THE ESSENTIAL NORMS OF COMPOSITION OPERATORS BETWEEN GENERALIZED BLOCH SPACES IN THE POLYDISC AND THEIR APPLICATIONS

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Let U^n be the unit polydisc of $\mathbb{C}^n$ and $\phi = (\phi_1,\ldots,\phi_n)$ a holomorphic self-map of U^n. $\mathcal{B}^p(U^n)$, $\mathcal{B}^p_0(U^n)$, and $\mathcal{B}^p_{0_*}(U^n)$ denote the p-Bloch space, little p-Bloch space, and little star p-Bloch space in the unit polydisc U^n, respectively, where $p,q > 0$. This paper gives the estimates of the essential norms of bounded composition operators C_ϕ induced by ϕ between $\mathcal{B}^p(U^n)$ ($\mathcal{B}^p_0(U^n)$ or $\mathcal{B}^p_{0_*}(U^n)$) and $\mathcal{B}^q(U^n)$ ($\mathcal{B}^q_0(U^n)$ or $\mathcal{B}^q_{0_*}(U^n)$). As their applications, some necessary and sufficient conditions for the (bounded) composition operators C_ϕ to be compact from $\mathcal{B}^p(U^n)$ ($\mathcal{B}^p_0(U^n)$ or $\mathcal{B}^p_{0_*}(U^n)$) into $\mathcal{B}^q(U^n)$ ($\mathcal{B}^q_0(U^n)$ or $\mathcal{B}^q_{0_*}(U^n)$) are obtained.

1. Introduction

The class of all holomorphic functions with domain Ω will be denoted by $H(\Omega)$, where Ω is a bounded homogeneous domain in $\mathbb{C}^n$. Let ϕ be a holomorphic self-map of Ω, the composition operator C_ϕ induced by ϕ is defined by

$$(C_\phi f)(z) = f(\phi(z)), \tag{1.1}$$

for z in Ω and $f \in H(\Omega)$.

Let $K(z,z)$ be the Bergman kernel function of Ω, the Bergman metric $H_z(u,u)$ in Ω is defined by

$$H_z(u,u) = \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \log K(z,z)}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k, \tag{1.2}$$

where $z \in \Omega$ and $u = (u_1,\ldots,u_n) \in \mathbb{C}^n$.

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Following Timoney [5], we say that $f \in H(\Omega)$ is in the Bloch space $\mathcal{B}(\Omega)$ if

$$\|f\|_{\mathcal{B}(\Omega)} = \sup_{z\in\Omega} Q_f(z) < \infty, \tag{1.3}$$

where

$$Q_f(z) = \sup\left\{ \frac{|\nabla f(z)u|}{H_z^{1/2}(u,u)} : u \in \mathbb{C}^n - \{0\} \right\}, \tag{1.4}$$

and $\nabla f(z) = (\partial f(z)/\partial z_1,\dots,\partial f(z)/\partial z_n)$, $\nabla f(z)u = \sum_{l=1}^n (\partial f(z)/\partial z_l)u_l$.

The little Bloch space $\mathcal{B}_0(\Omega)$ is the closure in the Banach space $\mathcal{B}(\Omega)$ of the polynomial functions.

Let $\partial\Omega$ denote the boundary of Ω. Following Timoney [6], for $\Omega = B_n$ the unit ball of $\mathbb{C}^n$, $\mathcal{B}_0(B_n) = \{f \in \mathcal{B}(B_n) : Q_f(z) \to 0, \text{ as } z \to \partial B_n\}$; for $\Omega = \mathcal{D}$ the bounded symmetric domain other than the ball B_n, $\{f \in \mathcal{B}(\mathcal{D}) : Q_f(z) \to 0, \text{ as } z \to \partial\mathcal{D}\}$ is the set of constant functions on $\mathcal{D}$. So if $\mathcal{D}$ is a bounded symmetric domain other than the ball, we denote the $\mathcal{B}_{0*}(\mathcal{D}) = \{f \in \mathcal{B}(\mathcal{D}) : Q_f(z) \to 0, \text{ as } z \to \partial^*\mathcal{D}\}$ and call it little star Bloch space; here $\partial^*\mathcal{D}$ means the distinguished boundary of $\mathcal{D}$. The unit ball is the only bounded symmetric domain $\mathcal{D}$ with the property that $\partial^*\mathcal{D} = \partial\mathcal{D}$.

Let U^n be the unit polydisc of $\mathbb{C}^n$. Timoney [5] shows that $f \in \mathcal{B}(U^n)$ if and only if

$$\|f\|_1 = |f(0)| + \sup_{z\in U^n} \sum_{k=1}^n \left|\frac{\partial f}{\partial z_k}(z)\right| \left(1 - |z_k|^2\right) < +\infty, \tag{1.5}$$

where $f \in H(U^n)$.

This definition was the starting point for introducing the p-Bloch spaces.

Let $p > 0$, a function $f \in H(U^n)$ is said to belong to the p-Bloch space $\mathcal{B}^p(U^n)$ if

$$\|f\|_p = |f(0)| + \sup_{z\in U^n} \sum_{k=1}^n \left|\frac{\partial f}{\partial z_k}(z)\right| \left(1 - |z_k|^2\right)^p < +\infty. \tag{1.6}$$

It is an easy exercise to show that $\mathcal{B}^p(U^n)$ is a Banach space with the norm $\|\cdot\|_p$ for $p \geq 1$; and for $0 < p < 1$, $\mathcal{B}^p(U^n)$ is a nonlocally convex topological vector space and $d(f,g) = \|f - g\|_p^p$ is a complete metric for it. Its proof idea is basic, we refer the reader to see the proof of Proposition 3.1 or the statement corresponding the Bloch-type space for the unit ball in [13].

Just like Timoney [6], if

$$\lim_{z\to\partial U^n} \sum_{k=1}^n \left|\frac{\partial f}{\partial z_k}(z)\right| \left(1 - |z_k|^2\right)^p = 0, \tag{1.7}$$

it is easy to show that f must be a constant. Indeed, for fixed $z_1 \in U$, $(\partial f/\partial z_1)(z)(1 - |z_1|^2)^p$ is a holomorphic function in $z' = (z_2,\dots,z_n) \in U^{n-1}$. If $z \to \partial U^n$, then $z' \to \partial U^{n-1}$, which implies that

$$\lim_{z'\to\partial U^{n-1}} \left|\frac{\partial f}{\partial z_1}(z)\right| \left(1 - |z_1|^2\right)^p = 0. \tag{1.8}$$

Hence, $(\partial f/\partial z_1)(z)(1-|z_1|^2)^p \equiv 0$ for every $z' \in \partial U^{n-1}$, and for each $z_1 \in U$, and consequently $(\partial f/\partial z_1)(z) = 0$ for every $z \in U^n$. Similarly, we can obtain that $(\partial f/\partial z_j)(z) = 0$ for every $z_j \in U^n$ and each $j \in \{2,\dots,n\}$; therefore $f \equiv$ const.

So, there is no sense to introduce the corresponding little p-Bloch space in this way. We will say that the little p-Bloch space $\mathcal{B}_0^p(U^n)$ is the closure of the polynomials in the p-Bloch space. If $f \in H(U^n)$ and

$$\sup_{z\in\partial^* U^n}\sum_{k=1}^{n}\left|\frac{\partial f}{\partial z_k}(z)\right|\left(1-|z_k|^2\right)^p = 0, \tag{1.9}$$

we say f belongs to little star p-Bloch space $\mathcal{B}_{0*}^p(U^n)$. Using the same methods as that of [6, Theorem 4.15], we can show that $\mathcal{B}_0^p(U^n)$ is a proper subspace of $\mathcal{B}_{0*}^p(U^n)$ and $\mathcal{B}_{0*}^p(U^n)$ is a nonseparable closed subspace of $\mathcal{B}^p(U^n)$.

For the unit disc $U \subset \mathbb{C}$, Madigan and Matheson [1] proved that C_ϕ is always bounded on $\mathcal{B}(U)$ and bounded on $\mathcal{B}_0(U)$ if and only if $\phi \in \mathcal{B}_0(U)$. They also gave the sufficient and necessary conditions that C_ϕ is compact on $\mathcal{B}(U)$ or $\mathcal{B}_0(U)$.

The analogues of these facts for the unit polydisc and classical symmetric domains were obtained by Zhou and Shi in [8–10]. They had already shown that C_ϕ is always bounded on the Bloch space of these domains, and also gave some sufficient and necessary conditions for C_ϕ to be compact on those spaces. For the results on the unit ball, we refer the reader to see [4, 12].

We recall that the essential norm of a continuous linear operator T is the distance from T to the compact operators, that is,

$$\|T\|_e = \inf\{\|T-K\| : K \text{ is compact}\}. \tag{1.10}$$

Notice that $\|T\|_e = 0$ if and only if T is compact, so that estimates on $\|T\|_e$ lead to conditions for T to be compact.

As we have known that C_ϕ is always bounded on the Bloch space in the unit disc and polydisc, in [2], Montes-Rodriguez gave the exact essential norm of a composition operator on the Bloch space in the disc and obtained a different proof for the corresponding compactness results in [1]. After that, Zhou and Shi generalized Alsonso's result to the polydisc in [11].

In [7], Zhou stated and proved the corresponding compactness characterization for $\mathcal{B}^p(U^n)$ for $0<p<1$, however, C_ϕ is not always bounded, and the test functions used in [7] are only suitable for handling the case $0<p<1$. It is therefore natural to wonder what results can be proven about boundedness and compactness of C_ϕ on p-Bloch spaces for an arbitrary positive number p or, more generally, between possibly different p- and q-Bloch spaces of multivariable domains. In this paper, we answer these questions completely for U^n with essential norm approach, we give some estimates of the essential norms of bounded composition operators C_ϕ between $\mathcal{B}^p(U^n)(\mathcal{B}_0^p(U^n)$ or $\mathcal{B}_{0*}^p(U^n))$ and $\mathcal{B}^q(U^n)(\mathcal{B}_0^q(U^n)$ or $\mathcal{B}_{0*}^q(U^n))$. Further, we apply these results to obtain some necessary and sufficient conditions for the composition operators C_ϕ to be compact from $\mathcal{B}^p(U^n)(\mathcal{B}_0^p(U^n)$ or $\mathcal{B}_{0*}^p(U^n))$ into $\mathcal{B}^q(U^n)(\mathcal{B}_0^q(U^n)$ or $\mathcal{B}_{0*}^q(U^n))$. The fundamental ideas of the proof are those used by Shapiro [3] to obtain the essential norm of a composition operator on Hilbert spaces of analytic functions (Hardy and weighted Bergman spaces) in terms of natural counting functions associated with ϕ. This paper generalizes the results on the Bloch space for the unit disc in [2] and the unit polydisc in [11].

Throughout the remainder of this paper C will denote a positive constant, the exact value of which will vary from one appearance to the next.

Our main results are the following.

Theorem 1.1. *Let $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ be a holomorphic self-map of U^n and $\|C_\phi\|_e$ the essential norm of a bounded composition operator $C_\phi : \mathcal{B}^p(U^n)(\mathcal{B}_0^p(U^n)$ or $\mathcal{B}_{0*}^p(U^n)) \to \mathcal{B}^q(U^n)(\mathcal{B}_0^q(U^n)$ or $\mathcal{B}_{0*}^q(U^n))$, then*

$$\begin{aligned}
&\frac{1}{n}\lim_{\delta\to 0}\sup_{\operatorname{dist}(\phi(z),\partial U^n)<\delta}\sum_{k,l=1}^{n}\left|\frac{\partial\phi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p}\\
&\qquad\le \|C_\phi\|_e \le 2\lim_{\delta\to 0}\sup_{\operatorname{dist}(\phi(z),\partial U^n)<\delta}\sum_{k,l=1}^{n}\left|\frac{\partial\phi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p}.
\end{aligned} \tag{1.11}$$

By Theorem 1.1 and the fact that $C_\phi : \mathcal{B}^p(U^n)$ (or $\mathcal{B}_0^p(U^n)$ or $\mathcal{B}_{0*}^p(U^n)) \to \mathcal{B}^q(U^n)$ (or $\mathcal{B}_0^q(U^n)$ or $\mathcal{B}_{0*}^q(U^n)$) is compact if and only if $\|C_\phi\|_e = 0$, we obtain Theorem 1.2 at once.

Theorem 1.2. *Let $\phi = (\phi_1, \dots, \phi_n)$ be a holomorphic self-map of U^n. Then the bounded composition operator $C_\phi : \mathcal{B}^p(U^n)(\mathcal{B}_0^p(U^n)$ or $\mathcal{B}_{0*}^p(U^n)) \to \mathcal{B}^q(U^n)(\mathcal{B}_0^q(U^n)$ or $\mathcal{B}_{0*}^q(U^n))$ is compact if and only if for any $\varepsilon > 0$, there exists a δ with $0 < \delta < 1$, such that*

$$\sup_{\operatorname{dist}(\phi(z),\partial U^n)<\delta}\sum_{k,l=1}^{n}\left|\frac{\partial\phi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p} < \varepsilon. \tag{1.12}$$

Remark 1.3. When $n = 1$, $p = q = 1$, on $\mathcal{B}(U)$ we obtain [1, Theorem 2]. Since $\partial U = \partial^* U$, $\mathcal{B}_0(U) = \mathcal{B}_{0*}(U)$, we can also obtain [1, Theorem 1].

Remark 1.4. When $n > 1, p = q = 1$, C_ϕ is always bounded on $\mathcal{B}(U^n)$, so we can obtain the corresponding results in [8, 11].

The remainder of the present paper is assembled as follows: in Section 2, we state some lemmas for the proof of Theorem 1.1. In terms of mapping properties of symbol ϕ, Lemmas 2.3, 2.4, and 2.6 will give some conditions for C_ϕ to be bounded between possibly different p- and q-Bloch spaces, "little" or "little star" p- and q-Bloch spaces, the methods used are different from that of [7], since the test functions used in [7] are only suitable for handling the p-Bloch space for the case $0 < p < 1$, not others. In Section 3, we give the proof of Theorem 1.1. In Section 4, as applications of Theorems 1.1 and 1.2, we give some corollaries for C_ϕ to be compact on those spaces.

2. Some lemmas

In order to prove Theorem 1.1, we need some lemmas.

Lemma 2.1. *Let* $f \in \mathcal{B}^p(U^n)$*, then*

(1) *if* $0 \le p < 1$*, then* $|f(z)| \le |f(0)| + (n/(1-p))\|f\|_p$;

(2) *if* $p = 1$*, then* $|f(z)| \le (1 + 1/n\ln 2)(\sum_{k=1}^n \ln(2/(1-|z_k|^2)))\|f\|_p$;

(3) *if* $p > 1$*, then* $|f(z)| \le (1/n + 2^{p-1}/(p-1))\sum_{k=1}^n (1/(1-|z_k|^2)^{p-1})\|f\|_p$.

Proof. This Lemma can be easily obtained by some integral estimates, so we omit the detail. □

Lemma 2.2. *For* $p > 0$*, set*

$$f_w(z) = \int_0^{z_l} \frac{dt}{(1-\overline{w}t)^p}, \tag{2.1}$$

where $w \in U$*. Then* $f \in \mathcal{B}_0^p(U^n) \subset \mathcal{B}_{0*}^p(U^n) \subset \mathcal{B}^p(U^n)$.

Proof. Since

$$\frac{\partial f_w}{\partial z_l} = (1-\overline{w}z_l)^{-p}, \quad \frac{\partial f_w}{\partial z_i} = 0, \quad i \ne l, \tag{2.2}$$

it follows that

$$|f(0)| + \sum_{k=1}^n \left|\frac{\partial f_w}{\partial z_k}(z)\right|\left(1-|z_k|^2\right)^p = \frac{\left(1-|z_l|^2\right)^p}{|1-\overline{w}z_l|^p} \le (1+|z_l|)^p \le 2^p. \tag{2.3}$$

Hence $f_w \in \mathcal{B}^p(U^n)$.

Now we prove that $f_w \in \mathcal{B}_0^p(U^n)$. Using the asymptotic formula

$$(1-\overline{w}t)^{-p} = \sum_{k=0}^{+\infty} \frac{p(p+1)\cdots(p+k-1)}{k!}(\overline{w})^k t^k, \tag{2.4}$$

we obtain

$$f_w(z) = \sum_{k=0}^{+\infty} \frac{p(p+1)\cdots(p+k-1)}{k!}(\overline{w})^k \int_0^{z_l} t^k dt. \tag{2.5}$$

Denoting $P_n(z) = \sum_{k=0}^n (p(p+1)\cdots(p+k-1)/k!)(\overline{w})^k \int_0^{z_l} t^k dt$, it is easy to see that

$$\left|\frac{\partial(f_w - P_n)}{\partial z_l}\right| \le \sum_{k=n+1}^{+\infty} \frac{p(p+1)\cdots(p+k-1)}{k!}|w|^k \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.6}$$

Thus

$$\begin{aligned}\|f_w - P_n\|_p &= |f_w(0) - P_n(0)| + \sup_{z\in U^n}\left|\frac{\partial(f_w - P_n)}{\partial z_l}\right|\left(1-|z_l|^2\right)^p \\ &\leq \sup_{z\in U^n}\left|\frac{\partial(f_w - P_n)}{\partial z_l}\right| \longrightarrow 0,\end{aligned} \tag{2.7}$$

which shows that $f_w \in \mathcal{B}_0^p(U^n)$. So $f \in \mathcal{B}_0^p(U^n) \subset \mathcal{B}_{0*}^p(U^n) \subset \mathcal{B}^p(U^n)$. □

Lemma 2.3. *Let $\phi = (\phi_1,\dots,\phi_n)$ be a holomorphic self-map of U^n, $p,q > 0$. Then C_ϕ : $\mathcal{B}^p(U^n)(\mathcal{B}_0^p(U^n)$ or $\mathcal{B}_{0*}^p(U^n)) \to \mathcal{B}^q(U^n)$ is bounded if and only if there exists a constant C such that*

$$\sum_{k,l=1}^n \left|\frac{\partial\phi_l}{\partial z_k}(z)\right| \frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p} \leq C, \tag{2.8}$$

for all $z \in U^n$.

Proof. First assume that condition (2.8) holds and let $f \in \mathcal{B}^p(U^n)$. By Lemma 2.1, we know the evaluation at $\phi(0)$ is a bounded linear functional on $\mathcal{B}^p(U^n)$, so $|f(\phi(0))| \leq C\|f\|_p$.

On the other hand we have

$$\begin{aligned}\sum_{k=1}^n\left|\frac{\partial(C_\phi f(z))}{\partial z_k}\right|\left(1-|z_k|^2\right)^q &= \sum_{k=1}^n\left|\sum_{l=1}^n \frac{\partial f}{\partial \phi_l}(\phi(z))\frac{\partial\phi_l}{\partial z_k}(z)\right|\left(1-|z_k|^2\right)^q \\ &\leq \sum_{k,l=1}^n \left|\frac{\partial f}{\partial \phi_l}(\phi(z))\frac{\partial\phi_l}{\partial z_k}(z)\right|\left(1-|z_k|^2\right)^q \\ &\leq \sum_{l=1}^n \left|\frac{\partial f}{\partial \phi_l}(\phi(z))\right|\left(1-|\phi_l(z)|^2\right)^p \sum_{k,l=1}^n\left|\frac{\partial\phi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p} \\ &\leq \|f\|_p \sum_{k,l=1}^n\left|\frac{\partial\phi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p} \leq C\|f\|_p.\end{aligned} \tag{2.9}$$

So $C_\phi : \mathcal{B}^p(U^n) \to \mathcal{B}^q(U^n)$ is bounded.

For the converse, assume that $C_\phi : \mathcal{B}^p(U^n) \to \mathcal{B}^q(U^n)$ is bounded, with

$$\|C_\phi f\|_q \leq C\|f\|_p \tag{2.10}$$

for all $f \in \mathcal{B}^p(U^n)$.

For fixed l $(1 \leq l \leq n)$, we will make use of a family of test functions $\{f_w : w \in \mathbb{C},\ |w| < 1\}$ defined in Lemma 2.2.

Since

$$f_w \in \mathcal{B}_0^p(U^n) \subset \mathcal{B}_{0*}^p(U^n) \subset \mathcal{B}^p(U^n), \tag{2.11}$$

it follows from (2.10) that for $z \in U^n$,

$$\sum_{k=1}^{n} \left| \sum_{l=1}^{n} \frac{\partial f_w(\phi(z))}{\partial \phi_l} \frac{\partial \phi_l}{\partial z_k}(z) \right| \left(1 - |z_k|^2\right)^q \leq C. \tag{2.12}$$

Let $w = \phi_l(z)$. Then

$$\sum_{k=1}^{n} \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \frac{\left(1 - |z_k|^2\right)^q}{\left(1 - |\phi_l(z)|^2\right)^p} \leq C. \tag{2.13}$$

The results are stated above for $\mathcal{B}^p(U^n)$, but they also hold with minor modifications for $\mathcal{B}_0^p(U^n)$ and $\mathcal{B}_{0*}^p(U^n)$. Now the proof of Lemma 2.3 is completed. □

Lemma 2.4. *Let $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ be a holomorphic self-map of U^n. Then $C_\phi : \mathcal{B}_{0*}^p(U^n)(\mathcal{B}_0^p(U^n)) \to \mathcal{B}_{0*}^q(U^n)$ is bounded if and only if $\phi_l \in \mathcal{B}_{0*}^q(U^n)$ for every $l = 1, 2, \ldots, n$ and (2.8) holds.*

Proof. If $C_\phi : \mathcal{B}_{0*}^p(U^n)(\mathcal{B}_0^p(U^n)) \to \mathcal{B}_{0*}^q(U^n)$ is bounded, it is clear that, for every $l = 1, 2, \ldots, n$, $f_l(z) = z_l \in \mathcal{B}_0^p(U^n) \subset \mathcal{B}_{0*}^q(U^n)$, so $C_\phi f_l = \phi_l \in \mathcal{B}_{0*}^q(U^n)$. Furthermore, (2.12) holds by Lemma 2.3.

In order to prove the converse, we first prove that if $\phi_l \in \mathcal{B}_{0*}^q(U^n)$, for every $l = 1, 2, \ldots, n$, then $f \circ \phi \in \mathcal{B}_{0*}^q(U^n)$ for any $f \in \mathcal{B}_{0*}^p(U^n)$.

Without loss of generality, we prove this result when $n = 2$.

For any sequence $\{z^j = (z_1^j, z_2^j)\} \subset U^n$ with $z^j \to \partial^* U^n$ as $j \to \infty$, then

$$|z_1^j| \longrightarrow 1, \qquad |z_2^j| \longrightarrow 1. \tag{2.14}$$

Since $|\phi_1(z^j)| < 1$ and $|\phi_2(z^j)| < 1$, there exists a subsequence $\{z^{j_s}\}$ in $\{z^j\}$ such that

$$|\phi_1(z^{j_s})| \longrightarrow \rho_1, \qquad |\phi_2(z^{j_s})| \longrightarrow \rho_2, \tag{2.15}$$

as $s \to \infty$.

It is clear that $0 \le \rho_1, \rho_2 \le 1$. Then for $k = 1,2$, we have

$$\begin{aligned}
&\left|\frac{\partial(f\circ\phi)}{\partial z_k}(z^{j_s})\right|\left(1-|z_k^{j_s}|^2\right)^q \\
&\quad\le \left|\frac{\partial f}{\partial w_1}(\phi(z^{j_s}))\right|\left|\frac{\partial\phi_1}{\partial z_k}(z^{j_s})\right|\left(1-|z_k^{j_s}|^2\right)^q \\
&\qquad+\left|\frac{\partial f}{\partial w_2}(\phi(z^{j_s}))\right|\left|\frac{\partial\phi_2}{\partial z_k}(z^{j_s})\right|\left(1-|z_k^{j_s}|^2\right)^q \\
&\quad= \left|\frac{\partial f}{\partial w_1}(\phi(z^{j_s}))\right|\left(1-|\phi_1(z^{j_s})|^2\right)^p\left|\frac{\partial\phi_1}{\partial z_k}(z^{j_s})\right|\frac{\left(1-|z_k^{j_s}|^2\right)^q}{\left(1-|\phi_1(z^{j_s})|^2\right)^p} \\
&\qquad+\left|\frac{\partial f}{\partial w_2}(\phi(z^{j_s}))\right|\left(1-|\phi_2(z^{j_s})|^2\right)^p\left|\frac{\partial\phi_2}{\partial z_k}(z^{j_s})\right|\frac{\left(1-|z_k^{j_s}|^2\right)^q}{\left(1-|\phi_2(z^{j_s})|^2\right)^p}.
\end{aligned}\tag{2.16}$$

Now we prove the left-hand side of (2.16) $\to 0$ as $s \to \infty$ according to four cases.

Case 1. If $\rho_1 < 1$ and $\rho_2 < 1$, there exist r_1 and r_2 such that $\rho_1 < r_1 < 1$ and $\rho_2 < r_2 < 1$, so as j is large enough, $|\phi_1(z^{j_s})| \le r_1$ and $|\phi_2(z^{j_s})| \le r_2$.

Since $\phi_1, \phi_2 \in \mathcal{B}^q_{0*}(U^n)$, by (2.16), we get

$$\begin{aligned}
\left|\frac{\partial(f\circ\phi)}{\partial z_k}(z^{j_s})\right|\left(1-|z_k^{j_s}|^2\right)^q &\le \|f\|_p\frac{1}{(1-r_1^2)^p}\left|\frac{\partial\phi_1}{\partial z_k}(z^{j_s})\right|\left(1-|z_k^{j_s}|^2\right)^q \\
&\quad+\|f\|_p\frac{1}{(1-r_2^2)^p}\left|\frac{\partial\phi_2}{\partial z_k}(z^{j_s})\right|\left(1-|z_k^{j_s}|^2\right)^q \longrightarrow 0
\end{aligned}\tag{2.17}$$

as $s \to \infty$.

Case 2. If $\rho_1 = 1$ and $\rho_2 = 1$, then $\phi(z^{j_s}) \to \partial^* U^n$, by (2.8) and, since $f \in \mathcal{B}^p_{0*}(U^n)$, (2.16) yields that

$$\begin{aligned}
&\left|\frac{\partial(f\circ\phi)}{\partial z_k}(z^{j_s})\right|\left(1-|z_k^{j_s}|^2\right)^q \\
&\quad\le C\left|\frac{\partial f}{\partial w_1}(\phi(z^{j_s}))\right|\left(1-|\phi_1(z^{j_s})|^2\right)^p+C\left|\frac{\partial f}{\partial w_2}(\phi(z^{j_s}))\right|\left(1-|\phi_2(z^{j_s})|^2\right)^p \longrightarrow 0
\end{aligned}\tag{2.18}$$

as $s \to \infty$.

Case 3. If $\rho_1 < 1$ and $\rho_2 = 1$, similarly to Case 1, we can prove that

$$\begin{aligned} &\left| \frac{\partial f}{\partial w_1}(\phi(z^{j_s})) \right| \left(1 - |\phi_1(z^{j_s})|^2\right)^p \left| \frac{\partial \phi_1}{\partial z_k}(z^{j_s}) \right| \frac{\left(1 - |z_k^{j_s}|^2\right)^q}{\left(1 - |\phi_1(z^{j_s})|^2\right)^p} \\ &\quad \le \|f\|_p \frac{1}{(1 - r_1^2)^p} \left| \frac{\partial \phi_1}{\partial z_k}(z^{j_s}) \right| \frac{\left(1 - |z_k^{j_s}|^2\right)^q}{\left(1 - |\phi_1(z^{j_s})|^2\right)^p} \longrightarrow 0 \end{aligned} \tag{2.19}$$

as $s \to \infty$.

On the other hand, for fixed s, let $w_2^{j_s} = \phi_2(z^{j_s})$. Then $|w_2^{j_s}| < 1$. Denote

$$F(w_1) = \frac{\partial f}{\partial w_2}(w_1, w_2^{j_s}). \tag{2.20}$$

It is clear that $F(w_1)$ is holomorphic on $|w_1| < 1$. Choosing $R_{j_s} \to 1$ with $r_1 \le R_{j_s} < 1$. $|\phi_1(z^{j_s})| \le r_1$, so

$$\left|F(\phi_1(z^{j_s}))\right| \le \max_{|w_1| \le r_1} |F(w_1)| \le \max_{|w_1| \le R_{j_s}} |F(w_1)| = \max_{|w_1| = R_{j_s}} |F(w_1)| = |F(w_1^{j_s})|, \tag{2.21}$$

where $w_1^{j_s}$ is a point of modulus R_{j_s} where maximum of $F(w_1)$ is attained. This means that $|(\partial f/\partial w_2)(\phi_1(z^{j_s}), \phi_2(z^{j_s}))| \le |(\partial f/\partial w_2)(w_1^{j_s}, w_2^{j_s})|$. Since $|w_1^{j_s}| \to 1$, $|w_2^{j_s}| \to \rho_2 = 1$ and $f \in \mathcal{B}_{0*}^p(U^n)$,

$$\left| \frac{\partial f}{\partial w_2}(w_1^{j_s}, w_2^{j_s}) \right| (1 - |w_2^{j_s}|^2)^p \longrightarrow 0 \tag{2.22}$$

as $s \to \infty$, so by (2.8),

$$\begin{aligned} &\left| \frac{\partial f}{\partial w_2}(\phi(z^{j_s})) \right| \left(1 - |\phi_2(z^{j_s})|^2\right)^p \left| \frac{\partial \phi_2}{\partial z_k}(z^{j_s}) \right| \frac{\left(1 - |z_k^{j_s}|^2\right)^q}{\left(1 - |\phi_2(z^{j_s})|^2\right)^p} \\ &\quad \le C \left| \frac{\partial f}{\partial w_2}(w_1^{j_s}, w_2^{j_s}) \right| \left(1 - |w_2^{j_s}|^2\right)^p \longrightarrow 0 \end{aligned} \tag{2.23}$$

as $s \to \infty$.

By (2.19) and (2.23), (2.16) yields

$$\left| \frac{\partial (f \circ \phi)}{\partial z_k}(z^{j_s}) \right| \left(1 - |z_k^{j_s}|^2\right)^q \longrightarrow 0, \tag{2.24}$$

as $s \to \infty$.

Case 4. If $\rho_1 = 1$ and $\rho_2 < 1$, similarly to Case 3, we can prove

$$\left| \frac{\partial (f \circ \phi)}{\partial z_k}(z^{j_s}) \right| \left(1 - |z_k^{j_s}|^2\right)^q \longrightarrow 0, \tag{2.25}$$

as $s \to \infty$.

Combining Cases 1, 2, 3, and 4, we know there exists a subsequence $\{z^{j_s}\}$ in $\{z^j\}$ such that

$$\left|\frac{\partial(f\circ\phi)}{\partial z_k}(z^{j_s})\right|\left(1-|z_k^{j_s}|^2\right)^q \longrightarrow 0, \tag{2.26}$$

as $s\to\infty$ for $k=1,2$. We claim that

$$\left|\frac{\partial(f\circ\phi)}{\partial z_k}(z^{j})\right|\left(1-|z_k^{j}|^2\right)^q \longrightarrow 0, \tag{2.27}$$

as $j\to\infty$. In fact, if it fails, then there exists a subsequence $\{z^{j_s}\}$ such that

$$\left|\frac{\partial(f\circ\phi)}{\partial z_k}(z^{j_s})\right|\left(1-|z_k^{j_s}|^2\right)^q \longrightarrow \varepsilon>0 \tag{2.28}$$

for $k=1$ or 2. But from the above discussion, we can find a subsequence in $\{z^{j_s}\}$; we still write $\{z^{j_s}\}$ with

$$\left|\frac{\partial(f\circ\phi)}{\partial z_k}(z^{j_s})\right|\left(1-|z_k^{j_s}|^2\right)^q \longrightarrow 0, \tag{2.29}$$

it contradicts with (2.28).

So for any sequence $\{z^j\}\subset U^n$ with $z^j\to\partial^* U^n$ as $j\to\infty$, we have

$$\left|\frac{\partial(f\circ\phi)}{\partial z_k}(z^{j})\right|\left(1-|z_k^{j}|^2\right)^q \longrightarrow 0 \tag{2.30}$$

for $k=1,2$. By (2.8) and Lemma 2.3, it is clear that $f\circ\phi\in\mathcal{B}^q(U^n)$, so $f\circ\phi\in\mathcal{B}^q_{0*}(U^n)$.

For any $f\in\mathcal{B}^p_0(U^n)$. Since $\mathcal{B}^p_0(U^n)\subset\mathcal{B}^p_{0*}(U^n)$, then $f\circ\phi\in\mathcal{B}^q_{0*}(U^n)$.

By closed graph theorem, we know that

$$C_\phi:\mathcal{B}^p_{0*}(U^n)\,(\mathcal{B}^p_0(U^n))\longrightarrow\mathcal{B}^q_{0*}(U^n) \tag{2.31}$$

is bounded. This ends the proof of Lemma 2.4. □

Remark 2.5. For the case $C_\phi:\mathcal{B}^p(U^n)\to\mathcal{B}^q_{0*}(U^n)$, the necessity also holds, but we cannot guarantee that the sufficiency holds because we cannot be sure that $C_\phi f\in\mathcal{B}^q_{0*}(U^n)$ for all $f\in\mathcal{B}^p(U^n)$.

LEMMA 2.6. *Let $\phi=(\phi_1,\phi_2,\dots,\phi_n)$ be a holomorphic self-map of U^n. Then*

$$C_\phi:\mathcal{B}^p_0(U^n)\longrightarrow\mathcal{B}^q_0(U^n) \tag{2.32}$$

is bounded if and only if $\phi^\gamma\in\mathcal{B}^q_0(U^n)$ for every multiindex γ, and (2.8) holds.

Proof (sufficiency). From (2.8) and by Lemma 2.3 we know that $C_\phi:\mathcal{B}^p(U^n)\to\mathcal{B}^q(U^n)$ is bounded, in particular

$$\|C_\phi f\|_q\le\|C_\phi\|_{\mathcal{B}^p(U^n)\to\mathcal{B}^q(U^n)}\|f\|_p,\quad\forall f\in\mathcal{B}^p_0(U^n). \tag{2.33}$$

The boundedness of $C_\phi : \mathcal{B}_0^p(U^n) \to \mathcal{B}_0^q(U^n)$ directly follows, if we prove $C_\phi f \in \mathcal{B}_0^q(U^n)$ whenever $f \in \mathcal{B}_0^p(U^n)$. So, let $f \in \mathcal{B}_0^p(U^n)$. By the definition of $\mathcal{B}_0^p(U^n)$ it follows that for every $\varepsilon > 0$ there is a polynomial p_ε such that $\|f - p_\varepsilon\|_p < \varepsilon$. Hence

$$||C_\phi f - C_\phi p_\varepsilon||_q \le ||C_\phi||_{\mathcal{B}^p(U^n)\to\mathcal{B}^q(U^n)}||f - p_\varepsilon||_p < \varepsilon||C_\phi||_{\mathcal{B}^p(U^n)\to\mathcal{B}^q(U^n)}. \tag{2.34}$$

Since $\phi^\gamma \in \mathcal{B}_0^q(U^n)$ for every multiindex γ, we obtain $C_\phi p_\varepsilon \in \mathcal{B}_0^q(U^n)$. From this and (2.34) the result follows.

If $C_\phi : \mathcal{B}_0^p(U^n) \to \mathcal{B}_0^q(U^n)$ is bounded, then (2.8) can be proved as in Lemma 2.3, since the test functions appearing there belong to $\mathcal{B}_0^p(U^n)$. Since the polynomials $z^\gamma \in \mathcal{B}_0^p(U^n)$ for every multiindex γ, we get $C_\phi z^\gamma \in \mathcal{B}_0^q(U^n)$, as desired. □

Remark 2.7. For the case $C_\phi : \mathcal{B}^p(U^n)(\mathcal{B}_{0*}^p(U^n)) \to \mathcal{B}_0^q(U^n)$, in analogy to Remark 2.5, the necessity also holds, but we cannot guarantee that the sufficiency holds.

LEMMA 2.8. *If $\{f_k\}$ is a bounded sequence in $\mathcal{B}^p(U^n)$, then there exists a subsequence $\{f_{k_l}\}$ of $\{f_k\}$ which converges uniformly on compact subsets of U^n to a holomorphic function $f \in \mathcal{B}^p(U^n)$.*

Proof. Let $\{f_k\}$ be a bounded sequence in $\mathcal{B}^p(U^n)$ with $\|f_k\|_p \le C$. By Lemma 2.1, $\{f_j\}$ is uniformly bounded on compact subsets of U^n and hence normal by Montel's theorem. So we may extract a subsequence $\{f_{j_k}\}$ which converges uniformly on compact subsets of U^n to a holomorphic function f. It follows that $\partial f_{j_k}/\partial z_l \to \partial f/\partial z_l$ for each $l \in \{1,2,\dots,n\}$, so

$$\sum_{l=1}^n \left|\frac{\partial f}{\partial z_l}\right|\left(1-|z_l|^2\right)^p = \lim_{k\to\infty}\sum_{l=1}^n \left|\frac{\partial f_{j_k}}{\partial z_l}\right|\left(1-|z_l|^2\right)^p \le \sup_k ||f_{j_k}||_p \le C, \tag{2.35}$$

which implies $f \in \mathcal{B}^p(U^n)$. The Lemma is proved. □

LEMMA 2.9. *Let Ω be a domain in $\mathbb{C}^n$, $f \in H(\Omega)$. If a compact set K and its neighborhood G satisfy $K \subset G \subset \overline{G} \subset \Omega$ and $\rho = \operatorname{dist}(K, \partial G) > 0$, then*

$$\sup_{z\in K}\left|\frac{\partial f}{\partial z_j}(z)\right| \le \frac{\sqrt{n}}{\rho}\sup_{z\in G}|f(z)|. \tag{2.36}$$

Proof. For any $a \in K$, the polydisc

$$P_a = \left\{(z_1,\dots,z_n) \in \mathbb{C}^n : |z_j - a_j| < \frac{\rho}{\sqrt{n}},\ j = 1,\dots,n\right\} \tag{2.37}$$

is contained in G. By Cauchy's inequality,

$$\left|\frac{\partial f}{\partial z_j}(a)\right| \le \frac{\sqrt{n}}{\rho}\sup_{z\in\partial^* P_a}|f(z)| \le \frac{\sqrt{n}}{\rho}\sup_{z\in G}|f(z)|. \tag{2.38}$$

Taking the supremum for a over K gives the desired inequality. □

3. The proof of Theorem 1.1

Now we turn to the proof of Theorem 1.1. In the following, we are dealing with the case for $C_\phi : \mathcal{B}^p(U^n) \to \mathcal{B}^q(U^n)$, but if we note that the test functions f_m introduced below belong to $\mathcal{B}_0^p(U^n) \subset \mathcal{B}_{0*}^p(U^n) \subset \mathcal{B}^p(U^n)$, the results in Theorem 1.1 also hold with minor modifications for the other cases.

We begin by proving the lower estimate. It is clear that $\{m^{p-1}z_1^m\} \subset \mathcal{B}_0^p(U^n) \subset \mathcal{B}_{0*}(U^n) \subset \mathcal{B}(U^n)$ for $m = 1,2,\dots$, and this sequence converges to zero uniformly on compact subsets of the unit polydisc U^n. Furthermore

$$\|m^{p-1}z_1^m\|_p = \sup_{z\in U^n} \left(1 - |z_1|^2\right)^p m^p |z_1|^{m-1}. \tag{3.1}$$

Let $p(x) = m^p(1-x^2)^p x^{m-1}$, then

$$p'(x) = -m^p x^{m-2}(1-x^2)^{p-1}[(2p+m-1)x^2 - (m-1)], \tag{3.2}$$

so

$$\begin{aligned} p'(x) \le 0 \quad &\text{for } x \in \left[\sqrt{(m-1)/(2p+m-1)},1\right], \\ p'(x) \ge 0 \quad &\text{for } x \in \left[0,\sqrt{(m-1)/(2p+m-1)}\right]. \end{aligned} \tag{3.3}$$

That is, $p(x)$ is a decreasing function for $x \in [\sqrt{(m-1)/(2p+m-1)},1]$ and $p(x)$ is an increasing function for $x \in [0,\sqrt{(m-1)/(2p+m-1)}]$. Hence

$$\max_{x\in[0,1]} p(x) = p\left(\sqrt{\frac{m-1}{2p+m-1}}\right). \tag{3.4}$$

It follows from (3.1) that

$$\|m^{p-1}z_1^m\|_p = p\left(\sqrt{\frac{m-1}{2p+m-1}}\right) = \left(\frac{2p}{2p+m-1}\right)^p m^p \left(\frac{m-1}{2p+m-1}\right)^{(m-1)/2} \longrightarrow \left(\frac{2p}{e}\right)^p, \tag{3.5}$$

as $m \to \infty$.

Therefore, the sequence $\{m^{p-1}z_1^m\}_{m\ge 2}$ is bounded away from zero. Now we consider the normalized sequence $\{f_m = m^{p-1}z_1^m/\|m^{p-1}z_1^m\|_p\}$ which also tends to zero uniformly on compact subsets of U^n. For each $m \ge 2$, we define

$$A_m = \{z = (z_1,\dots,z_n) \in U^n : r_m \le |z_1| \le r_{m+1}\}, \tag{3.6}$$

where $r_m = \sqrt{(m-1)/(2p+m-1)}$. So

$$\begin{aligned}
&\min_{A_m} \sum_{l=1}^{n} \left\{ \left| \frac{\partial f_m}{\partial z_l}(z) \right| \left(1 - |z_l|^2\right)^p \right\} \\
&\quad = \min_{A_m} \left| \frac{\partial f_m}{\partial z_1} \right| \left(1 - |z_1|^2\right)^p = \frac{\left(1 - r_{m+1}^2\right)^p m^p r_{m+1}^{m-1}}{\left\| m^{p-1} z_1^m \right\|_p} \qquad (3.7) \\
&\quad = \left(\frac{2p+m-1}{2p+m} \right) \left(\frac{m(2p+m-1)}{(m-1)(2p+m)} \right)^{((m-1)/2)} = c_m.
\end{aligned}$$

It is easy to show that c_m tends to 1 as $m \to \infty$. For the moment fix any compact operator $K : \mathcal{B}^p(U^n) \to \mathcal{B}^q(U^n)$. The uniform convergence on compact subsets of the sequence $\{f_m\}$ to zero and the compactness of K imply that $\|K f_m\|_q \to 0$. It is easy to show that if a bounded sequence that is contained in $\mathcal{B}_{0*}^p(U^n)$ converges uniformly on compact subsets of U^n, then it also converges weakly to zero in $\mathcal{B}_{0*}^p(U^n)$ as well as in $\mathcal{B}^p(U^n)$. Since $\|f_m\|_p = 1$, we have

$$\begin{aligned}
\|C_\phi - K\| &\geq \limsup_m \left\| (C_\phi - K) f_m \right\|_q \\
&\geq \limsup_m \left(\|C_\phi f_m\|_q - \|K f_m\|_q \right) = \limsup_m \|C_\phi f_m\|_q \\
&\geq \limsup_m \sup_{z \in U^n} \sum_{k=1}^{n} \left\{ \left| \frac{\partial (f_m \circ \phi)}{\partial z_k}(z) \right| \left(1 - |z_k|^2\right)^q \right\} \\
&= \limsup_m \sup_{z \in U^n} \sum_{k=1}^{n} \left| \frac{\partial f_m}{\partial w_1}(\phi(z)) \right| \left| \frac{\partial \phi_1}{\partial z_k}(z) \right| \left(1 - |z_k|^2\right)^q \\
&= \limsup_m \sup_{z \in U^n} \sum_{k=1}^{n} \left| \frac{\partial \phi_1}{\partial z_k}(z) \right| \frac{\left(1 - |z_k|^2\right)^q}{\left(1 - |\phi_1(z)|^2\right)^p} \left| \frac{\partial f_m}{\partial w_1}(\phi(z)) \right| \left(1 - |\phi_1(z)|^2\right)^p \\
&\geq \limsup_m \sup_{\phi(z) \in A_m} \sum_{k=1}^{n} \left| \frac{\partial \phi_1}{\partial z_k}(z) \right| \frac{\left(1 - |z_k|^2\right)^q}{\left(1 - |\phi_1(z)|^2\right)^p} \left| \frac{\partial f_m}{\partial w_1}(\phi(z)) \right| \left(1 - |\phi_1(z)|^2\right)^p \\
&\geq \limsup_m \sup_{\phi(z) \in A_m} \sum_{k=1}^{n} \left| \frac{\partial \phi_1}{\partial z_k}(z) \right| \frac{\left(1 - |z_k|^2\right)^q}{\left(1 - |\phi_1(z)|^2\right)^p} \\
&\quad \times \liminf_m \min_{\phi(z) \in A_m} \left| \frac{\partial f_m}{\partial w_1}(\phi(z)) \right| \left(1 - |\phi_1(z)|^2\right)^p
\end{aligned}$$

$$
\begin{aligned}
&\geq \limsup_{m} \sup_{\phi(z)\in A_m} \sum_{k=1}^{n} \left| \frac{\partial \phi_1}{\partial z_k}(z) \right| \frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_1(z)|^2\right)^p} \liminf_{m} c_m \\
&\geq \limsup_{m} \sup_{\phi(z)\in A_m} \sum_{k=1}^{n} \left| \frac{\partial \phi_1}{\partial z_k}(z) \right| \frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_1(z)|^2\right)^p}.
\end{aligned}
\tag{3.8}
$$

So

$$
\begin{aligned}
\|C_\phi\|_e &= \inf\{\|C_\phi - K\| : K \text{ is compact}\} \\
&\geq \limsup_{m} \sup_{\phi(z)\in A_m} \sum_{k=1}^{n} \left| \frac{\partial \phi_1}{\partial z_k}(z) \right| \frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_1(z)|^2\right)^p}.
\end{aligned}
\tag{3.9}
$$

For each $l = 1,2,\dots,n$, define

$$
a_l = \lim_{\delta\to 0} \sup_{\operatorname{dist}(\phi(z),\partial U^n)<\delta} \sum_{k=1}^{n} \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p}.
\tag{3.10}
$$

For any $\varepsilon > 0$, (3.10) shows that there exists a δ_0 with $0 < \delta_0 < 1$, such that

$$
\sum_{k=1}^{n} \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p} > a_l - \varepsilon,
\tag{3.11}
$$

whenever $\operatorname{dist}(\phi(z),\partial U^n) < \delta_0$ and $l = 1,2,\dots,n$.

Since $r_m \to 1$ as $m \to \infty$, we may choose m large enough so that $r_m > 1 - \delta_0$. If $\phi(z) \in A_m$, $r_m \leq |\phi_1(z)| \leq r_{m+1}$, so $1 - r_{m+1} < 1 - |\phi_1(z)| < 1 - r_m < \delta_0$; hence $\operatorname{dist}(\phi_1(z),\partial U) < \delta_0$. There exists w_1 with $|w_1| = 1$ such that $\operatorname{dist}(\phi_1(z),w_1) = \operatorname{dist}(\phi_1(z),\partial U) < \delta_0$.

Let $w = (w_1,\phi_2(z),\dots,\phi_n(z)) \in \partial U^n$. Then

$$
\operatorname{dist}\left(\phi(z),\partial U^n\right) \leq \operatorname{dist}\left(\phi(z),w\right) = \operatorname{dist}\left(\phi_1(z),w_1\right) < \delta_0.
\tag{3.12}
$$

By (3.11), (3.9) implies that

$$
\|C_\phi\|_e \geq a_1 - \varepsilon.
\tag{3.13}
$$

Similarly, if we choose $g_m(z) = m^{p-1}z_l^m / \|m^{p-1}z_l^m\|$, we have

$$
\|C_\phi\|_e \geq a_l - \varepsilon,
\tag{3.14}
$$

for every $l = 2\dots,n$. So

$$\begin{aligned}
||C_\phi||_e &\geq \frac{1}{n}\sum_{l=1}^{n}(a_l-\varepsilon)\\
&= \frac{1}{n}\sum_{l=1}^{n}\left(\lim_{\delta\to 0}\sup_{\operatorname{dist}(\phi(z),\partial U^n)<\delta}\sum_{k=1}^{n}\left|\frac{\partial\phi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p}-\varepsilon\right)\\
&\geq \frac{1}{n}\lim_{\delta\to 0}\sup_{\operatorname{dist}(\phi(z),\partial U^n)<\delta}\sum_{k,l=1}^{n}\left|\frac{\partial\phi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p}-\varepsilon.
\end{aligned} \tag{3.15}$$

Let $\varepsilon \to 0$, the low estimate follows.

To obtain the upper estimate we first prove the following proposition.

Proposition 3.1. *Let $\phi = (\phi_1,\dots,\phi_n)$ be a holomorphic self-map of U^n. Then for $m \geq 2$, the operator K_m on $H(U^n)$ defined by $K_m f(z) = f(((m-1)/m)z)$ has the following properties. For each $f \in H(U^n)$,*

(i) $K_m f \in \mathcal{B}_0^p(U^n) \subset \mathcal{B}_{0*}^p(U^n) \subset \mathcal{B}^p(U^n)$;
(ii) *if $C_\phi : \mathcal{B}^p(U^n) \to \mathcal{B}^q(U^n)$ is bounded, then $C_\phi K_m f \in \mathcal{B}^q(U^n)$;*
(iii) *for fixed m, the operator K_m is compact on $\mathcal{B}^p(U^n)$;*
(iv) *if $C_\phi : \mathcal{B}^p(U^n) \to \mathcal{B}^q(U^n)$ is bounded, then $C_\phi K_m f \in \mathcal{B}^q(U^n)$ is compact;*
(v) $\|I - K_m\| \leq 2$;
(vi) *$(I - K_m)f$ converges to zero uniformly on compacta in U^n.*

Proof. (i) Let $f \in H(U^n)$, $r_m = (m-1)/m$, and $f_m(z) = K_m f(z) = f(r_m z)$. First note that

$$\begin{aligned}
||f_m||_p &= |f(0)| + \sup_{z\in U^n}\sum_{k=1}^{n} r_m\left|\frac{\partial f}{\partial z_k}(r_m z)\right|\left(1-|z_k|^2\right)^p\\
&\leq |f(0)| + \sup_{z\in U^n}\sum_{k=1}^{n}\left|\frac{\partial f}{\partial z_k}(r_m z)\right|\left(1-|r_m z_k|^2\right)^p \leq \|f\|_p.
\end{aligned} \tag{3.16}$$

On the other hand, $f_m \in H((1/r_m)U^n)$, and observe that $(2/(1+r_m))\overline{U^n} \subset (1/r_m)U^n$ which implies that for fixed m, corresponding to each $j = 1,2,\dots$, there is a polynomial $P_m^{(j)}$ such that

$$\sup_{z\in(2/(1+r_m))\overline{U^n}}\left|f_m(z) - P_m^{(j)}(z)\right| < (1-r_m)^2\frac{1}{j}. \tag{3.17}$$

Let $K = \overline{U^n}$, $G = (2/(1+r_m))U^n$, $\Omega = (1/r_m)U^n$, then $K \subset G \subset \overline{G} \subset \Omega$ and $\rho = \operatorname{dist}(K,\partial G) = (1-r_m)/(1+r_m) > 0$, so for all $w \in U^n$, $k \in \{1,\dots,n\}$, it follows from

Lemma 2.9 that

$$\left|\frac{\partial\left(f_m - P_m^{(j)}\right)}{\partial w_k}(w)\right| \leq \sup_{w\in K}\left|\frac{\partial\left(f_m - P_m^{(j)}\right)}{\partial w_k}(w)\right|$$
$$\leq \frac{\sqrt{n}(1+r_m)}{1-r_m}\sup_{w\in G}\left|f_m(w) - P_m^{(j)}(w)\right| \tag{3.18}$$
$$\leq \frac{\sqrt{n}(1+r_m)}{1-r_m}\left(1-r_m^2\right)\frac{1}{j} \leq 4\sqrt{n}\frac{1}{j}.$$

Therefore

$$\sum_{k=1}^{n}\left|\frac{\partial\left(f_m - P_m^{(j)}\right)}{\partial w_k}(w)\right|\left(1-\left|w_k\right|^2\right)^p \leq 4n\sqrt{n}\frac{1}{j} \longrightarrow 0 \tag{3.19}$$

as $j \to \infty$, that is,

$$\left\|f_m - P_m^{(j)}\right\|_{\mathcal{B}^p} = \left|f_m(0) - P_m^{(j)}(0)\right| + \sup_{w\in U^n}\sum_{k=1}^{n}\left|\frac{\partial\left(f_m - P_m^{(j)}\right)}{\partial w_k}(w)\right|\left(1-\left|w_k\right|^p\right)^p \longrightarrow 0. \tag{3.20}$$

$P_m^{(j)}(w) \in \mathcal{B}_0^p(U^n)$ implies that $f_m \in \mathcal{B}_0^p(U^n)$.

(ii) follows immediately from (i).

(iii) For any sequence $\{f_j\} \subset \mathcal{B}^p(U^n)$ with $\|f_j\|_p \leq M$, by (i), $\{K_m f_j\} \in \mathcal{B}_0^p(U^n)$. By Lemma 2.8, there is a subsequence $\{f_{j_s}\}$ of $\{f_j\}$ which converges uniformly on compact subsets of U^n to a holomorphic function $f \in \mathcal{B}^p(U^n)$ and $\|f\|_p \leq M$. The sequence $\{\partial f_{j_s}/\partial z_i\}$, $i = 1,2,\ldots,n$, also converges uniformly on compact subsets of U^n to the holomorphic function $\partial f/\partial z_i$. So as s is large enough, for any $w \in E = \{((m-1)/m)z : z \in \overline{U^n}\} \subset U^n$,

$$\left|\frac{\partial(f_{j_s} - f)}{\partial w_l}(w)\right| < \varepsilon, \tag{3.21}$$

for every $l = 1,2,\ldots,n$. So

$$\|K_m f_{j_s} - K_m f\|_p = \left\|f_{j_s}\left(\frac{m-1}{m}z\right) - f\left(\frac{m-1}{m}z\right)\right\|_p$$
$$= \sup_{z\in U^n}\sum_{k=1}^{n}\left\{\left|\frac{\partial[(f_{j_s} - f)(((m-1)/m)z)]}{\partial z_k}\right|\left(1-\left|z_k\right|^2\right)^p\right\}$$
$$+ \left|f_{j_s}(0) - f(0)\right|$$

$$\leq \sup_{z\in U^n}\sum_{k=1}^{n}\sum_{l=1}^{n}\left|\frac{\partial(f_{j_s}-f)}{\partial w_l}\left(\frac{m-1}{m}z\right)\right|\frac{m-1}{m}+|f_{j_s}(0)-f(0)|$$
$$\leq n\sup_{w\in E}\frac{m-1}{m}\sum_{l=1}^{n}\left|\frac{\partial(f_{j_s}-f)}{\partial w_l}(w)\right|+|f_{j_s}(0)-f(0)|\longrightarrow 0, \tag{3.22}$$

as $s\to\infty$. This shows that $\{K_m f_{j_s}\}$ converges to $g=K_m f\in\mathcal{B}_0^p(U^n)\subset\mathcal{B}_{0*}^p(U^n)\subset\mathcal{B}^p(U^n)$. So K_m is compact on $\mathcal{B}^p(U^n)$.

(iv) follows immediately from (i) and (iii).

(v) follows from the fact that for any $f\in\mathcal{B}^p(U^n)$, $(I-K_m)f(0)=0$, so

$$\begin{aligned}
\|(I-K_m)f\|_p &= \sup_{z\in U^n}\sum_{k=1}^{n}\left|\frac{\partial(I-K_m)f}{\partial z_k}(z)\right|\left(1-|z_k|^2\right)^p\\
&= \sup_{z\in U^n}\sum_{k=1}^{n}\left|\frac{\partial f}{\partial z_k}(z)-\left(1-\frac{1}{m}\right)\frac{\partial f}{\partial z_k}\left(\left(1-\frac{1}{m}\right)z\right)\right|\left(1-|z_k|^2\right)^p\\
&\leq \sup_{z\in U^n}\sum_{k=1}^{n}\left|\frac{\partial f}{\partial z_k}(z)\right|\left(1-|z_k|^2\right)^p\\
&\quad+\left(1-\frac{1}{m}\right)\sup_{z\in U^n}\sum_{k=1}^{n}\left|\frac{\partial f}{\partial z_k}\left(\left(1-\frac{1}{m}\right)z\right)\right|\left(1-\left|\left(1-\frac{1}{m}\right)z_k\right|^2\right)^p\\
&\leq \|f\|_p+\|f\|_p=2\|f\|_p,
\end{aligned} \tag{3.23}$$

so $\|I-K_m\|\leq 2$.

(vi) For any compact subset $E\subset U^n$, there exists r, $0<r<1$ such that $E\subset rU^n\subset r\overline{U^n}\subset U^n$. For all $z\in E$,

$$\begin{aligned}
|(I-K_m)f(z)| &= |f(z)-f_m(z)| = |f(r)-f(r_m z)|\\
&\leq \sum_{k=1}^{n}\int_{r_m}^{1}\left|\frac{\partial f}{\partial w_k}(tz)\right|dt.
\end{aligned} \tag{3.24}$$

For $t\in[r_m,1]$ and $z\in E$, we have $|tz_k|=t|z_k|\leq|z_k|<r$, $tz\in rU^n$, so there exists $M>0$ such that $|(\partial f/\partial w_k)(tz)|\leq M$ for all $t\in[r_m,1]$ and $z\in E$. Thus

$$|(I-K_m)f(z)|\leq nM(1-r_m)\longrightarrow 0 \tag{3.25}$$

as $m\to\infty$, proving the results in Theorem 1.1.

Let us now return to the proof of the upper estimate. For convenience, we remove the subscript p from $\|f\|_p$,

$$\begin{aligned}
\|C_\phi\|_e &\le \|C_\phi - C_\phi K_m\| = \|C_\phi(I-K_m)\| = \sup_{\|f\|=1} \|C_\phi(I-K_m)f\|_q \\
&= \sup_{\|f\|=1}\left(\sup_{z\in U^n}\sum_{k=1}^n\left\{\left|\frac{\partial(I-K_m)(f\circ\phi)}{\partial z_k}\right|\left(1-|z_k|^2\right)^q\right\} + |(I-K_m)f(\phi(0))|\right) \\
&\le \sup_{\|f\|=1}\sup_{z\in U^n}\sum_{k=1}^n\sum_{l=1}^n\left|\frac{\partial(I-K_m)f}{\partial w_l}(\phi(z))\right|\left|\frac{\partial\phi_l}{\partial z_k}(z)\right|\left(1-|z_k|^2\right)^q \\
&\quad + \sup_{\|f\|=1}\left|f(\phi(0)) - f\left(\frac{m-1}{m}\phi(0)\right)\right| \\
&\le \sup_{\|f\|=1}\sup_{z\in U^n}\sum_{k,l=1}^n\left|\frac{\partial\phi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p}\left|\frac{\partial(I-K_m)f}{\partial w_l}(\phi(z))\right|\left(1-|\phi_l(z)|^2\right)^p \\
&\quad + \sup_{\|f\|=1}\left|f(\phi(0)) - f\left(\frac{m-1}{m}\phi(0)\right)\right|.
\end{aligned} \tag{3.26}$$

Fix $\delta>0$, let $G_1=\{z\in U^n : \operatorname{dist}(\phi(z),\partial U^n)<\delta\}$, $G_2=\{z\in U^n : \operatorname{dist}(\phi(z),\partial U^n)\ge\delta\}$, $G=\{w\in U^n : \operatorname{dist}(w,\partial U^n)\ge\delta\}$, and observe that G is a compact subset of $\mathbb{C}^n$.

Then by Lemmas 2.3, 2.4, and 2.6, and by Proposition 3.1, we deduce

$$\begin{aligned}
\|C_\phi\|_e &\le \sup_{\|f\|=1}\sup_{z\in G_1}\sum_{k,l=1}^n\left|\frac{\partial\phi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p}\left|\frac{\partial(I-K_m)f}{\partial w_l}(\phi(z))\right|\left(1-|\phi_l(z)|^2\right)^q \\
&\quad + C\sup_{\|f\|=1}\sup_{z\in G_2}\sum_{l=1}^n\left(1-|\phi_l(z)|^2\right)^p\left|\frac{\partial(I-K_m)f}{\partial w_l}(\phi(z))\right| \\
&\quad + \sup_{\|f\|=1}\left|f(\phi(0)) - f\left(\frac{m-1}{m}\phi(0)\right)\right| \\
&\le \|I-K_m\|\sup_{z\in G_1}\sum_{k,l=1}^n\left|\frac{\partial\phi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p} \\
&\quad + C\sup_{\|f\|=1}\sup_{z\in G_2}\sum_{l=1}^n\left(1-|\phi_l(z)|^2\right)^p\left|\frac{\partial(I-K_m)f}{\partial w_l}(\phi(z))\right| \\
&\quad + \sup_{\|f\|=1}\left|f(\phi(0)) - f\left(\frac{m-1}{m}\phi(0)\right)\right|
\end{aligned}$$

$$\le 2\sup_{z\in G_1}\sum_{k,l=1}^{n}\left|\frac{\partial\phi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p}$$
$$+C\sup_{\|f\|=1}\sup_{z\in G_2}\sum_{l=1}^{n}\left(1-|\phi_l(z)|^2\right)^p\left|\frac{\partial(I-K_m)f}{\partial w_l}(\phi(z))\right|$$
$$+\sup_{\|f\|=1}\left|f(\phi(0))-f\left(\frac{m-1}{m}\phi(0)\right)\right|. \tag{3.27}$$

Denoting the second term and third term of the right-hand side of (3.27) by I_1 and I_2, then Theorem 1.1 is proved if we can prove

$$\lim_{m\to\infty} I_1=0,\qquad \lim_{m\to\infty} I_2=0. \tag{3.28}$$

To do this, let $z\in G_2$ and $w=\phi(z)\in G$. Then

$$\begin{aligned}
I_1 &\le C\sup_{\|f\|=1}\sup_{w\in G}\sum_{l=1}^{n}\left(1-|w_l|^2\right)^p\left|\frac{\partial f}{\partial w_l}(w)-\left(1-\frac{1}{m}\right)\frac{\partial f}{\partial w_l}\left(\left(1-\frac{1}{m}\right)w\right)\right|\\
&\le C\sup_{\|f\|=1}\sup_{w\in G}\sum_{l=1}^{n}\left(1-|w_l|^2\right)^p\left|\frac{\partial f}{\partial w_l}(w)-\frac{\partial f}{\partial w_l}\left(\left(1-\frac{1}{m}\right)w\right)\right|\\
&\quad+\frac{C}{m}\sup_{\|f\|=1}\sup_{w\in G}\sum_{l=1}^{n}\left(1-|w_l|^2\right)^p\left|\frac{\partial f}{\partial w_l}\left(\left(1-\frac{1}{m}\right)w\right)\right|\\
&\le C\sup_{\|f\|=1}\sup_{w\in G}\sum_{l=1}^{n}\left(1-|w_l|^2\right)^p\left|\frac{\partial f}{\partial w_l}(w)-\frac{\partial f}{\partial w_l}\left(\left(1-\frac{1}{m}\right)w\right)\right|+\frac{C}{m}.
\end{aligned} \tag{3.29}$$

Letting $w=(w_1,w_2,\dots,w_{n-1},w_n)$, for m large enough, we have

$$\begin{aligned}
&\left|\frac{\partial f}{\partial w_l}(w)-\frac{\partial f}{\partial w_l}\left(\left(1-\frac{1}{m}\right)w\right)\right|\\
&\quad\le\sum_{j=1}^{n}\left|\frac{\partial f}{\partial w_l}\left(\left(1-\frac{1}{m}\right)w_1,\dots,\left(1-\frac{1}{m}\right)w_{j-1},w_j,\dots,w_n\right)\right.\\
&\qquad\qquad\left.-\frac{\partial f}{\partial w_l}\left(\left(1-\frac{1}{m}\right)w_1,\dots,\left(1-\frac{1}{m}\right)w_j,w_{j+1},\dots,w_n\right)\right|\\
&\quad=\sum_{j=1}^{n}\left|\int_{(1-(1/m))w_j}^{w_j}\frac{\partial^2 f}{\partial w_l\partial w_j}\left(\left(1-\frac{1}{m}\right)w_1,\dots,\left(1-\frac{1}{m}\right)w_{j-1},\zeta,w_{j+1},\dots,w_n\right)d\zeta\right|\\
&\quad\le\frac{1}{m}\sum_{j=1}^{n}\sup_{w\in G}\left|\frac{\partial^2 f}{\partial w_l\partial w_j}(w)\right|.
\end{aligned} \tag{3.30}$$

Denote G_3 by the set $\{w\in U^n:\operatorname{dist}(w,\partial U^n)>\delta/2\}$. Then $G\subset G_3\subset\overline{G_3}\subset U^n$.

Since $\mathrm{dist}(G,\partial G_3) = \delta/2$, then by Lemma 2.9, (3.30) gives

$$\left|\frac{\partial f}{\partial w_l}(w) - \frac{\partial f}{\partial w_l}\left(\left(1-\frac{1}{m}\right)w\right)\right| \le \frac{2n\sqrt{n}}{m\delta}\max_{z\in G_3}\left|\frac{\partial f}{\partial w_l}(w)\right|. \tag{3.31}$$

On the other hand, on the unit ball of $\mathcal{B}^p(U^n)$, we have

$$\sup_{z\in G_3}\left(1-|w_l|^2\right)^p\left|\frac{\partial f}{\partial w_l}(w)\right| = \sup_{\mathrm{dist}(w,\partial U^n)>\delta/2}\left(1-|w_l|^2\right)^p\left|\frac{\partial f}{\partial w_l}(w)\right| \le \|f\|_p = 1, \tag{3.32}$$

namely,

$$\sup_{z\in G_3}\left|\frac{\partial f}{\partial w_l}(w)\right| \le \frac{1}{\left(1-(\delta/2)^2\right)^p} = \frac{4^p}{(4-\delta^2)^p}. \tag{3.33}$$

Combining (3.29), (3.31), and (3.33)), it follows that

$$I_1 \le \frac{2n\sqrt{n}C}{m\delta}\frac{4^p}{\left(4-\delta^2\right)^p} + \frac{C}{m} \tag{3.34}$$

and $\lim_{m\to\infty} I_1 = 0$.

Now we can prove $\lim_{m\to\infty} I_2 = 0$. In fact,

$$f(\phi(0)) - f\left(\frac{m-1}{m}\phi(0)\right) = \int_{(m-1)/m}^1 \frac{df(t\phi(0))}{dt}dt = \sum_{l=1}^n\int_{(m-1)/m}^1 \phi_l(0)\frac{\partial f}{\partial \zeta_l}(t\phi(0))dt. \tag{3.35}$$

By Lemma 2.1, it follows that for any compact subset $K \subset U^n$, $|f(z)| \le C_K\|f\|_p = C_K$. Let $K = \{z \in U^n : |z_i| \le |\phi_i(0)|,\ i = 1,\dots,n\}$, So

$$\left|f(\phi(0)) - f\left(\frac{m-1}{m}\phi(0)\right)\right| \le \sum_{l=1}^n |\phi_l(0)|\int_{(m-1)/m}^1 C_K dt \le nC_K\left(1-\frac{m-1}{m}\right) = \frac{nC_K}{m}, \tag{3.36}$$

so $I_2 \le nC_K/m \to 0$. Thus letting first $m \to \infty$ and then $\delta \to 0$ in (3.27), we get the upper estimate of $\|C_\phi\|_e$:

$$\|C_\phi\|_e \le 2\lim_{\delta\to 0}\sup_{\mathrm{dist}(\phi(z),\partial U^n)<\delta}\sum_{k,l=1}^n\left|\frac{\partial \phi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p}. \tag{3.37}$$

Now the proof of Theorem 1.1 is finished. □

4. Some corollaries

The following three corollaries follow from Theorem 1.2.

Corollary 4.1. *Let* $\phi = (\phi_1,\dots,\phi_n)$ *be a holomorphic self-map of* U^n. *Then* $C_\phi : \mathcal{B}^p(U^n)(\mathcal{B}_0^p(U^n)$ *or* $\mathcal{B}_{0*}^p(U^n)) \to \mathcal{B}^q(U^n)$ *is compact if and only if*

$$\sum_{k,l=1}^{n} \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \frac{\left(1-|z_k|^2\right)^q}{\left(1-|\phi_l(z)|^2\right)^p} \le C \tag{4.1}$$

for all $z \in U^n$ *and (1.12) holds.*

Proof. By Lemma 2.3, we know $C_\phi : \mathcal{B}^p(U^n)(\mathcal{B}_0^p(U^n)$ or $\mathcal{B}_{0*}^p(U^n)) \to \mathcal{B}^q(U^n)$ is bounded. It follows from Theorem 1.2 that $C_\phi : \mathcal{B}^p(U^n)(\mathcal{B}_0^p(U^n)$ or $\mathcal{B}_{0*}^p(U^n)) \to \mathcal{B}^q(U^n)$ is compact.

Conversely, if $C_\phi : \mathcal{B}^p(U^n)(\mathcal{B}_0^p(U^n)$ or $\mathcal{B}_{0*}^p(U^n)) \to \mathcal{B}^q(U^n)$ is compact, it is clear that $C_\phi : \mathcal{B}^p(U^n)(\mathcal{B}_0^p(U^n)$ or $\mathcal{B}_{0*}^p(U^n)) \to \mathcal{B}^q(U^n)$ is bounded, by Theorem 1.2, (1.12) holds. □

Corollary 4.2. *Let* $\phi = (\phi_1,\dots,\phi_n)$ *be a holomorphic self-map of* U^n. *Then* $C_\phi : \mathcal{B}_{0*}^p(U^n)(\mathcal{B}_0^p(U^n)) \to \mathcal{B}_{0*}^q(U^n)$ *is compact if and only if* $\phi_l \in \mathcal{B}_{0*}^q(U^n)$ *for every* $l = 1, 2,\dots,n$ *and (1.12) holds.*

The proof follows from Lemma 2.4.

Corollary 4.3. *Let* $\phi = (\phi_1,\dots,\phi_n)$ *be a holomorphic self-map of* U^n. *Then* $C_\phi : \mathcal{B}_0^p(U^n) \to \mathcal{B}_0^q(U^n)$ *is compact if and only if* $\phi_l \in \mathcal{B}_0^q(U^n)$ *for every* $l = 1,2,\dots,n$ *and (1.12) holds.*

The proof follows from Lemma 2.6.

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