defined by $h(x)$ = $h(x) = \lim_{k\to\infty} n^k f(\frac{x}{n^k})$, such that

$$||f(x) - h(x)|| \le \frac{M}{n}[\Psi(nx, 0, -x) + n^p\Psi(x, -x, 0)]^{\frac{1}{p}}$$

for all $x \in G$.

In particular, if a mapping $f : X \to Y$ satisfies the following functional inequality:

$$||f(x) + f(y) + nf(z)|| \le \left\|nf\left(\frac{x+y}{n} + z\right)\right\| + \theta_1(||x||^q + ||y||^r + ||z||^s)$$

for all x,y, z in a normed space X, where $q,r,s > 1$, $\theta_1 > 0$, then there exists a unique additive mapping $h : X \to Y$ such that

$$||f(x) - h(x)|| \le M\theta_1\left(\frac{(n^{pq}+n^p)||x||^{pq}}{n^{pq}-n^p} + \frac{||x||^{ps}}{n^{ps}-n^p} + \frac{n^p||x||pr}{n^{pr}-n^p}\right)^{\frac{1}{p}}$$

for all $x \in X$.

We may obtain more simple and sharp approximation than that of Theorem 2.6 for the stability result under the oddness condition.

Remark 2.10. Let $\phi : G^3 \to \mathrm{R}^+$ and $f : G \to Y$ satisfy the assumptions of Theorem 2.6. If the mapping f is odd, then there exists a unique additive mapping $h : G \to Y$, defined by $h(x) = \lim_{k\to\infty} n^k f(\frac{x}{n^k})$, such that

$$||f(x) - h(x)|| \le \frac{1}{n}\Psi(nx, 0, -x)^{\frac{1}{p}}$$

for all $x \in G$.

3 Alternative generalized Hyers-Ulam stability of (c)

From now on, we investigate the generalized Hyers-Ulam stability of the functional inequality (c).

Theorem 3.1. Suppose that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the functional inequality

$$||f(x) + f(y) + nf(z)|| \le \left\|nf\left(\frac{x+y}{n} + z\right)\right\| + \varphi(x, y, z)$$

for all $x,y,z \in G$ and there exists a constant L with $0 < L < 1$ for which the perturbing function $\phi : G^3 \to \mathbf{R}^+$ satisfies

$$\varphi(nx, ny, nz) \leq nL\varphi(x, y, z) \tag{12}$$

for all $x,y,z \in G$. Then, there exists a unique additive mapping $h : G \to Y$, defined as $h(x) = \lim_{k\to\infty} \frac{f(n^k x) - f(-n^k x)}{2n^k}$, such that

$$||f(x) - h(x)|| \leq \frac{M^2}{2n\sqrt[p]{1 - L^p}}[\varphi(nx, 0, -x) + \varphi(-nx, 0, x)] + \frac{M}{2}\varphi(x, -x, 0)$$

for all $x \in G$.

Proof. It follows from (7) and (12) that

$$\begin{aligned} &\left\| \frac{g(n^1 x)}{n^1} - \frac{g(n^m x)}{n^m} \right\|^p \\ &\leq \sum_{k=1}^{m-1} \frac{M^p}{2^p n^{(k+1)p}}[\varphi(n^{k+1}x, 0, -n^k x) + \varphi(-n^{k+1}x, 0, n^k x)]^p \\ &\leq \sum_{k=1}^{m-1} \frac{M^p L^{kp}}{2^p n^p}[\varphi(nx, 0, -x) + \varphi(-nx, 0, x)]^p \end{aligned}$$

for all nonnegative integers m and l with $m > l \geq 0$ and $x \in G$,where $g(x) = \frac{f(x) - f(-x)}{2}$. Since the sequence $\left\{ \frac{g(n^m x}{n^m} \right\}$ is Cauchy for all $x \in G$, we can define a function $h : G \to Y$ by

$$h(x) = \lim_{m\to\infty} \frac{g(n^m x)}{n^m} = \lim_{m\to\infty} \frac{f(n^m x) - f(-n^m x)}{2n^m}, \quad x \in G.$$

Moreover, letting $l = 0$ and $m \to \infty$ in the last inequality yields

$$\left\| \frac{f(x) - f(-x)}{2} - h(x) \right\| \leq \frac{M}{2n\sqrt[p]{1 - L^p}}[\varphi(nx, 0, -x) + \varphi(-nx, 0, x)] \tag{13}$$

for all $x \in G$. It follows from (3) and (13) that

$$||f(x) - h(x)|| \leq \frac{M^2}{2n\sqrt[p]{1 - L^p}}[\varphi(nx, 0, -x) + \varphi(-nx, 0, x)] + \frac{M}{2}\varphi(x, -x, 0)$$

for all $x \in G$.

The remaining proof is similar to the corresponding proof of Theorem 2.1. This completes the proof.

Remark 3.2. Let $\phi : G^3 \to \mathbf{R}^+$ and $f : G \to Y$ satisfy the assumptions of Theorem 3.1. Then, there exists a unique additive mapping $h : G \to Y$, defined by $h(x) = \lim_{k\to\infty} \frac{f(n^k x)}{n^k}$, such that

$$||f(x) - h(x)|| \leq \frac{M}{n\sqrt[p]{1 - L^p}}[\varphi(nx, 0, -x) + n\varphi(x, -x, 0)]$$

for all $x \in G$ using the similar argument to Theorem 3.1.

In particular, if a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following functional inequality:

$$||f(x) + f(y) + nf(z)|| \le \left\| nf\left(\frac{x+y}{n} + z\right) \right\| + \theta_1(||x||^r + ||y||^r + ||z||^r) + \theta_2$$

for all x, y, z in a normed space X, where $0 < r < 1$, θ_1, $\theta_2 > 0$, then there exists a unique additive mapping $h : X \to Y$ such that

$$||f(x) - h(x)|| \le \frac{M}{\sqrt[p]{n^p - n^{pr}}}((n^r + 2n + 1)\theta_1||x||^r + (n+1)\theta_2)$$

for all $x \in X$, by considering $L := n^{r-1}$.

Theorem 3.3. Suppose that a mapping $f : G \to Y$ satisfies the functional inequality

$$||f(x) + f(y) + nf(z)|| \le \left\| nf\left(\frac{x+y}{n} + z\right) \right\| + \varphi(x, y, z)$$

for all $x,y,z \in G$ and there exists a constant L with $0 < L < 1$ for which the perturbing function $\phi : G^3 \to \mathbf{R}^+$ satisfies

$$\varphi\left(\frac{x}{n}, \frac{y}{n}, \frac{z}{n}\right) \le \frac{L}{n}\varphi(x, y, z) \tag{14}$$

for all $x,y,z \in G$. Then, there exists a unique additive mapping $h : G \to Y$, defined as $h(x)\lim_{k\to\infty}\frac{n^k}{2}\left(f(\frac{x}{n^k}) - f(-\frac{x}{n^k})\right)$, such that

$$||f(x) - h(x)|| \le \frac{M^2L}{2n\sqrt[p]{1 - L^p}}[\varphi(nx, 0, -x) + \varphi(-nx, 0, x)] + \frac{M}{2}\varphi(x, -x, 0)$$

for all $x \in G$.

Proof. We observe that $f(0) = 0$ because $\phi(0,0,0) = 0$, which follows from the condition $\varphi(0, 0, 0) \le \frac{L}{n}\varphi(0, 0, 0)$. It follows from the inequality (11) and (14) that

$$\begin{aligned}\left\| g(x) - n^m g\left(\frac{x}{n^m}\right) \right\|^p &\le \frac{M^p}{2^p}\sum_{i=0}^{m-1} n^{ip}\left[\varphi\left(\frac{x}{n^i}, 0, -\frac{x}{n^{i+1}}\right) + \varphi\left(-\frac{x}{n^i}, 0, \frac{x}{n^{i+1}}\right)\right]^p \\ &\le \frac{M^p}{2^p n^p}\sum_{i=0}^{m-1} L^{(i+1)p}[\varphi(nx, 0, -x) + \varphi(-nx, 0, x)]^p\end{aligned}$$

for all $x \in G$, where $g(x) = \frac{f(x) - f(-x)}{2}$, $x \in G$.

The remaining proof is similar to the corresponding proof of Theorem 2.1. This completes the proof.

Remark 3.4. Let $\phi : G^3 \to \mathbf{R}^+$ and $f : G \to Y$ satisfy the assumptions of Theorem 3.3. Then, there exists a unique additive mapping $h : G \to Y$, defined by $h(x) = \lim_{k\to\infty} n^k f(\frac{x}{n^k})$, such that

$$||f(x) - h(x)|| \le \frac{ML}{n\sqrt[p]{1 - L^p}}[\varphi(nx, 0, -x) + n\varphi(x, -x, 0)]$$

for all $x \in G$ using the similar argument to Theorem 3.3.

In particular, if a mapping $f : X \to Y$ satisfies the following functional inequality:

$$||f(x) + f(y) + nf(z)|| \le \left\| nf\left(\frac{x+y}{n} + z\right) \right\| + \theta_1(||x||^r + ||y||^r + ||z||^r)$$

for all x, y, z in a normed space X, where $r > 1$, $\theta_1 > 0$, then there exists a unique additive mapping $h : X \to Y$ such that

$$||f(x) - h(x)|| \leq \frac{M}{\sqrt[p]{n^{pr} - n^p}}(n^r + 2n + 1)\theta_1 ||x||^r$$

for all $x \in X$, by considering $L := n^{1-r}$.

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Authors' contributions
All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

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