TURAN'S INEQUALITY FOR APPELL POLYNOMIALS

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We give some necessary and sufficient conditions for the class of Appell polynomials to satisfy well-known Turan's inequality. Among the other corollaries, we apply our results to some classes of orthogonal polynomials.

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1. Introduction

A real sequence $\{a_n\}$, $n = 0,1,2,\ldots$ generates a sequence of Appell polynomials defined as:

$$A_n(x) := \sum_{k=0}^{n} a_k \binom{n}{k} x^{n-k}; \quad n = 0,1,2,\ldots. \tag{1.1}$$

This class of polynomials is of importance in real and combinatorial analysis [1, 2]. For example, the classical Bernoulli and Laguerre polynomials belong to Appell class.

It is said that the sequence of polynomials $\{C_n(x)\}$ have T property if it satisfies Turan's inequality

$$C_n^2(x) - C_{n-1}(x)C_{n+1}(x) \geq 0, \quad x \in [a,b],\ n \in N. \tag{1.2}$$

We will also consider inverse Turan's inequality and say that $\{C_n(x)\}$ have T^- property if it satisfies

$$C_n^2(x) - C_{n-1}(x)C_{n+1}(x) \leq 0, \quad x \in [a,b],\ n \in N. \tag{1.3}$$

For $x \in [-1,1]$, T property was proved by Turán for Legendre's polynomials [5]; in [3] this property proved Szegö for Laguerre polynomials ($x \in R$).

In 1964, Ikeda proved T property for Bernoulli polynomials and $x \in [0,1]$.

Nowadays, the possibility of T (T^-) property for various classes of polynomials still attracts the attention of mathematicians.

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2. Results

Our task in this paper is to investigate T property of Appell polynomials in two cases. In the first case we suppose that the generating sequence $\{a_n\}$ consists of positive numbers. In the second case we assume that all polynomials from the sequence $\{A_n(x)\}$ have real zeros only.

As concerns the first case, we can formulate the next theorem.

Theorem 2.1. *If the generating sequence $\{a_n\}$ consists of positive numbers only, then*

(i) *the sequence $\{A_n(x)\}$ have T (T^-) property for $x \in (0,b]$ if and only if the sequence $\{a_n\}$ have T (T^-) property;*

(ii) *the sequence $\{B_n(x)\}$ defined by*

$$B_n(x) := \left(A_n(x)/a_n\right)^{1/n}; \quad B_0(x) := 1, \quad n \in N, \tag{2.1}$$

is monotone nondecreasing (nonincreasing) for $x \in (0,b]$ if and only if the sequence $\{a_n\}$ have T (T^-) property.

From the above theorem follow some T property criteria.

Proposition 2.2. *If the sequence $\{a_n\}$ of positive numbers have not T (T^-) property then also $\{A_n(x)\}$ have not this property for $x \in [a,b]$, $a \le 0 < b$.*

Proposition 2.3. *If the sequence $\{A_n(x)/a_n\}$ have T^- (T) property for $x \in (0,b]$ then the sequence $\{A_n(x)\}$ have T (T^-) property for $x \in [0,\infty)$.*

Proposition 2.4. *Define*

$$A_n^{(\sigma)}(x) := \sum_{k=0}^{n} a_k^{\sigma} \binom{n}{k} x^{n-k}, \quad \sigma \in R; \qquad A_n^{(1)}(x) = A_n(x). \tag{2.2}$$

If $\{A_n(x)\}$ have T property for $x > 0$ then $\{A_n^{\sigma}(x)\}$ have T property for $\sigma \ge 0$ and T^- property for $\sigma < 0$.

Analogous statement takes place if $\{A_n(x)\}$ have T^- property.

T property implies bounds for $A_n(x)$. Namely,

Proposition 2.5. *If $\{A_n(x)\}$ ($A_0(x) = a_0 = 1$, $a_n > 0$, $n \in N$) have T property for $x \in (0,b]$, then*

$$\frac{a_n}{a_1^n} \le \frac{A_n(x)}{A_1^n(x)} \le 1, \quad x \in [0,b],\ n \in N. \tag{2.3}$$

If $\{A_n(x)\}$ have T^- property then the reverse inequalities hold.

In the sequel we will not assume positivity of the sequence $\{a_n\}$, but

Theorem 2.6. *If all the zeros of polynomials $A_n(x)$, $n \in N$ are real, then*

(i) *the generating sequence $\{a_n\}$ have T property;*

(ii) *the sequence $\{A_n(x)\}$ have T property for $x \in R$.*

This theorem is particularly useful in the case of sequences of orthogonal polynomials. Because of orthogonality, all their zeros are real and, as a corollary of Theorem 2.6., we get

Proposition 2.7. *The sequence of Hermite polynomials have T property for $x \in R$.*

Denote by $\{L_n^{(a)}(x)\}$ the sequence of generalized Laguerre polynomials of order $a > -1$. Then

Proposition 2.8. *The sequence $\{L_n^{(a)}(x)/\binom{n+a}{n}\}$ have T property for $x \in R$.*

For $a = 0$ we obtain Szego's theorem concerning Laguerre polynomials, mentioned above.

If the sequence of polynomials $\{Q_n(x)\}$ is not from Appell class, then the following assertion can be useful.

Theorem 2.9. *If, for fixed $t \in [a,b]$, polynomials $R_n(x)$ defined by*

$$R_n(x) := \sum_{k=0}^{n} Q_k(t)\binom{n}{k}x^{n-k}, \quad n \in N, \tag{2.4}$$

have all its zeros real, then the sequence $\{Q_n(x)\}$ have T property for $x \in [a,b]$.

As a corollary we have the next

Proposition 2.10. *If $\{P_n^{(\lambda)}(x)\}$ denotes the sequence of ultraspherical polynomials with parameter $\lambda > -1/2$, then the sequence $\{P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)\}$ have T property for $x \in [-1,1]$.*

For $\lambda = 1/2$ we obtain Turan's assertion on Legendre's polynomials and for $\lambda = 1$ we proved T property for the sequence $\{U_n(x)/(n+1)\}$, $x \in [-1,1]$, where $U_n(x) = P_n^{(1)}(x)$ is a Tchebychef polynomial of second kind.

3. Proofs

To prove part (i) of Theorem 2.1, suppose first T (T^-) property of the sequence $\{A_n(x)\}$ for $x \in (0,b]$. Then the polynomial $P(x)$,

$$\begin{aligned} P(x) &:= A_n^2(x) - A_{n-1}(x)A_{n+1}(x) \\ &= a_n^2 - a_{n-1}a_{n+1} + (n-1)(a_na_{n-1} - a_{n+1}a_{n-2})x + \cdots + (a_1^2 - a_0a_2)x^{2n-2} \end{aligned} \tag{3.1}$$

is nonnegative (nonpositive) for $x \in (0,b]$.

Using the identity

$$A_n'(x) = nA_{n-1}(x), \tag{3.2}$$

we obtain

$$P'(x) = (n-1)(A_n(x)A_{n-1}(x) - A_{n+1}(x)A_{n-2}(x)). \tag{3.3}$$

Hence, polynomials $P(x)$ and $P'(x)$ are of the same sign, that is, $P(x)$ is either nonnegative and nondecreasing or nonpositive and nonincreasing for $x \in (0,b]$.

Since it is also continuous in x, it follows that $P(0) = a_n^2 - a_{n-1}a_{n+1}$ has the same sign as $P(x)$, $x \in (0,b]$.

Suppose now that the sequence $\{a_n\}$ has T (T^-) property.

Putting $c_n = c_n(x) := a_n x^{-n}$, $x > 0$; $n = 0,1,2,\dots$, we have to prove that T (T^-) property of $\{c_n\}$ implies the same property for the sequence $\{C_n\}$, where

$$C_n := \sum_{k=0}^{n} \binom{n}{k} c_k; \quad n = 0,1,2,\dots. \tag{3.4}$$

It is easy to check that T (T^-) property of $\{c_n\}$ implies this property for the sequence $\{c_n^{(1)}\}$, defined by $c_n^{(1)} := c_n + c_{n-1}$.

By induction, the same is valid for sequences $\{c_n^{(m)}\}$, where

$$c_n^{(m+1)} := c_n^{(m)} + c_{n-1}^{(m)}, \quad m = 1,2,\dots. \tag{3.5}$$

It is only left to note that $c_n^{(n)} = C_n$.

Remark 3.1. It follows that T (T^-) property of the sequence $\{a_n\}$ implies the same property of $\{A_n(x)\}$ for $x \in [0,\infty)$.

To prove part (ii) of Theorem 2.1, assume first that $\{a_n\}$ have T property. By (i), the sequence $\{A_n(t)\}$ also have this property for $t > 0$, that is,

$$\frac{A_n(t)}{A_{n+1}(t)} \geq \frac{A_{n-1}(t)}{A_n(t)} \tag{3.6}$$

or, by (3.2),

$$\frac{1}{n+1}\frac{A'_{n+1}(t)}{A_{n+1}(t)} \geq \frac{1}{n}\frac{A'_n(t)}{A_n(t)}. \tag{3.7}$$

Integrating (3.7) over $t \in [0,x]$, we get

$$\left(A_{n+1}(x)/a_{n+1}\right)^{1/(n+1)} \geq \left(A_n(x)/a_n\right)^{1/n}; \quad n = 1,2,\dots. \tag{3.8}$$

This means that the sequence $\{B_n(x)\}$ is monotone nondecreasing for each fixed $x > 0$.

T^- case can be treated similarly.

Suppose now that $\{B_n(x)\}$ is monotone and consider the polynomial $Q(x)$ defined by

$$Q(x) := \left(A_{n+1}(x)/a_{n+1}\right)^n - \left(A_n(x)/a_n\right)^{n+1}. \tag{3.9}$$

By assumption, $Q(x)$ is nonnegative (nonpositive) for $x \in (0,b]$, $b > 0$.

We have $Q(0) = 0$ and, by (3.2)

$$Q'(0) = n(n+1)\left(a_n/a_{n+1} - a_{n-1}/a_n\right); \quad n = 1,2,\dots. \tag{3.10}$$

Therefore,

$$Q(x) = n(n+1)\left(a_n/a_{n+1} - a_{n-1}/a_n\right)x + \cdots + \left(\left(a_0/a_{n+1}\right)^n - \left(a_0/a_n\right)^{n+1}\right)x^{n(n+1)}. \tag{3.11}$$

Since x is independent of n, we see from (3.11) that, for sufficiently small positive x, the signs of $Q(x)$ and $Q'(0)$ have to be the same, that is, the part (ii) is also proved.

Proof of Proposition 2.2. This is a consequence of the assertion (i) of Theorem 2.1. □

Proof of the next proposition needs the following lemma.

LEMMA 3.2. *If the sequence* $\{b_n\}$, $b_0 := 1$ *of positive numbers have* T (T^-) *property, then the sequence* $\{b_n^{1/n}\}$ *is nonincreasing (nondecreasing).*

Proof. T property implies $b_n^2 \geq b_{n-1}b_{n+1}$, $n \in N$. Hence

$$(b_0b_2)(b_1b_3)^2(b_2b_4)^3\cdots(b_{n-1}b_{n+1})^n \leq b_1^2b_2^4b_3^6\cdots b_n^{2n}. \tag{3.12}$$

This gives $b_{n+1}^n \leq b_n^{n+1}$, that is, $\{b_n^{1/n}\}$ is nonincreasing.

Proof of T^- case goes along the same lines. □

Proof of Proposition 2.3. Assume that $\{A_n(x)/a_n\}$ have T^- property for $x \in (0,b]$. Then Lemma 3.2 asserts $\{(A_n(x)/a_n)^{1/n}\}$ nondecreasing for $x \in (0,b]$. By part (ii) of Theorem 2.1 this implies T property for $\{a_n\}$ which in turn, by part (i) and Remark 3.1, gives T property of $\{A_n(x)\}$ for $x \in [0,\infty)$. □

Proof of Proposition 2.4. This is a consequence of Theorem 2.1, part (i) and the fact that, if $\{a_n\}$ have T property, then $\{a_n^\sigma\}$ have T property for $\sigma \geq 0$ and T^- property for $\sigma < 0$. □

Proof of Proposition 2.5. T property and Lemma 3.2 imply $\{(A_n(x))^{1/n}\}$ nonincreasing. Therefore $A_n(x)^{1/n} \leq A_1(x)$.

On the other hand, by Theorem 2.1 part (i), T property of $\{A_n(x)\}$ implies T property for $\{a_n\}$ which in turn, by part (ii), implies that the sequence $\{(A_n(x)/a_n)^{1/n}\}$ is nondecreasing. Hence

$$\left(A_n(x)/a_n\right)^{1/n} \geq A_1(x)/a_1; \quad x > 0,\ n = 1,2,\ldots. \tag{3.13}$$

This is exactly the left-hand side of the inequality from Proposition 2.4.

The other case can be treated similarly. □

Proof of Theorem 2.6. We will prove first the part (ii) of this theorem. For this purpose, let

$$A_{n+1}(x) = a_0\prod_{i=0}^{n}(x - x_i), \quad x_i \in R,\ n = 0,1,2,\ldots. \tag{3.14}$$

Taking logarithmic derivative, by (3.2), we get

$$(n+1)\frac{A_n(x)}{A_{n+1}(x)} = \frac{A'_{n+1}(x)}{A_{n+1}(x)} = \sum_{i=0}^{n}\frac{1}{x - x_i}. \tag{3.15}$$

Analogously,

$$(n+1)\frac{nA_{n-1}(x)A_{n+1}(x) - (n+1)A_n^2(x)}{A_{n+1}^2(x)} = \left(\frac{A'_{n+1}(x)}{A_{n+1}(x)}\right)' = -\sum_{i=0}^{n}\frac{1}{(x - x_i)^2}. \tag{3.16}$$

Therefore, by Cauchy's inequality, we obtain

$$\begin{aligned}\frac{A_n^2(x)}{A_{n+1}^2(x)} &= \left(\frac{1}{n+1}\sum_{i=0}^{n}\frac{1}{x-x_i}\right)^2 \leq \frac{1}{n+1}\sum_{i=0}^{n}\frac{1}{(x-x_i)^2}\\ &= \frac{(n+1)A_n^2(x)-nA_{n-1}(x)A_{n+1}(x)}{A_{n+1}^2(x)},\end{aligned} \tag{3.17}$$

which is equivalent to T property for the sequence $\{A_n(x)\}$, $x\in R$. □

Since $a_n = A_n(0)$, the assertion (i) follows from (ii) for $x=0$.

Proof of Proposition 2.7. The classical Hermite polynomials $\{H_n(x)\}$ are defined by [4, page 105],

$$\frac{H_n(x)}{n!} = \sum_{i=0}^{[n/2]}\frac{(-1)^i}{i!}\frac{(2x)^{n-2i}}{(n-2i)!}, \quad n=0,1,2,\dots. \tag{3.18}$$

To see that the sequence $\{H_n(x/2)\}$ belongs to the Appell class, write (3.18) in the form

$$H_n(x/2) = \sum_{i=0}^{n} h_i\binom{n}{i}x^{n-i}, \tag{3.19}$$

where

$$h_{2k} = \frac{(-1)^k(2k)!}{k!}; \quad h_{2k+1}=0; \quad k=0,1,2,\dots. \tag{3.20}$$

Since all their zeros are real [4, page 110], by Theorem 2.6, part (ii), T property follows for $x\in R$. □

Proof of Proposition 2.8. The class of generalized Laguerre polynomials $\{L_n^{(a)}(x)\}$ of order $a>-1$ is defined by [4, page 100]

$$L_n^{(a)}(x) = \sum_{i=0}^{n}\binom{n+a}{n-i}\frac{(-x)^i}{i!}. \tag{3.21}$$

All its zeros are real and positive [4, page 110]. For $x\neq 0$, an elementary transform gives

$$x^n L_n^{(a)}(1/x)\Big/\binom{n+a}{n} = \Gamma(a+1)\sum_{i=0}^{n}\frac{(-1)^i}{\Gamma(i+a+1)}\binom{n}{i}x^{n-i}. \tag{3.22}$$

Now, we can apply Theorem 2.6, part (ii). □

Proof of Theorem 2.9. This assertion is a consequence of Theorem 2.6, part (i). □

Proof of Proposition 2.10. An explicit form of ultraspherical polynomials of order λ is the following [4, page 85]

$$P_n^{(\lambda)}(x) = \sum_{i=0}^{[n/2]}(-1)^i\frac{\Gamma(n-i+\lambda)}{\Gamma(\lambda)\Gamma(i+1)\Gamma(n-2i+1)}(2x)^{n-2i}. \tag{3.23}$$

All its zeros are real and belong to the interval $(-1,1)$. We also have

$$P_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n} = (-1)^n P_n^{(\lambda)}(-1). \tag{3.24}$$

[4, page 81].

For the proof we will use the following identity [4, page 384]:

$$\sum_{k=0}^{n} \frac{P_k^{(\lambda)}(t)}{P_k^{(\lambda)}(1)} \binom{n}{k} x^{n-k} = \left(1+2tx+x^2\right)^{n/2} \frac{P_n^{(\lambda)}\left[\left(1+2tx+x^2\right)^{-1/2}(t+x)\right]}{P_n^{(\lambda)}(1)}. \tag{3.25}$$

Taking $t \in (-1,1)$ we can see that all the zeros $\{x_i\}$ of the polynomial on the right-hand side of (3.25) are real and given by

$$x_i = t_i\sqrt{\frac{1-t^2}{1-t_i^2}} - t, \tag{3.26}$$

where $\{t_i\}$ are corresponding zeros of $P_n^{(\lambda)}(t)$.

Hence, by (3.24) and Theorem 2.9, we obtain the assertion of Proposition 2.10. □

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